From earthquakes to understanding financial risks via football: around statistical modeling with Hawkes processes

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Definition

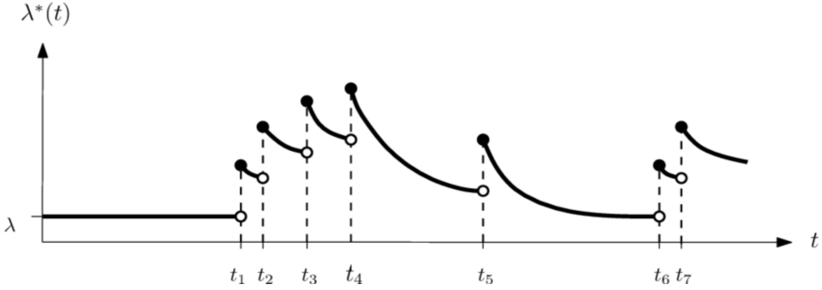
Hawkes process

- A Hawkes process $(N_t)_{t \geq 0}$ is a self-exciting point process, whose intensity at time t, denoted by λ_t, is of the form

$$\lambda_t = \mu + \sum_{0 < J_i < t} \phi(t - J_i) = \mu + \int_{(0,t)} \phi(t - s) dN_s,$$

  where μ is a positive real number, ϕ a regression kernel and the J_i are the points of the process before time t.
- These processes have been introduced in 1971 by Hawkes in the purpose of modeling earthquakes and their aftershocks.
- Used in neurosciences, network analysis, criminology...
- First introduction in finance : Chavez-Demoulin *et al.* (2005), Bowsher (2007).

Example of Hawkes process intensity



Popularity of Hawkes processes in finance

Two main reasons for the popularity of Hawkes processes

- In finance for example, it is nowadays classical to model the order flow (number of trades) thanks to Hawkes processes.
- These processes represent a very natural and tractable extension of Poisson processes. In fact, comparing point processes and conventional time series, Poisson processes are often viewed as the counterpart of iid random variables whereas Hawkes processes play the role of autoregressive processes.
- Another explanation for the appeal of Hawkes processes is that it is often easy to give a convincing interpretation to such modelling. To do so, the branching structure of Hawkes processes is quite helpful.

Hawkes processes as a population model

Poisson cluster representation

- Under the assumption $\|\phi\|_1 < 1$, where $\|\phi\|_1$ denotes the L^1 norm of ϕ, Hawkes processes can be represented as a population process where migrants arrive according to a Poisson process with parameter μ.
- Then each migrant gives birth to children according to a non homogeneous Poisson process with intensity function ϕ, these children also giving birth to children according to the same non homogeneous Poisson process, see Hawkes (74).
- Now consider for example the classical case of buy (or sell) market orders. Then migrants can be seen as exogenous orders whereas children are viewed as orders triggered by other orders.

Stability condition

The condition $\|\phi\|_1 < 1$

- The assumption $\|\phi\|_1 < 1$ is crucial in the study of Hawkes processes.
- If one wants to get a stationary intensity with finite first moment, then the condition $\|\phi\|_1 < 1$ is required (similar condition as for the AR(1) process).
- This condition is also necessary in order to obtain classical ergodic properties for the process.
- For these reasons, this condition is often called stability condition in the Hawkes literature.

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Main classes of volatility models

Prices are often modeled as continuous semi-martingales of the form

$$dP_t = P_t(\mu_t dt + \sigma_t dW_t).$$

The volatility process σ_s is the most important ingredient of the model. Practitioners consider essentially three classes of volatility models :

- Deterministic volatility (Black and Scholes 1973),
- Local volatility (Derman and Kani, Dupire 1994)
- Stochastic volatility (Hull and White 1987, Heston 1993, Hagan et al. 2002,...).

In term of regularity, in these models, the volatility is either very smooth or with a smoothness similar to that of a Brownian motion.

Fractional Brownian motion (I)

- To allow for a wider range of smoothness, one can use the fractional Brownian motion in volatility modelling.
- Idea introduced by Comte and Renault in 1998 in the context of long memory modelling with $H > 1/2$.

Definition

The fractional Brownian motion (fBm) with Hurst parameter H is the only process W^H to satisfy :

- Self-similarity : $(W^H_{at}) \overset{\mathcal{L}}{=} a^H (W^H_t)$.
- Stationary increments : $(W^H_{t+h} - W^H_t) \overset{\mathcal{L}}{=} (W^H_h)$.
- Gaussian process with $\mathbb{E}[W^H_1] = 0$ and $\mathbb{E}[(W^H_1)^2] = 1$.

