On derivation and analysis of traffic and lubrication models

From fluids to crowds

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Compressible and incompressible NS systems

- Compressible Navier-Stokes equations (constant temperature):

$$\left\{ \begin{array}{l} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) \underbrace{-\mu \Delta \mathbf{u} - \lambda \nabla \operatorname{div} \mathbf{u}}_{:= \operatorname{div} \mathbb{S}(\mathbf{u})} + \nabla p(\rho) = \mathbf{0}, \end{array} \right.$$

$\rightsquigarrow$ The unknowns: $\rho, \mathbf{u}$.

$\rightsquigarrow$ The pressure: $p = a\rho^\gamma$.

- Incompressibe Navier-Stokes equations:

$$\left\{ \begin{array}{l} \operatorname{div} \mathbf{u} = 0, \\ \rho^*(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) - \mu \Delta \mathbf{u} + \nabla \pi = \mathbf{0}, \end{array} \right.$$

$\rightsquigarrow$ The unknowns: $\rho, \mathbf{u}$ and π.

Fluid equations with constrained density

- Free boundary pb: free domain $\{\rho < \rho^*\}$ / congested domain $\{\rho = \rho^*\}$.
- Constrained compressible / incompressible Navier-Stokes equations

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho \mathbf{u}) = 0 \\ \partial_t(\rho \mathbf{u}) + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) + \nabla \pi + \mathrm{div}\, \mathbb{S}(\mathbf{u}) = \mathbf{0}, \\ 0 \leq \rho \leq \rho^*, \quad (\rho^* - \rho)\pi = 0, \quad \pi \geq 0. \end{cases}$$

Unknowns: ρ – density, $\mathbf{u}$ – velocity, $\nabla \pi$ – constraining force

Outline

- Motivation: crowds, floating structures, lubrication, traffic;
- Hard congestion limit explained in 1D;
- Analysis of Aw-Rascle system in multi-D.

Motivation: Applications

1/4 Modelling of crowd

Let $\mathbf{U}_{pref}$ - preferred velocity, ρ - pedestrian population density

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho \mathbf{u}) = 0 \\ \mathbf{u} = P_{C_\rho}(\mathbf{U}_{pref}), \end{cases}$$

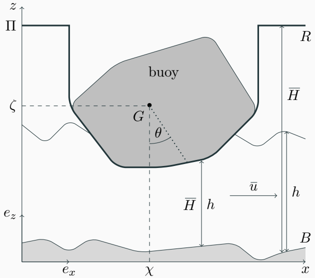
- cone of admis. vel.: $C_\rho = \{\mathbf{v} \in L^2(\Omega) : \int_\Omega \mathbf{v} \cdot \nabla q \leq 0 \quad \forall q \in H^1_\rho\}$
- space of pressures $H^1_\rho = \{q \in H^1, \; q \geq 0 \; a.e., q = 0 \; a.e. \; in \; \{\rho < 1\}\}$.

Find $(\mathbf{u}, p) \in L^2 \times H^1_\rho$ such that

$$\begin{cases} \mathbf{u} + \nabla p = \mathbf{U}_{pref} \\ \int_\Omega \mathbf{u} \cdot \nabla q \leq 0 \quad \forall q \in H^1_\rho. \end{cases}$$

Maury, Roudneff-Chupin, Santambrogio '10.

2/4 Floating structures/flows through the channels

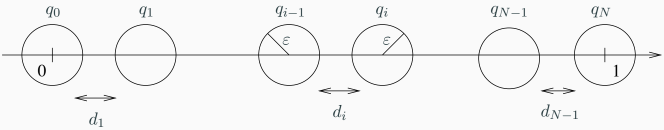


The (inviscid) fluid is described by shallow water approximation

$$\begin{cases} \partial_t h + \partial_x(h\bar{u}) = 0 \\ \partial_t(h\bar{u}) + \partial_x(h\bar{u}^2) = -h\partial_x \left(g(h+B) + \frac{\bar{p}}{\rho} \right), \\ \min\{\bar{H} - h, \bar{p} - P\} = 0. \end{cases}$$

Godlewski et al. '20, Lannes'17.

