Helicity in dispersive continuum mechanics

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5th Workshop on Compressible Mutiphase Flows
IRMA Strasbourg, 5–7 June, 2023

Barotropic Euler equations

- Kinetic energy $T = \int_{D_t} \rho \frac{\|\boldsymbol{u}\|^2}{2} dD,$
- Potential energy $U = \int_{D_t} W(\rho) dD, \quad W = \rho\, \varepsilon(\rho)$
- Lagrangian $L = T - U,$
- Hamilton's action $a = \int_{t_0}^{t_1} L dt.$

Constrained Hamilton's principle

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{u}) = 0$$

Barotropic Euler equations

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{u}) = 0,$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} \boldsymbol{u} + \nabla h(\rho) = 0, \quad dh = \frac{dp(\rho)}{\rho},$$

Conservation laws of momentum and energy

$$\frac{\partial (\rho \boldsymbol{u})}{\partial t} + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u} + p(\rho) \boldsymbol{I}) = 0,$$

$$\frac{\partial (\rho e)}{\partial t} + \operatorname{div}(\rho e \boldsymbol{u} + p(\rho) \boldsymbol{u}) = 0, \quad e = \varepsilon + \frac{||\boldsymbol{u}||^2}{2}, \quad d\varepsilon = \frac{p(\rho)}{\rho^2} d\rho.$$

Helicity for the barotropic Euler equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\boldsymbol{u} + \nabla h(\rho) = 0, \quad dh = \frac{dp(\rho)}{\rho},$$

Helmholtz equation for vorticity

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \frac{\partial \boldsymbol{\omega}}{\partial \boldsymbol{x}}\boldsymbol{u} + \boldsymbol{\omega}\,\mathrm{div}\,\boldsymbol{u} - \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\boldsymbol{\omega} = 0, \quad \boldsymbol{\omega} = \mathrm{curl}\boldsymbol{u}.$$

Helicity for the barotropic Euler equations

Local helicity conservation law

$$\frac{\partial}{\partial t}\left(\boldsymbol{u}^T\boldsymbol{\omega}\right)+\mathrm{div}\left((\boldsymbol{u}^T\boldsymbol{\omega})\boldsymbol{u}+\left(h(\rho)-\frac{|\boldsymbol{u}|^2}{2}\right)\boldsymbol{\omega}\right)=0,$$

Helicity

$$\mathcal{H}=\iiint_{D_t}\boldsymbol{u}^T\boldsymbol{\omega}\,dD$$

Here D_t is the material domain such that the vorticity field is initially tangent to its boundary (then, for any time, this property is conserved).

Jean Jacques Moreau and Henry Keith Moffatt

1. Moreau J J 1961 Constantes d'un îlot tourbillonnaire en fluide parfait barotrope *C. R. Acad. Sci. Paris* **252** 2810–2812

2. Moffatt H K 1969 The degree of knottedness of tangled vortex lines *J. Fluid Mech.* **35** 117–129

"It so often happens that results we publish turn out to have been proved much earlier, although their true significance may have escaped attention (as did Moreau's paper until I had sight of it in 1979 and cited it in 1981)". (H K Moffatt)

Relation to the relabeling symmetry group (H Gouin (1976), R Salmon (1982), V E Zakharov and E K Kuznetsov (1997)).

Further study of helicity and other generalized integrals for classical hydrodynamic equations and MHD equations (ideal or not) (A V Tur and V V Yanovsky (1993), V I Arnold and B A Khesin (1998), C C Cotter and D D Holm (2013), G M Webb *et al* (2014), A F Cheviakov and M Oberlack (2014), P A Davidson and A Ranjan (2018), ...)

What the helicity is in dispersive hydrodynamics ?

Classes of the dispersive models : gradient type models

- Kinetic energy $T = \int_{D_t} \rho \frac{\|\boldsymbol{u}\|^2}{2} dD$,
- Potential energy $U = \int_{D_t} W(\rho, \|\nabla \rho\|^2) dD$,
- Lagrangian $L = T - U$,
- Hamilton's action $a = \int_{t_0}^{t_1} L dt$.

Constrained Hamilton's principle

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho \boldsymbol{u}) = 0$$

Examples : Euler–Korteweg–Van de Waals equations (P Casal, M Eglit, H Gouin, M Slemrod, L Truskinovsky, S Benzoni-Gavage, M Rodrigues, P Noble, R Danchin, Ch Rohde, D Bresch, B Haspot, ...), thin film flows (F Dhaouadi, J P Vila, SG, ...), ...