Fractional Brownian motion (II)

Proposition

For all $\varepsilon > 0$, W^H is $(H - \varepsilon)$-Hölder a.s.

Proposition

The absolute moments satisfy

$$\mathbb{E}[|W_{t+h}^H - W_t^H|^q] = K_q h^{Hq}.$$

Mandelbrot-van Ness representation

$$W_t^H = \int_0^t \frac{dW_s}{(t-s)^{\frac{1}{2}-H}} + \int_{-\infty}^0 \left(\frac{1}{(t-s)^{\frac{1}{2}-H}} - \frac{1}{(-s)^{\frac{1}{2}-H}} \right) dW_s.$$

The log-volatility

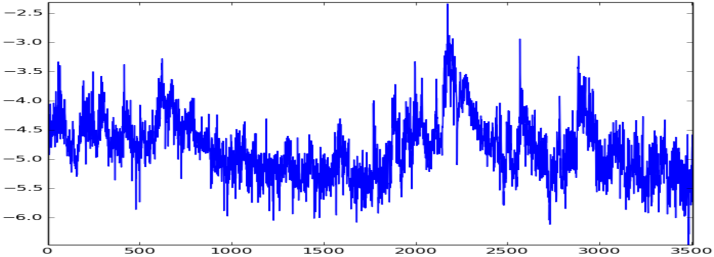

FIGURE – The log volatility $\log(\sigma_t)$ as a function of t, S&P.

Measure of the regularity of the log-volatility

The starting point of this work is to consider the scaling of the moments of the increments of the log-volatility. Thus we study the quantity

$$m(\Delta, q) = \mathbb{E}[|\log(\sigma_{t+\Delta}) - \log(\sigma_t)|^q],$$

or rather its empirical counterpart.

The behavior of $m(\Delta, q)$ when Δ is close to zero is related to the smoothness of the volatility (in the Hölder or even the Besov sense). Essentially, the regularity of the signal measured in l^q norm is s if $m(\Delta, q) \sim c\Delta^{qs}$ as Δ tends to zero.

Scaling of the moments

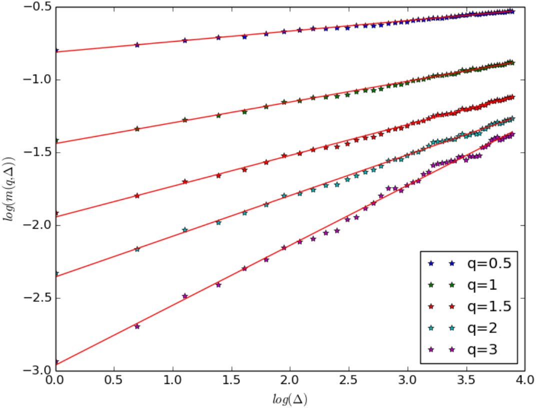

FIGURE – $\log(m(q,\Delta)) = \zeta_q \log(\Delta) + C_q$. The scaling is not only valid as Δ tends to zero, but holds on a wide range of time scales.

Monofractality of the log-volatility

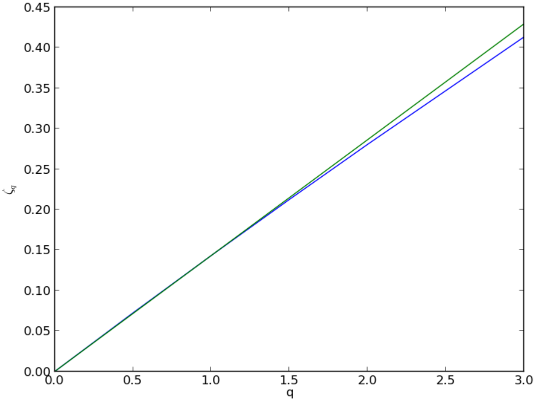

FIGURE – Empirical ζ_q and $q \to Hq$ with $H = 0.14$ (similar to a fBm with Hurst parameter H).

Distribution of the log-volatility increments

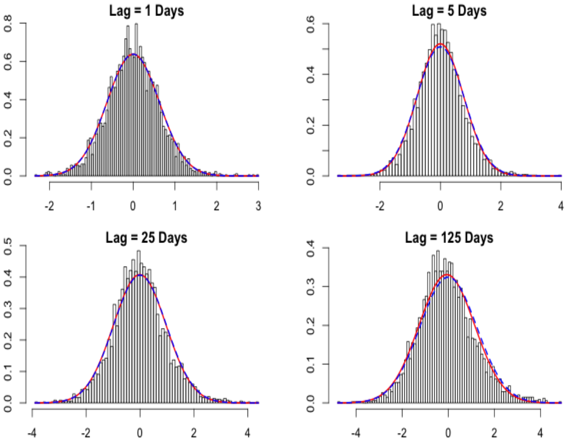

FIGURE – The distribution of the log-volatility increments is close to Gaussian.