3/4 Lubrication model



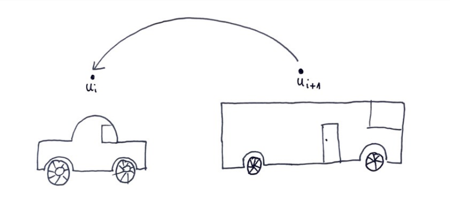
When number of masses goes to infinity, the flow is described by:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) - \partial_x \left(\frac{\varepsilon}{1-\rho} \partial_x u \right) = \rho f, \end{cases}$$

Conjecture: when $\varepsilon \to 0$ we obtain the system:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x p = f, \\ \partial_t \gamma + \partial_x(\gamma u) = -p, \\ \gamma \leq 0, \ \rho \leq 1, \ \gamma(1-\rho) = 0. \end{cases}$$

A. Lefebvre-Lepot and B. Maury '11, Chaudhuri, Navoret, Perrin, Z. '23.



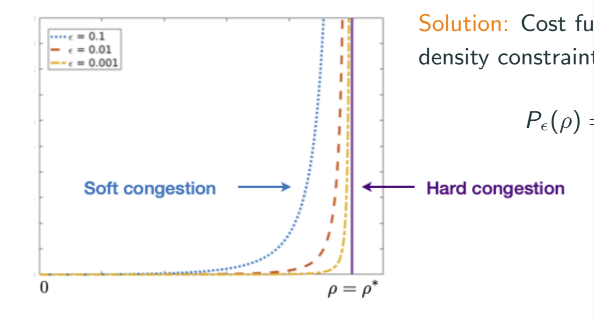
From Follow the Leader to fluid-like Aw-Rascle system:

$$\begin{cases} \dot{x}_i = u_i, \\ \dot{u}_i = C \frac{u_{i+1} - u_i}{(x_{i+1} - x_i)^{\gamma+1}}, \end{cases} \longrightarrow \begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho w) + \partial_x(\rho w u) = 0, \\ w = u + P(\rho) \end{cases}$$

where $P(\rho) = \rho^\gamma$ is the cost (offset) function.

Aw, Klar, Rascle, Materne. *Derivation of continuum traffic flow models from microscopic follow-the-leader models.* SIAM J. Math. Anal., 2002.

Problem: Maximal velocity and maximal density constraints not preserved.

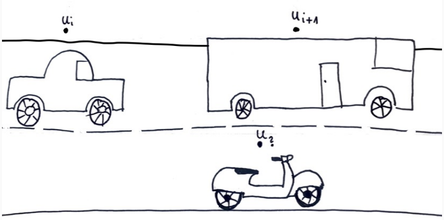


Solution: Cost function with maximal density constraint $\rho^* > 0$

$$P_\epsilon(\rho) = \epsilon \left(\frac{\rho}{\rho^* - \rho} \right)^\gamma .$$

F. Berthelin, P. Degond, M. Delitata, and M. Rascle. A Model for the Formation and Evolution of Traffic Jams. *ARMA*, 2008.

Who is the leader now?



One dimension $\longrightarrow$ Several dimensions

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho w) + \partial_x(\rho w u) = 0, \end{cases} \longrightarrow \begin{cases} \partial_t \rho + \mathrm{div}(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{w}) + \mathrm{div}(\rho \mathbf{w} \otimes \mathbf{u}) = 0. \end{cases}$$

Problem: Dimension incompatibility: $\underbrace{\mathbf{w}}_{\text{vector}} = \underbrace{\mathbf{u}}_{\text{vector}} + \underbrace{P(\rho)}_{\text{scalar}}.$

Solutions to dimension incompatibility problem

Either:

$$\mathbf{w} = \mathbf{u} + \mathbf{P}(\rho),$$

where $\mathbf{P}(\rho) = [P_1(\rho), P_2(\rho)]$.

M. Herty, S. Moutari, G. Visconti. Macroscopic modeling of multilane motorways using a two-dimensional second-order model of traffic flow. *SIAM J. Appl. Math.*, 2018.

Or:

$$\mathbf{w} = \mathbf{u} + \nabla p(\rho),$$

where $p(\rho)$ is a scalar function.

A.Tosin, P. Degond, E. Zatorska Students' theses 2016-2017.