Lagrangian for the defocusing NLS equation

$$i\psi_t + \frac{1}{2}\Delta\psi - |\psi|^2\,\psi = 0.$$

Madelung's transform

$$\psi(\mathrm{x}, t) = \sqrt{\rho(\mathrm{x}, t)}e^{i\theta(\boldsymbol{x},t)} \qquad \boldsymbol{u} = \nabla\theta$$

Lagrangian

$$L = \int_{D_t} \left(\rho \frac{|\boldsymbol{u}|^2}{2} - W(\rho, \|\nabla\rho\|) \right) dD$$

$$W(\rho, \|\nabla\rho\|) = \frac{\rho^2}{2} + \frac{1}{4\rho}\frac{\|\nabla\rho\|^2}{2}$$

Lagrangian for thin film flows (F Dhaouadi, SG and J P Vila, 2022)

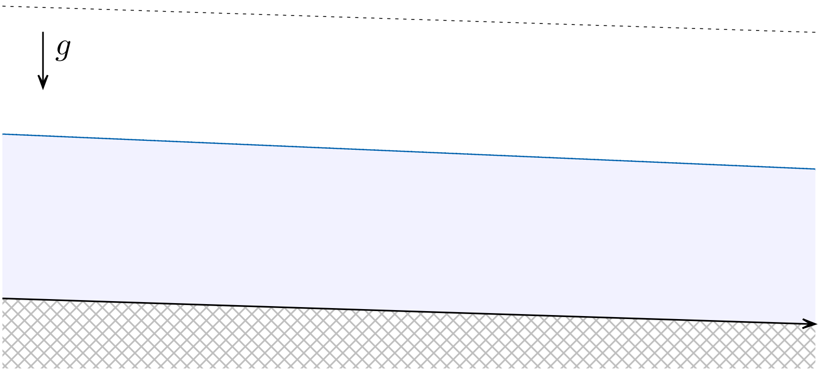
$$L = \int_{D_t} \left(h\frac{|\boldsymbol{u}|^2}{2} - W(h, \|\nabla h\|) \right) dD$$

$$W(h, \|\nabla h\|) = f(h) + \frac{\sigma}{\rho_l}\frac{\|\nabla h\|^2}{2}, \quad \rho_l = const$$

σ is the surface tension coefficient, $\rho_l = const$ is the homogeneous fluid density, $f(h) = \frac{gh^2}{2}$ (not only).

Thin film flows : F Dhaouadi, SG and J P Vila (2022)

Let us replace ρ by h (the fluid depth).



Classes of the dispersive models : fluids with internal inertia

- Kinetic energy $T = \int_{D_t} \rho \frac{\|\boldsymbol{u}\|^2}{2} dD$,
- Potential $U = \int_{D_t} W\left(\rho, \frac{D\rho}{Dt}\right) dD$, $\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u}^T \nabla$.
- Lagrangian $L = T - U$,
- Hamilton's action $a = \int_{t_0}^{t_1} L dt$.

Examples : fluids containing gas bubbles (S V Iordanskii (1960), B Kogarko (1961), L van Wijngaarden (1968), S V Iordanskii and A G Kulikovskii (1977), O V Voinov and A G Petrov (1975)), Serre-Green-Naghdi equations (R Salmon, 1988, SG and V Teshukov (2001))

Serre–Green–Naghdi (SGN) equations

Here ρ is replaced by the fluid depth denoted by $h(t, \boldsymbol{x})$, $\boldsymbol{x} = (x, y)^T$, $\boldsymbol{u}(t, \boldsymbol{x})$ is the depth averaged horizontal velocity. Potential

$$W\left(h, \frac{Dh}{Dt}\right) = \frac{gh^2}{2} - \frac{h}{6}\left(\frac{Dh}{Dt}\right)^2. \tag{1}$$