Properties of the rough volatility models

Statistical analysis of rough volatility models

- The log-volatility behaves essentially as a fractional Brownian motion with Hurst parameter of order 0.1.
- More precisely, basically all the statistical stylized facts of volatility are retrieved when modelling it by a rough fractional Brownian motion.
- Such model also enables us to reproduce very well the behavior of the implied volatility surface, in particular the at-the-money skew (without jumps).
- Very relevant approach for risk management of derivatives.
- The phenomenon is universal, see G. Szymanski's thesis for theoretical foundations of the statistical analysis of rough volatility.

In this presentation

What we want to understand :

- Why is volatility rough ?
- Something universal in finance$\rightarrow$ should be related to some no arbitrage concept.
- Can we make this link ?

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Market impact

Some definitions

- Market impact is the link between the volume of an order (either market order or metaorder) and the price moves during and after the execution of this order.
- We focus here on the impact function of metaorders, which is the expectation of the price move with respect to time during and after the execution of the metaorder.
- We call permanent market impact of a metaorder the limit in time of the impact function (that is the average price move between the start of the metaorder and a long time after its execution).

Market impact in practice

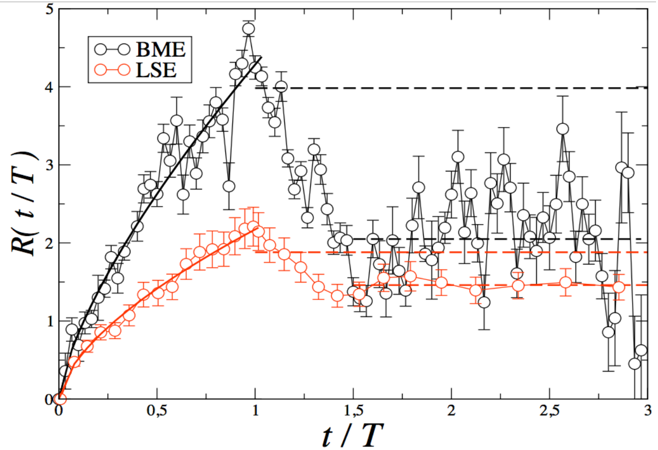

FIGURE – Market impact curves.

Market impact

Linear permanent impact

- Let P_t be the asset price at time t. Consider a metaorder with total volume V.
- $$PMI(V) = \lim_{s \to +\infty} \mathbb{E}[P_s - P_0 | V].$$
- Price manipulation is a roundtrip with negative average cost.
- From Huberman and Stanzl and Gatheral : Only linear permanent market impact can prevent price manipulation : $PMI(V) = kV$.

Dynamics

Assumptions

- All market orders are part of metaorders.
- Let $[0, S]$ be the time during which metaorders are being executed (which can be thought of as the trading day). Let v_i^a (resp. v_i^b) be the volume of the i-th buy (resp. sell) metaorder and N_S^a (resp. N_S^b) be the number of buy (resp. sell) metaorders up to time S. Finally, write V_S^a and V_S^b for cumulated buy and sell order flows up to time S.
- We assume

$$P_S = P_0 + k\Big(\sum_{i=1}^{N_S^a} v_i^a - \sum_{i=1}^{N_S^b} v_i^b\Big) + Z_S = P_0 + k(V_S^a - V_S^b) + Z_S,$$

with Z a martingale term that we neglect.

Dynamics

Martingale assumption

- We furthermore assume that the price P_t is a martingale. We obtain

$$P_t = P_0 + \mathbb{E}\big[k(V_S^a - V_S^b)|F_t\big].$$

- We suppose that $\lim_{S \to +\infty} \mathbb{E}\big[k(V_S^a - V_S^b)|F_t\big]$ is well defined. This means

$$\mathbb{E}\big[(V_{S+h}^a - V_{S+h}^b) - (V_S^a - V_S^b)|F_t\big] \to 0,$$

  that is the order flow imbalance between S and $S+h$ is asymptotically (in S) not predictable at time t.