Observations about the model

- Taking the offset function $P(\rho) = \partial_x p(\rho) = \frac{\lambda(\rho)}{\rho^2}\partial_x \rho$, we get pressureless, compressible, degenerate Navier-Stokes equations:

$$\partial_t \rho + \partial_x(\rho u) = 0,$$
$$\partial_t(\rho u) + \partial_x(\rho u^2) - \partial_x\left(\lambda(\rho)\partial_x u\right) = 0.$$

- In more dimensions this dissipative effect looks differently

$$\partial_t(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) = \nabla_x(\rho Q'(\rho)\operatorname{div}\mathbf{u}) + \mathcal{L}[\nabla_x Q(\rho), \nabla_x \mathbf{u}],$$

where $Q'(\rho) = \rho p'(\rho)$ and

$$\mathcal{L}[\nabla_x Q(\rho), \nabla_x \mathbf{u}] = \nabla_x(\nabla_x Q(\rho) \cdot \mathbf{u}) - \operatorname{div}(\nabla_x Q(\rho) \otimes \mathbf{u}),$$

which is a lower order term

$$(\mathcal{L}[\nabla_x Q(\rho), \nabla_x \mathbf{u}])_j = \sum_{i=1}^{3}\left(\partial_{x_i} Q(\rho)\partial_{x_j} u_i - \partial_{x_j} Q(\rho)\partial_{x_i} u_i\right), \quad j = 1, 2, 3.$$

Analysis of hard congestion limit in 1D

Setup of the problem

The starting point is the following system on 1D tourus:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) - \partial_x\left(\lambda_\varepsilon(\rho)\partial_x u\right) = 0, \end{cases}$$

where

$$\lambda_\varepsilon(\rho) = \rho^2 p_\varepsilon'(\rho), \qquad p_\varepsilon(\rho) = \varepsilon \frac{\rho^\gamma}{(1-\rho)^\beta}, \quad \gamma \geq 0, \quad \beta > 1.$$

Taking $w = u + \partial_x p_\varepsilon(\rho)$ we formally rewrite the momentum equation as:

$$\partial_t(\rho w) + \partial_x(\rho u w) = 0,$$

or as

$$\partial_t\left(\rho u + \partial_x \pi_\varepsilon(\rho)\right) + \partial_x\left(u\left(\rho u + \partial_x \pi_\varepsilon(\rho)\right)\right) = 0$$

where $\partial_x \pi_\varepsilon(\rho) = \rho \partial_x p_\varepsilon(\rho)$.

We consider the following approximation:

$$\begin{cases} \partial_t \rho_\varepsilon + \partial_x(\rho_\varepsilon u_\varepsilon) = 0, \\ \partial_t(\rho_\varepsilon u_\varepsilon) + \partial_x(\rho_\varepsilon u_\varepsilon^2) - \partial_x\left(\lambda_\varepsilon(\rho_\varepsilon)\partial_x u_\varepsilon\right) = 0, \\ \rho_\varepsilon|_{t=0} = \rho_\varepsilon^0, \quad u_\varepsilon|_{t=0} = u_\varepsilon^0, \end{cases}$$

with λ_ε re-defined as

$$\lambda_\varepsilon(\rho_\varepsilon) = \rho_\varepsilon^2 p_\varepsilon'(\rho_\varepsilon) + \underbrace{\rho_\varepsilon^2 \varphi_\varepsilon'(\rho_\varepsilon)}_{\text{approximation}}$$

where

$$p_\varepsilon(\rho_\varepsilon) = \varepsilon \frac{\rho_\varepsilon^\gamma}{(1-\rho_\varepsilon)^\beta}, \quad \varphi_\varepsilon(\rho_\varepsilon) = \frac{\varepsilon}{\alpha - 1}\rho^{\alpha-1}, \quad \gamma \geq 0, \quad \beta > 1, \quad \alpha \in \left(0, \frac{1}{2}\right).$$

Theorem (Chaudhuri, Navoret, Perrin, Z. '22)
Let $\varepsilon > 0$ fixed, $T > 0$ arbitrary, $\rho_\varepsilon^0, u_\varepsilon^0 \in H^3(\mathbb{T})$, with $0 < \rho_\varepsilon^0 < 1$.
$\exists!$ global solution $(\rho_\varepsilon, u_\varepsilon)$ s. t. $0 < \rho_\varepsilon(t,x) < 1$, and

$$\rho_\varepsilon \in \mathcal{C}([0,T]; H^3(\mathbb{T})), \qquad u_\varepsilon \in \mathcal{C}([0,T]; H^3(\mathbb{T})) \cap L^2(0,T; H^4(\mathbb{T})).$$

Constantin, Drivas, Nguyen, Pasqualotto '20, Mehmood '23.