Solitary wave interaction with an island

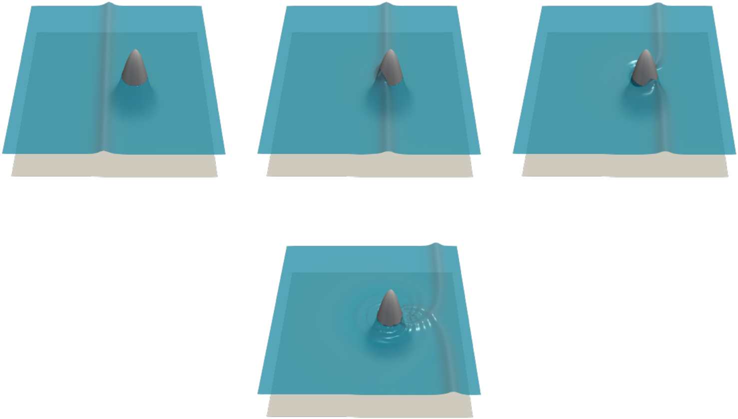


Figure – Interaction of a solitary wave with an island (S. Busto, C. Escalante, M. Dumbser, N. Favrie and SG (JSC 2021)).

Lagrangian fluid mechanics

The motion of a continuous media is

$$\boldsymbol{x} = \varphi\left(t, \boldsymbol{X}\right),$$

where t denotes the time. The transformation φ possesses an inverse and has continuous derivatives up to the second order. The deformation gradient is :

$$\boldsymbol{F} = \frac{\partial \varphi(t, \boldsymbol{X})}{\partial \boldsymbol{X}} \equiv \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}.$$

The evolution equation for $\boldsymbol{F}$:

$$\frac{D\boldsymbol{F}}{Dt} = \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\boldsymbol{F} \quad \text{with} \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u}^T \nabla, \tag{2}$$

where $\boldsymbol{u}(t, \boldsymbol{x}) = \left.\dfrac{\partial \varphi(t, \boldsymbol{X})}{\partial t}\right|_{\boldsymbol{X} = \varphi^{-1}(t, \boldsymbol{x})}$ is the velocity field.

Lemma
Let $\boldsymbol{e}_i$ be a natural local basis (with lower indexes $i = 1, 2, 3$),
$\boldsymbol{e}_i = \dfrac{\partial \boldsymbol{x}}{\partial X^i} = \dfrac{\partial \boldsymbol{\varphi}(t, \boldsymbol{X})}{\partial X^i}$, expressed in Eulerian coordinates, and
$\boldsymbol{e}^i = \nabla X^i(t, \boldsymbol{x})$ (with upper indexes $i = 1, 2, 3$) be the corresponding cobasis (dual basis). Then

$$\frac{\boldsymbol{e}_k}{\det \boldsymbol{F}} = \boldsymbol{e}^i \wedge \boldsymbol{e}^j,$$

where $\{i, j, k\}$ forms an even permutation of $\{1, 2, 3\}$. Moreover,

$$\operatorname{div}\left(\frac{\boldsymbol{e}_k}{\det \boldsymbol{F}}\right) = 0.$$

Proof

The proof of the first formula comes from the identity

$$\delta^i_j = \frac{\partial X^i}{\partial X^j} = \frac{\partial X^i}{\partial \boldsymbol{x}} \frac{\partial \boldsymbol{x}}{\partial X^j} = \boldsymbol{e}^{i\,T} \boldsymbol{e}_j,$$

where δ^i_j is the Kronecker symbol. For the second formula one has :

$$\left(\boldsymbol{e}^i \wedge \boldsymbol{e}^j\right)^T \boldsymbol{e}^k = \frac{\boldsymbol{e}_k^T \boldsymbol{e}^k}{\det \boldsymbol{F}} = \frac{1}{\det \boldsymbol{F}},$$

Finally,

$$\mathrm{div}\left(\frac{\boldsymbol{e}_k}{\det \boldsymbol{F}}\right) = \mathrm{div}\left(\boldsymbol{e}^i \wedge \boldsymbol{e}^j\right),$$

$$= \boldsymbol{e}^{j\,T} \mathrm{curl}\, \boldsymbol{e}^i - \boldsymbol{e}^{i\,T} \mathrm{curl}\, \boldsymbol{e}^j = 0.$$

Equations for the deformation gradient $\boldsymbol{F}$

The following identities are satisfied :

$$\mathrm{div}\left(\frac{\boldsymbol{F}}{\det \boldsymbol{F}}\right) = 0, \tag{3}$$

$$\frac{D}{Dt}\left(\frac{\boldsymbol{F}}{\det \boldsymbol{F}}\right) = \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} - (\mathrm{div}\,\boldsymbol{u})\,\boldsymbol{I}\right)\frac{\boldsymbol{F}}{\det \boldsymbol{F}} \tag{4}$$

In particular, (19) implies

$$\frac{D\boldsymbol{E}_i}{Dt} = \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} - (\mathrm{div}\,\boldsymbol{u})\,\boldsymbol{I}\right)\boldsymbol{E}_i, \quad \boldsymbol{E}_i = \frac{\boldsymbol{e}_i}{\det \boldsymbol{F}}, \quad \mathrm{div}\boldsymbol{E}_i = 0.$$

Proof Use the equation for the deformation gradient and Euler's equation for its determinant.