Dynamics

Price dynamics

- Under the preceding assumptions, we finally get

$$P_t = P_0 + k \lim_{S \to +\infty} \mathbb{E}\big[(V_S^a - V_S^b)|F_t\big].$$

- Martingale price.
- Linear permanent impact, independent of execution mode.
- The price process only depends on the global market order flow and not on the individual executions of metaorders. We thus do not need to assume that the market sees the execution of metaorders as it is usually done.
- Market orders move the price because they change the anticipation that market makers have about the future of the order flow.

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Hawkes specification

Hawkes propagator

- We now assume that buy and sell order flows are modeled by independent Hawkes processes N^a and N^b with same parameters μ and ϕ. All orders have same unit volume.
- Later on we will consider an asymptotic setting so that the flows are defined on $[0, T]$ with $T \to +\infty$.
- To be very general, we allow the parameters to depend on T (but do not assume they do). So we write $N^{a,T}$, $N^{b,T}$, μ^T, $\phi^T = a^T \phi$ with $a^T < 1$ and $\int \phi = 1$ (stability condition).
- Note that the average intensity of our processes is essentially $\beta^T = \mu^T (1 - a^T)^{-1}$ (stationary case).

Price dynamic under Hawkes specification

Price equation

- In this case, the general equation above rewrites as the following propagator dynamic

$$P_t = P_0 + \int_0^t \zeta^T(t-s)(dN_s^{a,T} - dN_s^{b,T}),$$

with $\zeta^T(t) = \big(1 + \int_t^{+\infty} \psi^T(u) - \int_0^t \psi^T(u-s)\phi^T(s)dsdu\big)$.

- The propagator kernel compensates the correlation of the order flow implied by the Hawkes dynamics to recover a martingale price. Note that the kernel does not tend to 0 since there is permanent impact.

Adding our own transactions

Labeled order

- In the above framework, $N^{a,T}$ and $N^{b,T}$ are the flows of anonymous market orders.
- Now assume we arrive on the market, executing our own (buy) metaorder. Our flow is a Poisson process n on $[0, T]$ (can be generalized) with intensity $I^T = \gamma\beta^T$, $\gamma < 1$ (proportion γ of the total flow).
- According to the propagator approach, we get

$$P_t = P_0 + \int_0^t \zeta^T(t-s)(dN_s^{a,T} - dN_s^{b,T}) + \int_0^t \zeta^T(t-s)dn_s.$$

Impact function

Explicit market impact

- We get that the impact function of a metaorder executed between 0 and T is for $0 \leq t \leq T$

$$MI(t) := \mathbb{E}[P_t - P_0] = I^T \int_0^t \zeta^T(t-s)ds.$$

- We define

$$\overline{MI}^T(t) = \frac{MI_{tT}^T}{T\beta^T} = \int_0^t \chi^T(t-s)\mathrm{d}s,$$

with

$$\chi^T(s) = \gamma \frac{\zeta^T(Ts)}{1-a^T}.$$

Decomposing the impact

Transient and permanent market impact

- We have

$$\overline{MI}^T(t) = \int_0^t \chi^T(t-s)\mathrm{d}s,$$

$$\chi^T(s) = \gamma\big(1 + (1-a^T)^{-1}\int_{Ts}^{+\infty}\phi\big).$$

- The market impact kernel is the sum of a linear market impact representing the permanent component and of a transient term vanishing after the metaorder completion.
- Existence of transient part is equivalent (asymptotically) to the existence of a limit for $(1-a^T)^{-1}\int_0^t\int_{T(t-s)}^{+\infty}\phi(u)\mathrm{d}u\mathrm{d}s$.

Shape of the market impact

Power-law market impact

Assume the transient part of the market impact exists. Then for $t < 1$,

$$\lim_{T \to +\infty} \overline{MI}^T(t) - \gamma t = \gamma K t^{1-\alpha}$$

for some $K > 0$ and $\alpha \in (0,1)$. Furthermore, we necessarily have $a^T \to 1$ (highly endogenous market) and the tail of the Hawkes kernel is power-law of order $x^{-(1+\alpha)}$.

Note that the celebrated square-root law (Bouchaud et al., Farmer et al., Pohl et al.) corresponds to $\alpha = 1/2$.