Basic a priori estimates:

- $\|\rho_\varepsilon\|_{L^1_x}(t) = \|\rho^0_\varepsilon\|_{L^1_x}$,
- $\|\sqrt{\rho_\varepsilon}u_\varepsilon\|_{L^\infty_t L^2_x} + \|\sqrt{\lambda_\varepsilon(\rho_\varepsilon)}\partial_x u_\varepsilon\|_{L^2_{t,x}} \leq C$ and $\|\sqrt{\rho_\varepsilon}w_\varepsilon\|_{L^\infty_t L^2_x} \leq C$,
  (classical energy) (BD estimate)
- $\|H_\varepsilon(\rho_\varepsilon)\|_{L^\infty_t L^1_x} + \|\sqrt{\rho_\varepsilon}\partial_x\big(p_\varepsilon(\rho_\varepsilon) + \varphi_\varepsilon(\rho_\varepsilon)\big)\|^2_{L^2_t L^2_x} \leq C(T)$,
  (porous medium structure)

where $H'_\varepsilon(\rho_\varepsilon) := p_\varepsilon(\rho_\varepsilon) + \varphi_\varepsilon(\rho_\varepsilon)$.

$$\implies \rho_\varepsilon(t,x) \leq 1 - C\left(\frac{\varepsilon}{1+\sqrt{T}}\right)^{\frac{1}{\beta-1}}, \quad {\rho_\varepsilon}^{-1} \leq C\varepsilon^{-\frac{2}{1-2\alpha}}\big(1+T\big)^{\frac{1}{1-2\alpha}}.$$

⚠ No estimates on u_ε (or w_ε) independent of ρ_ε;

⚠ ⚠ The BD estimate not uniform w.r.t. ε;

⚠ ⚠ ⚠ No uniform estimates for $p_\varepsilon(\rho_\varepsilon)$, not even L^1.

Idea: Duality solutions of Bouchut and James '98, '99, Boudin '00

Definition A solution $b \in \mathrm{Lip}_{loc}([0,T] \times \mathbb{R})$ to

$$\partial_t b + u_\varepsilon \partial_x b = 0, \qquad b_{|t=T} = b_T \tag{1}$$

is said to be reversible if there exist two solutions $b_1, b_2 \in \mathrm{Lip}_{loc}([0,T] \times \mathbb{R})$ of (1) such that $\partial_x b_1 \geq 0$, $\partial_x b_2 \geq 0$ and $b = b_1 - b_2$.

Remark Bouchut and James showed that the backward problem (1) is well-posed in the class of reversible solutions provided $u_\varepsilon \in L^\infty([0,T] \times \mathbb{R})$, and if u_ε satisfies the *Oleinik entropy condition*, i.e. $\partial_x u_\varepsilon \leq 1/t$.

Definition We say that $\mu \in \mathcal{C}([0,T], \mathcal{M}_{loc,x})$ is a duality solution to

$$\partial_t \mu + \partial_x(\mu u) = 0 \quad \text{in }]0,T[\times \mathbb{T}$$

if, for any $0 < \tau \leq T$, and any reversible solution b, the function

$$t \mapsto \int_{\mathbb{T}} b(t,x)\mu(t,dx)$$

is constant on $[0,\tau]$.

Further estimates on u_ε

Idea: Duality solutions of Bouchut and James '98, '99, Boudin '00.
$\rightsquigarrow$ we prove the one-sided Lipschitz condition on $\partial_x u_\varepsilon$.

Proposition

Let $\rho_\varepsilon, u_\varepsilon$ be a regular solution, and set $A_\varepsilon := \max(\mathrm{ess\,sup}\,(\lambda_\varepsilon(\rho_\varepsilon^0)\partial_x u_\varepsilon^0), 0)$. Then

$$V_\varepsilon = \lambda_\varepsilon(\rho_\varepsilon)\partial_x u_\varepsilon \leq A_\varepsilon.$$

In particular:

- *If $A_\varepsilon \to 0$ as $\varepsilon \to 0$, then*

$$(\lambda_\varepsilon(\rho_\varepsilon)\partial_x u_\varepsilon)_+ \to 0 \quad \text{as} \quad \varepsilon \to 0;$$