Example

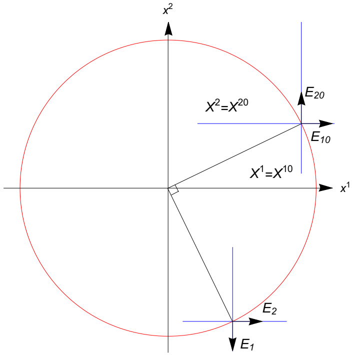


Figure – The stream lines (circles) of the velocity field $\boldsymbol{u} = (x^2, -x^1)^T$ are shown in the (x^1, x^2) plane. The initial material lines $X^i = const$ having tangent vectors $\boldsymbol{E}_{i0},\ i = 1, 2$ represent at each time instant straight lines rotating with a constant angular velocity. The material lines positions at time $t = \pi/2$ and their tangent vectors $\boldsymbol{E}_i,\ i = 1, 2$ are shown.

Basic theorem (part 1)

1^o) Consider the divergence–free field $\boldsymbol{L}$ satisfying the Helmholtz equation

$$\frac{D\boldsymbol{L}}{Dt} + \boldsymbol{L}\,\mathrm{div}\,\boldsymbol{u} - \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\boldsymbol{L} = 0,$$

and the field $\boldsymbol{K}$ satisfying

$$\frac{D\boldsymbol{K}}{Dt} + \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\right)^T \boldsymbol{K} + \nabla\left(G - \frac{|\boldsymbol{u}|^2}{2}\right) = 0.$$

Then,

$$\frac{\partial}{\partial t}\left(\boldsymbol{K}^T\,\boldsymbol{L}\right) + \mathrm{div}\left\{\left(\boldsymbol{u}\,\boldsymbol{K}^T + \left(G - \frac{|\boldsymbol{u}|^2}{2}\right)\boldsymbol{I}\right)\boldsymbol{L}\right\} = 0. \qquad (5)$$

Basic theorem (part 2)

2^o) Consider a material domain D_t of boundary S_t. If at $t = 0$ the divergence–free vector field $\boldsymbol{L}$ is tangent to S_0, then for any time t the vector field $\boldsymbol{L}$ is tangent to S_t. Moreover, the quantity

$$\mathcal{H} = \iiint_{D_t} \boldsymbol{K}^T \boldsymbol{L} \, dD, \tag{6}$$

we call *generalized helicity*, keeps a constant value along the motion.

Euler–van der Waals–Korteweg's fluids : $W(\rho, \|\nabla\rho\|)$

Governing equations

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{u}) = 0,$$

$$\frac{\partial \rho\, \boldsymbol{u}^T}{\partial t} + \operatorname{div}(\rho\, \boldsymbol{u} \otimes \boldsymbol{u} + \Pi) + \rho\, \frac{\partial V}{\partial \boldsymbol{x}} = 0^T, \quad \Pi = P\, \boldsymbol{I} + \frac{\partial W}{\partial \left(\dfrac{\partial \rho}{\partial \boldsymbol{x}}\right)} \frac{\partial \rho}{\partial \boldsymbol{x}},$$

$$P = \rho \frac{\delta W}{\delta \rho} - W \quad \text{with} \quad \frac{\delta W}{\delta \rho} = \frac{\partial W}{\partial \rho} - \operatorname{div}\left(\frac{\partial W}{\partial \nabla \rho}\right).$$

Energy conservation law :

$$\frac{\partial e}{\partial t} + \operatorname{div}\left(e\boldsymbol{u} + \Pi\boldsymbol{u} - \frac{\partial \rho}{\partial t} \frac{\partial W}{\partial \left(\dfrac{\partial \rho}{\partial \boldsymbol{x}}\right)}\right) - \rho\, \frac{\partial V}{\partial t} = 0, \quad e = \frac{\rho |\boldsymbol{u}|^2}{2} + W + \rho\, V$$