Market impact decomposition

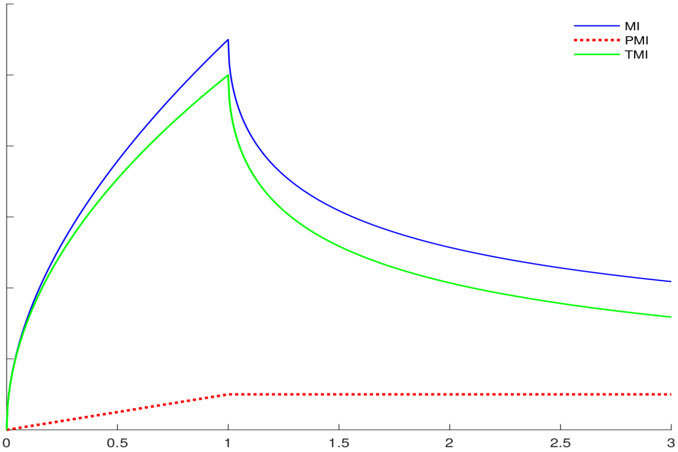

FIGURE – Permanent and temporary market impact

Limiting price process

Emergence of (hyper-)rough processes

Let $\bar{P}_t^T = \frac{1}{T\beta^T} P_t^T$ and assume $\mu^T(1-a^T)T$ tends to δ. As T goes to infinity, the limit P_t of $\bar{P}_t^T$ satisfies

$$P_t = B_{X_t},$$

$$X_t = \frac{2}{\delta}\int_0^t F^{\alpha,\lambda}(s)\mathrm{d}s + \frac{1}{\delta\sqrt{\lambda}}\int_0^t F^{\alpha,\lambda}(t-s)\mathrm{d}W_{X_s},$$

where B and W are Brownian motions, $\lambda = K\Gamma(1-\alpha)^{-1}$ and $F^{\alpha,\lambda}(t) = \int_0^t f^{\alpha,\lambda}(s)\mathrm{d}s$ with $f^{\alpha,\lambda}$ the density of the Mittag-Leffler distribution. Furthermore, X has Hölder regularity $\min(2\alpha, 1) - \varepsilon$.

The case $\alpha > 1/2$

Rough Heston limit

When $\alpha > \frac{1}{2}$, the rescaled price process variance is almost surely differentiable. Furthermore

$$P_t = \int_0^t \sqrt{Y_s}\mathrm{d}B_s,$$

$$Y_t = \frac{\lambda}{\Gamma(\alpha)}\Big(\int_0^t (t-s)^{\alpha-1}\big(\frac{2}{\delta} - \lambda Y_s\big)\mathrm{d}s + \frac{1}{\delta\sqrt{\lambda}}\int_0^t (t-s)^{\alpha-1}\sqrt{Y_s}\mathrm{d}W_s\Big).$$

Therefore we have a rough Heston model with $H = \alpha - 1/2$.

Characteristic function of rough Heston models

We write :

$$I^{1-\alpha}f(x) = \frac{1}{\Gamma(1-\alpha)}\int_0^x \frac{f(t)}{(x-t)^\alpha}dt, \;\; D^\alpha f(x) = \frac{d}{dx}I^{1-\alpha}f(x).$$

Theorem

The characteristic function at time t for the rough Heston model is given by

$$\exp\Big(\int_0^t g(a,s)ds + \frac{V_0}{\theta\lambda}I^{1-\alpha}g(a,t)\Big),$$

with $g(a,)$ the unique solution of the fractional Riccati equation :

$$D^\alpha g(a,s) = \frac{\lambda\theta}{2}(-a^2 - ia) + \lambda(ia\rho\nu - 1)g(a,s) + \frac{\lambda\nu^2}{2\theta}g^2(a,s).$$

Comments on the theorem

The rough Heston formula

- The formula is the very same as the celebrated Heston formula, up to the replacement of a classical time derivative by a fractional derivative.
- This formula allows for fast derivatives pricing and risk management.
- Thanks to this approach, we can derive the infinite dimensional Markovian structure underlying rough Heston models, leading to explicit hedging formulas.
- Other probabilistic aspects of rough volatility models : Large and moderate deviation principles, connections with Hairer's regularity structures : works by P. Gassiat and co-authors. Also links with log-correlated Gaussian field.

Summary

From no-arbitrage to volatility

- We made two assumptions : Linear permanent impact and martingale price.
- Only modeling assumption : Hawkes dynamics for the order flow (reasonable...).
- This leads to rough volatility.

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Further aspects of volatility

Zumbach effect (Zumbach *et al.*) : description

- Feedback of price returns on volatility.
- Price trends induce an increase of volatility.
- In the literature (notably works by J.P. Bouchaud and co-authors), a way to reinterpret the Zumbach effect is to consider that the predictive power of past squared returns on future volatility is stronger than that of past volatility on future squared returns.
- To check this on data, one typically shows that the covariance between past squared price returns and future realized volatility (over a given duration) is larger than that between past realized volatility and future squared price returns.
- We refer to this version of Zumbach effect as *weak Zumbach effect*.