- *If $A_\varepsilon \leq \lambda_\varepsilon(\underline{\rho}_\varepsilon) \leq \bar{C}\varepsilon^{\frac{1}{1-2\alpha}}$, for some $\bar{C}$ independent of ε, then*

$$\partial_x u_\varepsilon \leq \bar{C}.$$

Consequences

We have

$$\|\partial_x u_\varepsilon\|_{L^\infty_t L^1_x} \leq C \quad \rightsquigarrow \quad \|u_\varepsilon\|_{L^\infty_{t,x}} \leq C$$

for a constant C independent of ε. As a consequence of it, we deduce

$$\|\pi_\varepsilon(\rho_\varepsilon)\|_{L^\infty_t L^1_x} + \|\partial_x \pi_\varepsilon(\rho_\varepsilon)\|_{L^\infty_t L^2_x} \leq C,$$

where $\pi'_\varepsilon(\rho_\varepsilon) = \rho_\varepsilon p'_\varepsilon(\rho_\varepsilon) + \rho_\varepsilon \varphi'_\varepsilon(\rho_\varepsilon)$.

Idea: Testing the momentum equation with

$$\psi(t,x) = \int_0^x \big(\rho_\varepsilon(t,y) - <\rho_\varepsilon>\big)\, dy,$$

to obtain the bound

$$\left| \int_0^t \int_{\mathbb{T}} \rho_\varepsilon^2 p'_\varepsilon(\rho_\varepsilon) \partial_x u_\varepsilon \, dx\, dt \right| \leq C.$$

This is then used to bound the r.h.s. of the renormalised continuity equation

$$\partial_t(\rho_\varepsilon p_\varepsilon(\rho_\varepsilon)) + \partial_x(\rho_\varepsilon p_\varepsilon(\rho_\varepsilon) u_\varepsilon) = -\rho_\varepsilon^2 p'_\varepsilon(\rho_\varepsilon) \partial_x u_\varepsilon.$$

The limit passage

Recall the system once more:

$$\partial_t \rho_\varepsilon + \partial_x(\rho_\varepsilon u_\varepsilon) = 0,$$
$$\partial_t \left(\rho_\varepsilon u_\varepsilon + \partial_x \pi_\varepsilon\right) + \partial_x\Big(\left(\rho_\varepsilon u_\varepsilon + \partial_x \pi_\varepsilon\right) u_\varepsilon\Big) = 0.$$

From the a-priori estimates:

$$\rho_\varepsilon \rightharpoonup \rho, \quad u_\varepsilon \rightharpoonup u \quad \text{weakly-* in} \quad L^\infty_{t,x}, \qquad \pi_\varepsilon(\rho_\varepsilon) \rightharpoonup \pi \quad \text{weakly in} \quad L^2_t H^1_x,$$

and also

$$\rho_\varepsilon \varphi_\varepsilon(\rho_\varepsilon) \to 0 \quad \text{strongly in} \quad L^\infty_{t,x}, \quad (1-\rho_\varepsilon)\pi_\varepsilon(\rho_\varepsilon) \to 0 \quad \text{strongly in } L^q_{t,x}.$$

Using the standard compensated compactness arguments we then show:

$$(1-\rho_\varepsilon)\pi_\varepsilon(\rho_\varepsilon) \rightharpoonup (1-\rho)\pi, \quad \rho_\varepsilon u_\varepsilon \to \rho u \quad \text{and} \qquad \rho_\varepsilon u_\varepsilon^2 \to \rho u^2 \quad \text{in } \mathcal{D}'_{t,x}.$$

Hence, passing to the limit in the system, we verify that:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u + \partial_x \pi) + \partial_x\big((\rho u + \partial_x \pi) u\big) = 0 \\ 0 \le \rho \le 1, \quad (1-\rho)\pi = 0, \quad \pi \ge 0 \end{cases}$$

Analysis of Aw-Rascle system in multi-D

The set up of the problem

Let $\mathbf{w} = \mathbf{u} + \nabla p(\rho)$ we can either solve:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{w}) + \operatorname{div}(\rho \mathbf{w} \otimes \mathbf{u}) = 0, \end{cases}$$

or equivalently:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{w}) - \operatorname{div}(\sqrt{\rho}\nabla Q) = 0, \\ \partial_t(\rho \mathbf{w}) + \operatorname{div}(\rho \mathbf{w} \otimes \mathbf{w}) = \operatorname{div}(\sqrt{\rho}\nabla Q \otimes \sqrt{\rho}\mathbf{w}), \end{cases}$$

where $Q'(\rho) = \sqrt{\rho}p'(\rho)$.