The momentum equation can be written as

$$\frac{D\boldsymbol{u}}{Dt} + \nabla\left(\frac{\delta W}{\delta \rho} + V\right) = 0,$$

where $W = W(\rho, \|\nabla \rho\|)$ and $V(t, \boldsymbol{x})$ is a given external potential. We use the notation

$$H = \frac{\delta W}{\delta \rho} + V$$

for the total specific enthalpy. The momentum equation can be written in the form :

$$\frac{D\boldsymbol{u}}{Dt} + \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\right)^T \boldsymbol{u} + \nabla\left(H - \frac{|\boldsymbol{u}|^2}{2}\right) = 0. \qquad (7)$$

The vorticity of the capillary fluid $\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{u}$ satisfies the Helmholtz equation :

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \frac{\partial \boldsymbol{\omega}}{\partial \boldsymbol{x}}\boldsymbol{u} + \boldsymbol{\omega} \operatorname{div} \boldsymbol{u} - \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\boldsymbol{\omega} = 0.$$

Theorem

1^o) *Equations of capillary fluids (7) admit the following conservation law*

$$\frac{\partial}{\partial t}\left(\boldsymbol{u}^T\,\omega\right)+\operatorname{div}\left\{\left(\boldsymbol{u}^T\omega\right)\boldsymbol{u}+\left(H-\frac{|\boldsymbol{u}|^2}{2}\right)\omega\right\}=0$$

2^o) *Consider a material domain D_t of boundary S_t. If at $t=0$ the vector field $\boldsymbol{\omega}$ is tangent to S_0, then for any time t the vector field $\boldsymbol{\omega}$ is tangent to S_t. Moreover, the quantity (called helicity)*

$$\mathcal{H}=\iiint_{D_t}\boldsymbol{u}^T\omega\,dD$$

keeps a constant value along the motion. The results remain true if we replace $\boldsymbol{\omega}$ by vectors $\boldsymbol{E}_i$, $i=1,2,3$.

Application to the NLS equation

For the NLS equation $\boldsymbol{\omega} = 0$. Thus we have only conservation laws related with the vectors $\boldsymbol{E}_i$, $i = 1, 2, 3$.

Fluids with internal inertia

Mathematical model of bubbly fluids with incompressible liquid phase at small volume concentration of gas bubbles (Iordansky–Kogarko–van Wijngarden model), Serre–Green–Naghdi (SGN) equations describing long surface gravity waves. These models can be obtained as Euler-Lagrange equations of the Hamilton action (SG, V M Teshukov, 2001).

$$\mathcal{L} = \frac{\rho\|\boldsymbol{u}\|^2}{2} - W\left(\rho, \frac{D\rho}{Dt}\right) - \rho\, V(t, \boldsymbol{x}).$$

Here $W\left(\rho, \frac{D\rho}{Dt}\right)$ represents a potential depending not only on the density, but also on the material time derivative of the density.

Governing equations

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{u}) = 0,$$

$$\frac{\partial \rho\, \boldsymbol{u}^T}{\partial t} + \operatorname{div}(\rho\, \boldsymbol{u} \otimes \boldsymbol{u} + p\, \boldsymbol{I}) + \rho \frac{\partial V}{\partial \boldsymbol{x}} = 0^T,$$

$$p = \rho\, \frac{\delta W}{\delta \rho} - W$$

The energy conservation law

$$\frac{\partial e}{\partial t} + \mathrm{div}\left((e+p)\, \boldsymbol{u}\right) - \rho\, \frac{\partial V}{\partial t} = 0,$$

with the definition of the total volume energy e

$$e = \frac{\rho |\boldsymbol{u}|^2}{2} + E + \rho V \quad \text{where} \quad E = W - \frac{D\rho}{Dt}\left(\frac{\partial W}{\partial\left(\frac{D\rho}{Dt}\right)}\right).$$

We introduce the vector field $\boldsymbol{K}$:

$$\boldsymbol{K} = \boldsymbol{u} + \frac{\nabla \sigma}{\rho} \quad \text{where} \quad \sigma = -\rho \left(\frac{\partial W}{\partial \left(\frac{D\rho}{Dt} \right)} \right).$$

Let us introduce the volume internal energy $E(\rho, \tau)$ as a partial Legendre transform of $W(\rho, \dot{\rho})$:

$$E(\rho, \tau) = W + \frac{D\rho}{Dt} \tau \quad \text{where} \quad \tau = -\left(\frac{\partial W}{\partial \left(\frac{D\rho}{Dt} \right)} \right).$$