Further aspects of volatility

Weak and strong Zumbach effect

- It is shown in Gatheral *et al.* that the rough Heston model reproduces the weak form of Zumbach effect.
- However, it is not obtained through feedback effect, which is the motivating phenomenon in the original paper by Zumbach. It is only due to the dependence between price and volatility induced by the correlation of the Brownian motions driving their dynamics.
- In particular in the rough Heston model, the conditional law of the volatility depends on the past dynamic of the price only through the past volatility.
- We speak about *strong Zumbach effect* when the conditional law of future volatility depends not only on past volatility trajectory but also on past returns.

A convenient microscopic model encoding Zumbach effect

Quadratic Hawkes processes

- Inspired by Blanc *et al.*, we model high frequency prices using quadratic Hawkes processes.
- Jump sizes of the price P_t are i.i.d taking values -1 and 1 with probability $1/2$ and jump times are those of a point process N_t with intensity

$$\lambda_t = \mu + \int_0^t \phi(t-s)\mathrm{d}N_s + Z_t^2, \text{ with } Z_t = \int_0^t k(t-s)\mathrm{d}P_s.$$

- The component Z_t is a moving average of past returns.
- If the price has been trending in the past, Z_t is large leading to high intensity. On the contrary if it has been oscillating, Z_t is close to zero and there is no feedback from the returns on the volatility. So Z_t is a (strong) Zumbach term.

One particular scaling limit

Quadratic rough Heston model

$$\mathrm{d}S_t = S_t \sqrt{V_t} dW_t, \;\; V_t = a(Z_t - b)^2 + c,$$

where a, b and c some positive constants and Z_t follows

$$Z_t = \int_0^t f^{\alpha,\lambda}(t-s)\theta_0(s)\mathrm{d}s + \int_0^t f^{\alpha,\lambda}(t-s)\sqrt{V_s}\mathrm{d}W_s,$$

with $\alpha \in (1/2, 1)$, $\lambda > 0$ and θ_0 a deterministic function.

- Z_t is *path-dependent* : a weighted average of past returns.
- c : minimal instantaneous variance.
- $b > 0$: asymmetry of the feedback effect.
- a : sensitivity of the volatility feedback.
- **A log-normal rough volatility model with strong Zumbach effect.**

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The Volatility Index

Definition of the VIX

- Introduced in 1993 by the CBOE.
- VIX is the square root of the price of a specific basket of options on the S&P 500 Index (SPX) with maturity $\Delta = 30$ days such that

$$\mathrm{VIX}_t = \frac{2}{\Delta}\sqrt{-\mathbb{E}[\log(S_{t+\Delta}/S_t)|\mathcal{F}_t]} \times 100,$$

  with S the SPX index.
- VIX futures and VIX options exist.

The joint calibration problem

VIX options

- More than 500,000 VIX options traded each day.
- Quite wide spreads for VIX options : non-mature market.
- VIX is by definition a derivative of the SPX, any reasonable methodology must necessarily be consistent with the pricing of SPX options.
- Designing a model that jointly calibrates SPX and VIX options prices is known to be extremely challenging.
- This problem is sometimes considered to be *the holy grail of volatility modeling*.
- We simply refer to it as the *joint calibration problem*.

The joint calibration problem

Attempts to solve the joint calibration problem

- Theoretical approch by J. Guyon : the joint calibration problem is interpreted as a model-free constrained martingale transport problem. Perfect calibration of VIX options smile at time T_1 and SPX options smiles at T_1 and $T_2 = T_1 + 30$ days. Hard to be extended to any set of maturities and high computational cost.
- Models with jumps : most of them fail to reproduce VIX smiles for maturities shorter than one month.
- Continuous models : Unsuccessful so far. Interpretation : the very large negative skew of short-term SPX options, which in continuous models implies a very large volatility of volatility, seems inconsistent with the comparatively low levels of VIX implied volatilities

The VIX conjecture

The joint calibration problem and continuous models

- "So far all the attempts at solving the joint SPX/VIX smile calibration problem [using a continuous time model] only produced imperfect, approximate fits".
- "Joint calibration seems out of the reach of continuous-time models with continuous SPX paths".
- Investigating Guyon's work one can realise the following : a necessary condition for a continuous model to fit simultaneously SPX and VIX smiles is the inversion of convex ordering between volatility and the local volatility implied by option prices.
- The intuition behind this condition could be reinterpreted as some kind of strong Zumbach effect.
- Natural for us to investigate the ability of super-Heston rough volatility models to solve the joint calibration problem.