We consider $\Omega = \mathbb{T}^d$ with the initial data $\rho(0,x) = \rho_0 \geq 0$, $(\rho \mathbf{w})(0,x) = \mathbf{m}_0$, satisfying the energy bound

$$E_0 = \int_\Omega \left(\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\rho_0} + E(\rho_0) \right) \mathrm{d}x < \infty, \quad \text{where} \quad E(\rho) = \int_0^\rho p(s)\, ds.$$

The uniform estimates are:

$$\|\sqrt{\rho_n}\mathbf{w}_n\|_{L^\infty(0,T;L^2(\mathbb{T}^d))} \leq C,$$
$$\|E(\varrho_n)\|_{L^\infty(0,T;L^1(\mathbb{T}^d))} \leq C,$$
$$\|Q(\varrho_n)\|_{L^2(0,T;W^{1,2}(\mathbb{T}^d))} \leq C,$$

where $E(\rho) = \int_0^\rho p(s)\ ds,\ Q'(\rho) = \sqrt{\rho}p'(\rho)$.

Remarks:

1. There is no uniform bound on $\mathbf{w}_n$.
2. The estimates for ρ_n are quite strong.

- The continuity equation

$$\partial_t \rho_n + \operatorname{div}(\underbrace{\sqrt{\rho_n}\sqrt{\rho_n}\mathbf{w}_n}_{L^\infty(L^p)}) - \operatorname{div}(\underbrace{\sqrt{\rho_n}\nabla Q(\rho_n)}_{L^2(L^p)}) = 0,$$

- The momentum equation

$$\partial_t(\sqrt{\rho_n}\sqrt{\rho_n}\mathbf{w}_n) + \operatorname{div}(\underbrace{\sqrt{\rho_n}\mathbf{w}_n \otimes \sqrt{\rho_n}\mathbf{w}_n}_{L^\infty(L^1)}) = \operatorname{div}(\underbrace{\nabla Q(\rho_n) \otimes \sqrt{\rho_n}\mathbf{w}_n}_{L^2(L^1)}).$$

Young measures

$$\mathcal{V} : Q \subset \mathbb{R}^k \to \mathcal{P}(\mathbb{R}^N),$$

in the sense that

$$z \in Q \to \langle \mathcal{V}_z; g(\xi) \rangle = \int_{\mathbb{R}^N} g(\xi) d\mathcal{V}_z(\xi)$$

is Borel measurable $\forall g \in C_0(\mathbb{R}^N)$.

Any measurable function $\mathbf{u}_n : Q \to \mathbb{R}^N$ generates a measure

$$\mathbf{u}_n : z \in Q \to \delta_{u_n(z)} \in \mathcal{P}(\mathbb{R}^N),$$

moreover $\mathbf{u}_n \to \mathcal{V}$ in the natural topology $L^\infty_{waek*}(Q; \mathcal{M}(\mathbb{R}^N))$, meaning that

$$\langle \mathbf{u}_n; g(\xi) \rangle \to \langle \mathcal{V}; g(\xi) \rangle \quad \text{weakly}^* \quad \text{in} \quad L^\infty(Q), \quad \forall g \in C_0(\mathbb{R}^N).$$

Definition: *$\mathcal{V}$ is called the Young measure generated by $\{\mathbf{u}_n\}_{n \in \mathbb{N}}$.*

Oscillations and concentrations

$\{\mathbf{u}_n\}_{n\in\mathbb{N}}$ such that

$$\|\mathbf{u}_n\|_{L^1(Q)} \leq C \quad \text{and} \quad \|b(\mathbf{u}_n)\|_{L^p(Q)} \leq C, \quad p > 1$$

then $\lim_{n\to\infty} b(\mathbf{u}_n)$ can be characterised by $\mathcal{V}$, i.e.

$$\int_Q \phi(z) b(\mathbf{u}_n(z))\, dz \to \int_Q \phi(z) \langle \mathcal{V}_z; b(\xi)\rangle\, dz, \quad \forall \phi \in L^{p'}(Q).$$

But if $\|b(\mathbf{u}_n)\|_{L^1(Q)} \leq C$ <u>only</u>, then

$$b(\mathbf{u}_n) \to \overline{b(u)} \in \mathcal{M}(Q).$$

Remark: Only the oscillations are captured by the Young measure, the concentrations are not!

Definition: *We call*

$$\mathcal{R}_b = \overline{b(u)} - \langle \mathcal{V}_z; b(\xi)\rangle$$

a <u>defect measure</u> for function b.