Instead of $E(\rho, \tau)$ we define $\tilde{E}(\rho, \sigma) = E(\rho, \sigma/\rho)$. The momentum equation becomes (SG, Teshukov, 2001) :

$$\frac{D\boldsymbol{K}}{Dt} + \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} \right)^T \boldsymbol{K} + \nabla \left(\tilde{E}_\rho + V - \frac{|\boldsymbol{u}|^2}{2} \right) = 0$$

Theorem

1^o) The equations of fluids with internal inertia admit the conservation law

$$\frac{\partial}{\partial t}\left(\boldsymbol{K}^T\,\Omega\right)+\operatorname{div}\left\{\left(\boldsymbol{K}^T\Omega\right)\boldsymbol{u}+\left(\tilde{E}_\rho+V-\frac{|\boldsymbol{u}|^2}{2}\right)\Omega\right\}=0$$

2^o) Consider a material domain $\mathcal{D}_t$ of boundary S_t. If at $t=0$ the vector field Ω is tangent to S_0, then for any time t the vector field Ω is tangent to S_t. Moreover, the quantity

$$\mathcal{H}=\iiint_{D_t}\boldsymbol{K}^T\Omega\,dD$$

we call *generalized helicity*, keeps a constant value along the motion. The results remain true if we replace Ω by the vectors $\boldsymbol{E}_i$, $i=1,2,3$.

Applications to Serre-Green-Naghdi equations

$$W(h,\dot{h}) = \frac{g\,h^2}{2} - \frac{h}{6}\left(\frac{Dh}{Dt}\right)^2 \quad \text{with} \quad \frac{Dh}{Dt} = \frac{\partial h}{\partial t} + \nabla h \cdot \boldsymbol{u}.$$

The water depth h and depth averaged velocity $\boldsymbol{u}$ are functions of time t and of the horizontal coordinates $\boldsymbol{x} = (x^1, x^2)^T$.

$$\boldsymbol{K} = \boldsymbol{u} + \frac{1}{3h}\nabla\left(h^3\,\mathrm{div}\boldsymbol{u}\right).$$

The physical meaning of $\boldsymbol{K}$: it is the fluid velocity tangent to the free surface (SG, Z Khorsand and H Kalisch (2015), Y Matsuno (2016)). The equation for $\boldsymbol{K}$ is :

$$\frac{D\boldsymbol{K}}{Dt} + \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}\right)^T \boldsymbol{K} + \nabla\left(g\,h - \frac{1}{2}\left(\frac{Dh}{Dt}\right)^2 - \frac{|\boldsymbol{u}|^2}{2}\right) = 0.$$

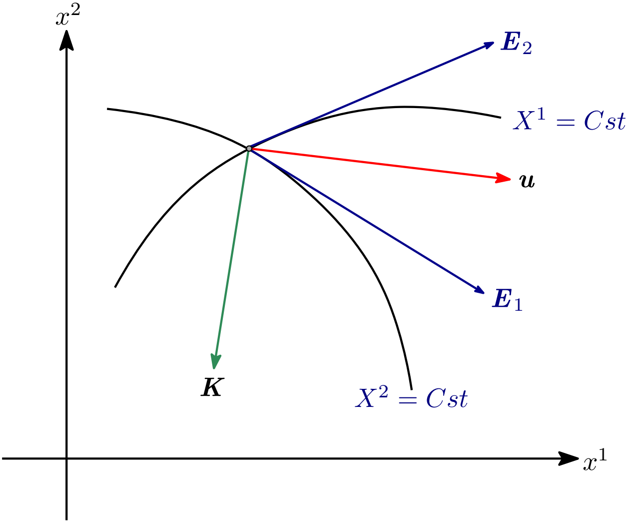


Figure – In the (x^1, x^2) the material curves $X^1(t, x^1, x^2) = const$ and $X^2(t, x^1, x^2) = const$ are shown. At any point, these curves are tangent to the vectors $\boldsymbol{E}_2$ and $\boldsymbol{E}_1$. *A priori*, the vectors $\boldsymbol{u}$ and $\boldsymbol{K}$ are not collinear.

Conclusion

An analog of helicity integrals is found for two classes of dispersive systems of equations coming from Hamilton's principle of stationary action.

Références :

S. Gavrilyuk and H. Gouin,
https ://hal.archives-ouvertes.fr/hal-03867002/document