Calibration for one day in history 19 May 2017

Parameters calibration with Deep Learning

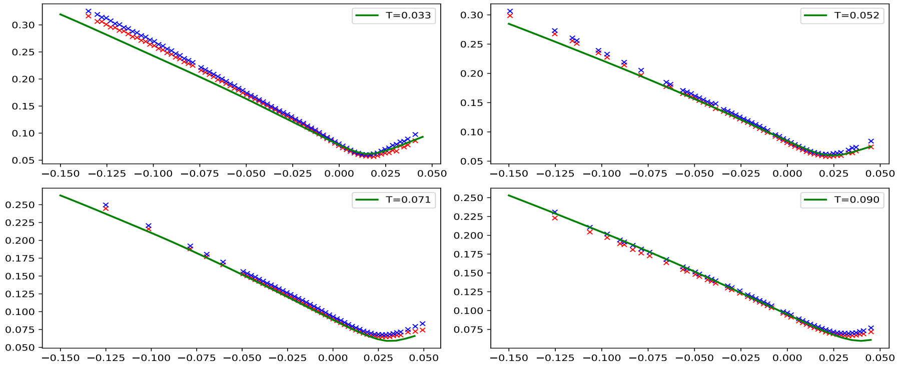

FIGURE – Implied volatility on SPX options for 19 May 2017. Blue and red points are bid and ask of market implied volatilities. Model implied volatility smiles from the model are in green. Strikes are in log-moneyness, maturity in year.

Calibration for one day in history 19 May 2017

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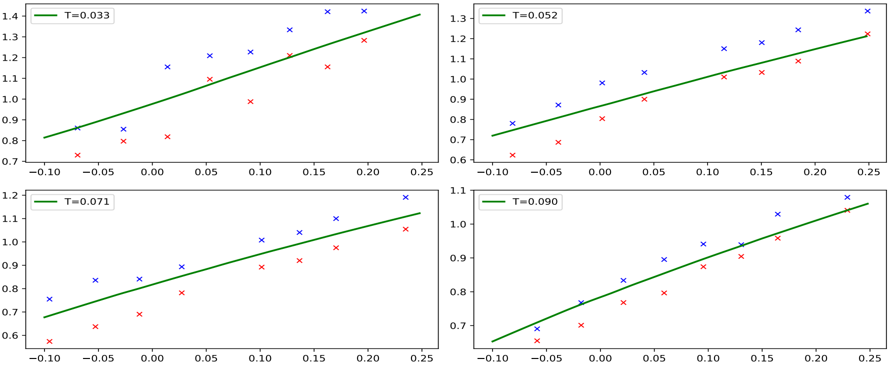

FIGURE – Implied volatility on VIX options for 19 May 2017. Blue and red points are bid and ask of market implied volatilities. Model implied volatility smiles from the model are in green. Strikes are in log-moneyness, maturity in year.

Take home message for the joint calibration problem

Thanks to the quadratic rough Heston model

- 6 parameters.
- VIX smiles in the bid-ask spread.
- Global shape of the implied volatility surface of the SPX very well reproduced
- Very accurate SPX skews of orders -1.5 (shortest maturites), -1 (longer maturities), as for market data.

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Event based data

Event based football data

- A time-coded feed that lists all events with the ball within a game with a player, team, event type and timestamp for each action.

Counting process

A 12-dimensional counting process is constructed as follows :

- Every time a player touches the ball, there is a jump in his assigned dimension $d \in \{1, \ldots, 11\}$ at the corresponding timestamp.
- Every time the ball is in the opponent danger area, there is a jump in the twelfth dimension at the corresponding timestamp.
- Once a danger state is triggered, no event are recorded until the ball goes out of the danger area $(+\epsilon)$.
- Once the ball is lost, no event is recorded until the ball is won again.

Modelling with Hawkes process

Modelling and estimation

- We fit a 12-dimensional exponential Hawkes process to the jump data through Maximum Likelihood Estimation.

$$\lambda_i(t) = \mu_i + \sum_{j=1}^{d} \int_0^t \phi_{i,j}(t-s) N_j(ds).$$

  where $\phi_{i,j}(s) = \alpha_{i,j} e^{-\beta_i s}$.
- The goal is to detect correlations between the event times related to each player and the danger state.
- The estimation is complex in large dimensions as the likelihood function is not convex in β. This is why we reduce the number of parameters and consider the rate of decay of the kernel is the same for each player. $\beta_{i,j} = \beta_i$ for all i, j.
- We use the algorithm of Bonnet *et al.* (2022).