$\{\rho_n, \sqrt{\rho_n}\mathbf{w}_n, \nabla Q(\rho_n)\}_{n\in\mathbb{N}}$, and so we consider $\{\mathcal{V}_{t,x}\}_{(t,x)\in(0,T)\times\mathbb{T}^d}$, and

$$\mathcal{V} \in L^\infty_{\text{weak-(*)}}\big((0,T)\times\mathbb{T}^d;\mathcal{P}(\mathcal{F})\big),$$

on the phase space

$$\mathcal{F} = \left\{ \left(\tilde{\rho}, \widetilde{\sqrt{\varrho}\mathbf{w}}, \widetilde{D_Q}\right) \mid \tilde{\rho}\in[0,\infty),\ \widetilde{\sqrt{\varrho}\mathbf{w}}\in\mathbb{R}^d,\ \widetilde{D_Q}\in\mathbb{R}^d \right\}.$$

Our convergence results allow us to identify

$$\rho = \langle \mathcal{V}_{t,x};\tilde{\varrho}\rangle, \quad \sqrt{\rho}\left\langle \mathcal{V}_{t,x};\widetilde{\sqrt{\varrho}\mathbf{w}}\right\rangle = \left\langle \mathcal{V}_{t,x};\sqrt{\tilde{\varrho}}\widetilde{\sqrt{\varrho}\mathbf{w}}\right\rangle,$$
$$Q(\rho) = \langle \mathcal{V}_{t,x}; Q(\tilde{\rho})\rangle, \quad \nabla_x Q(\rho) = \left\langle \mathcal{V}_{t,x};\widetilde{D_Q}\right\rangle.$$

In particular, we have

$$\mathcal{V}_{t,x} = \delta_{\{\rho(t,x)\}}\otimes Y_{t,x} \quad \text{for a.a. } (t,x)\in(0,T)\times\mathbb{T}^d,$$

where $Y \in L^\infty_{\text{weak-(*)}}\big((0,T)\times\mathbb{T}^d;\mathcal{P}(\mathbb{R}^d\times\mathbb{R}^d)\big)$.

Weak formulation

1. The continuity equation

$$\partial_t \rho + \mathrm{div}(\sqrt{\rho} \left\langle \mathcal{V}_{t,x}; \widetilde{\sqrt{\varrho}\mathbf{w}} \right\rangle) - \mathrm{div}(\sqrt{\rho}\nabla_x Q) = 0$$

2. The momentum equation

$$\partial_t \left(\sqrt{\rho} \left\langle \mathcal{V}_{t,x}; \widetilde{\sqrt{\varrho}\mathbf{w}} \right\rangle \right) + \mathrm{div} \left(\left\langle \mathcal{V}_{t,x}; \widetilde{\sqrt{\varrho}\mathbf{w}} \otimes \widetilde{\sqrt{\varrho}\mathbf{w}} \right\rangle \right)$$

$$- \mathrm{div} \left(\left\langle \mathcal{V}_{t,x}; \widetilde{\sqrt{\varrho}\mathbf{w}} \otimes \widetilde{D_Q} \right\rangle \right) + \mathrm{div}(r^M) = 0.$$

are satisfied in the sense of distributions, where

$$r^M \in L^\infty_{\text{weak-(*)}}(0,T; \mathcal{M}(\mathbb{T}^d; \mathbb{R}^{d\times d})) + \mathcal{M}([0,T]\times \mathbb{T}^d; \mathbb{R}^{d\times d}).$$

3. The energy inequality

$$\int_{\mathbb{T}^d} \left\langle \mathcal{V}_{\tau,x}; \frac{1}{2} \left| \widetilde{\sqrt{\varrho}\mathbf{w}} \right|^2 + E(\tilde{\varrho}) \right\rangle \mathrm{d}x + \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; |\widetilde{D_Q}|^2 \right\rangle \, \mathrm{d}x\mathrm{d}t + \mathcal{D}(\tau)$$

$$\leq \int_{\mathbb{T}^d} \left\langle \mathcal{V}_{0,x}; \frac{1}{2} \left| \widetilde{\sqrt{\varrho}\mathbf{w}} \right|^2 + E(\tilde{\varrho}) \right\rangle \mathrm{d}x + \int_0^\tau \int_\Omega \left\langle \mathcal{V}_{t,x}; \widetilde{\sqrt{\varrho}\mathbf{w}} \cdot \widetilde{D_Q} \right\rangle \, \mathrm{d}x\mathrm{d}t + \int_{(0,\tau)\times\mathbb{T}^d} \mathrm{d}\mathcal{R},$$

where

$$\mathcal{R} \in L^\infty_{\text{weak-(*)}}(0,T; \mathcal{M}(\mathbb{T}^d)) + \mathcal{M}([0,T]\times \mathbb{T}^d).$$

4. The weights are compatible, i.e. $\mathcal{D} \equiv 0 \implies \mathcal{R}, r^M \equiv 0$.

Theorem (Gwiazda, Chaudhuri, Zatorska '22)
Let $(\mathcal{V}, \mathcal{D})$ *be a measure valued solution in* $(0, T) \times \mathbb{T}^d$ *of the Aw-Rascle system. Let* $(\bar{\varrho}, \bar{\mathbf{w}})$ *be a strong solution to the same system in* $(0, T) \times \mathbb{T}^d$ *with initial data* $(\bar{\varrho}_0, \bar{\mathbf{w}}_0) \in (C^2(\mathbb{T}^d), C^2(\mathbb{T}^d; \mathbb{R}^d))$ *satisfying* $\bar{\varrho}_0 > 0$. *We assume that the strong solution belongs to the class*

$$\bar{\varrho} \in C^1(0, T; C^2(\mathbb{T}^d)), \; \bar{\mathbf{w}} \in C^1(0, T; C^2(\mathbb{T}^d); \mathbb{R}^d) \text{ with } \bar{\varrho} > 0.$$

If the initial states coincide, i.e.

$$\mathcal{V}_{0,x} = \delta_{\{\bar{\varrho}_0(x), \bar{\mathbf{w}}_0(x)\}}, \text{ for a.e. } x \in \mathbb{T}^d$$

then $\mathcal{D} = 0$, *and*

$$\mathcal{V}_{\tau,x} = \delta_{\{\bar{\varrho}(\tau,x), \sqrt{\bar{\varrho}}\bar{\mathbf{w}}(\tau,x), \nabla_x Q(\bar{\varrho})(\tau,x)\}}, \text{ for a.e. } (\tau, x) \in (0, T) \times \mathbb{T}^d.$$

However...

$(\rho_0, \mathbf{u}_0) = (\rho(0,\cdot), \mathbf{u}(0,\cdot))$ can connect to arbitrary terminal state $(\rho_T, \mathbf{u}_T) = ((\rho(T,\cdot), \mathbf{u}(T,\cdot))$ via a weak solution.

More specifically, we consider

$$\rho_0, \rho_T \in C^2(\mathbb{T}^d), \quad \inf_{\mathbb{T}^d} \rho_0 > 0, \quad \inf_{\mathbb{T}^d} \rho_T > 0,$$

$$\int_{\mathbb{T}^d} \rho_0 \,\mathrm{d}x = \int_{\mathbb{T}^d} \rho_T \,\mathrm{d}x,$$

together with

$$\mathbf{u}_0, \mathbf{u}_T \in C^3(\mathbb{T}^d; R^d),$$

$$\int_{\mathbb{T}^d} \rho_T \mathbf{u}_T \,\mathrm{d}x - \int_{\mathbb{T}^d} \rho_0 \mathbf{u}_0 \,\mathrm{d}x = \int_{\mathbb{T}^d} \rho_0 \mathbf{P}(\rho_0) \,\mathrm{d}x - \int_{\mathbb{T}^d} \rho_T \mathbf{P}(\rho_T) \,\mathrm{d}x.$$

Theorem (Chaudhuri, Feireisl, Zatorska '22)
Let $d = 2, 3$. Suppose that

$$\mathbf{P} \in C^2((0,\infty); R^d), \ p \in C^2((0,\infty)).$$

Let $(\rho_0, \mathbf{u}_0)$, $(\rho_T, \mathbf{u}_T)$ satisfy assumptions above.

Then, the Aw-Rascle system, endowed with the periodic boundary conditions admits infinitely many weak solutions in the class

$$\rho \in C^2([0,T] \times \mathbb{T}^d), \quad \mathbf{u} \in L^\infty((0,T) \times \mathbb{T}^d; R^d)$$

such that

$$\rho(0,\cdot) = \rho_0, \ \rho(T,\cdot) = \rho_T, \ (\rho\mathbf{u})(0,\cdot) = \rho_0\mathbf{u}_0, \ (\rho\mathbf{u})(T,\cdot) = \rho_T\mathbf{u}_T.$$

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Thank you!