Estimated parameters

- The integrated kernel $K_{i,j} = \int_0^\infty \phi_{i,j}$ can be interpreted as the expected number of touches of player i **directly** generated by player j touch. It can be estimated directly from the estimated parameters :

$$\hat{K}_{i,j} = \hat{\alpha}_{i,j}/\hat{\beta}_i.$$

- It is also relevant to consider the interactions between two states across multiple steps (generations). This is the case for defenders that rarely generate Danger directly but contribute to Danger creation by passing the ball to advanced positions.
- $M_{i,j}$ represents the expected total number of touches of the player i generated by player j directly but also indirectly through other players.

$$M = K + K^2 + K^3 + \cdots = K\left(I - K\right)^{-1}.$$

where $K = \int_0^\infty \phi$.

Estimated interactions

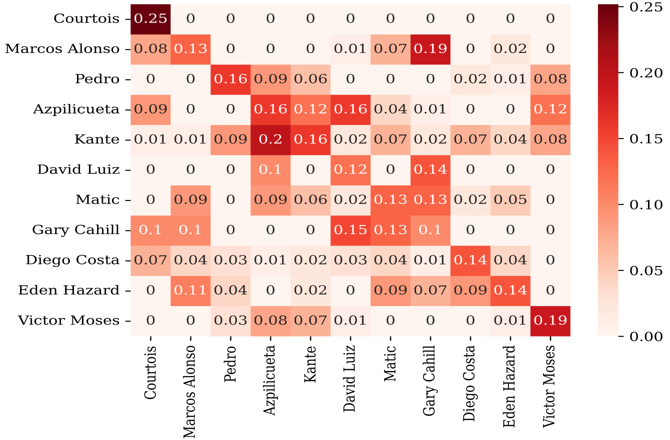

FIGURE – Graph of estimated interactions $\hat{K}_{i,j}$ between players for Chelsea 2016/2017.

Danger generated

Player Name	Danger directly generated	Total danger generated
Eden Hazard	0.22	0.28
Pedro	0.12	0.17
Victor Moses	0.08	0.15
Kante	0.06	0.13
Diego Costa	0.07	0.13
Marcos Alonso	0.04	0.10
Azpilicueta	0.02	0.10
Matic	0.03	0.10
Cahill	0	0.07
Courtois	0	0.05
David Luiz	0	0.04

TABLE – Danger directly and indirectly generated by one player touch. Direct danger is represented by $\hat{K}_{\text{danger},player}$ and Total danger by $\hat{M}_{\text{danger},player}$.

Some observations

Remarks

- Factoring in the indirect contribution in danger creation is important for defenders and midfielders.
- Kante is responsible for as many intrusions to the danger area per touch as the striker Diego Costa, while having more touches per game.
- The contribution of the central defender David Luiz in danger creation is minimal. This is not surprising as the flat 3-4-3 system relies heavily on the wings. David Luiz naturally passes the ball to either Cahill or Azpilicueta in build-up to spread the play.

A better visualization scheme

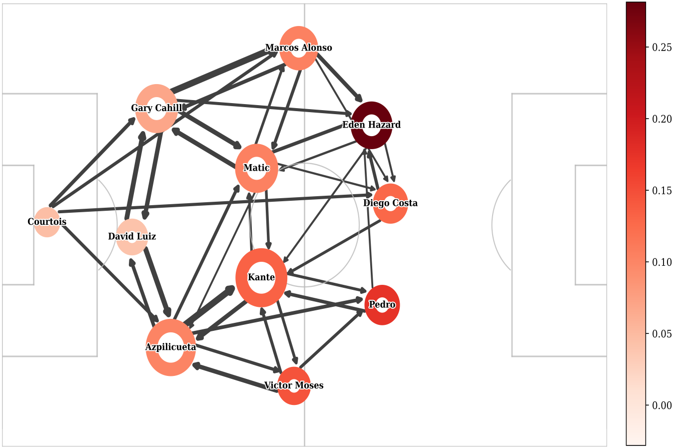

FIGURE – Graph of estimated interactions between players and with Danger state for Chelsea 2016/2017. The color of the circles represents the total danger created by the player. The size of arrows represents the parameters $\hat{K}_{player_1,player_2}$.

Comments

More remarks

- Azpilicueta, Kante and Victor Moses witness a considerable increase in Danger creation when considering the total contribution rather than the direct one. The right side of Chelsea combines a lot for danger generation and should be disrupted from the root.
- The left side relies a lot more on the huge offensive output of Eden Hazard. The links Marcos Alonso/Matic $\rightarrow$ Hazard should be controlled.
- Goalkeeper Courtois is successful in targeting Marcos Alonso and Diego Costa when playing long balls.