Machine learning using Hawkes processes and concentration for matrix martingales

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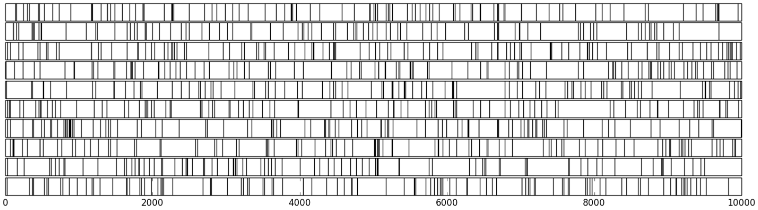
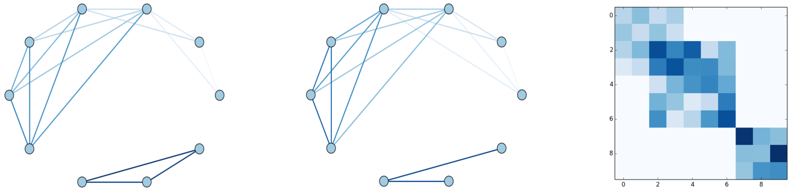
- You have users of a system (a social network, an e-commerce platform, etc.)
- You want to quantify the level of interaction between users
- You don't want to use only *declared* interactions, such as "friendship" or "likes". This information is often deprecated, and not really related to the activity of users
- You want levels of interaction driven by user's actions, using the timestamps' patterns of actions

Introduction

From:



We want to quantify interactions between users:



Model: Multivariate Hawkes Process (MHP)

- A d-dimensional counting process $N = [N_1, \ldots, N_d]^\top$
- d is "large"
- Observed on $[0, T]$. "Asymptotics" in $T \to +\infty$
- N_j has intensity λ_j, namely

$$\mathbb{P}\big(N_j \text{ has a jump in } [t, t + dt] \mid \mathcal{F}_t\big) = \lambda_j(t)dt$$

for $j = 1, \ldots, d$ where $\mathcal{F}_t$ some filtration

Model: Multivariate Hawkes Process (MHP)

- MHP assumes the following autoregressive structure:

$$\lambda_j(t) = \mu_j(t) + \int_{(0,t)} \sum_{k=1}^{d} \varphi_{j,k}(t-s) dN_k(s),$$

- $\mu_j(t) \geq 0$ baseline intensity of the j-th coordinate
- $\varphi_j : \mathbb{R}^+ \to \mathbb{R}^+$ self-exciting component
- Write this in matrix form

$$\lambda(t) = \boldsymbol{\mu} + \int_{(0,t)} \boldsymbol{\varphi}(t-s) dN(s),$$

with $\boldsymbol{\mu} = [\mu_1, \ldots, \mu_d]^\top$ and $\boldsymbol{\varphi}(t) = [\varphi_{j,k}(t)]_{1 \leq j,k \leq d}$.
- Notation:

$$\int_{(0,t)} \varphi(t-s) dN_k(s) = \sum_{i: 0 < T_{i,k} < t} \varphi(t - T_{i,k})$$

Introduced by Hawkes in 1971

- **Earthquakes and geophysics** : Kagan and Knopoff (1981), Zhuang, Harte, Werner, Hainzl and Zhou (2012)
- **Genomics** : Reynaud-Bouret and Schbath (2010)
- **High-frequency Finance** : Bacry Delattre Hoffmann and Muzy (2013)
- **Terrorist activity** : Porter and White (2012)
- **Neurobiology** : Hansen, Reynaud-Bouret and Rivoirard (2012)
- **Social networks** : Carne and Sornette (2008), Simma and Jordan (2010), Zhou Song and Zha (2013)
- And even **FPGA-based implementation** : Guo and Luk (2013)

A brief history of MHP

THE GENESIS BLOCK

Digital currency research and data

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Analyzing Trade Clustering To Predict Price Movement In Bitcoin Trading

Sep 19, 2013 Posted By *Jonathan Heusser* In *Bitcoin 201*, *Economics*, *Featured*, *News*, *Trading* Tagged *Analysis*, *Bitcoin Trading*, *Hawkes Process*, *Jonathan Heusser*, *London*, *Price*, *Trading*

Parametric estimation (Maximum likelihood)

- First work : Ogata 78
- Simma and Jordan (2010), Zhou Song and Zha (2013)
  $\rightarrow$ Expected Maximization (EM) algorithms, with priors

Non parametric estimation

- Marsan Lengliné (2008), generalized by Lewis, Mohler (2010)
  $\rightarrow$ EM for penalized likelihood function
  $\rightarrow$ Monovariate Hawkes processes, Small amount of data, No theoretical results
- Reynaud-Bouret and Schbath (2010)
  $\rightarrow$ Developed for small amount of data (Sparse penalization)
- Bacry and Muzy (2014)
  $\rightarrow$ Larger amount of data

What do we want to do with this?

- Do inference directly from **actions** of users
- Understand the community structure of users underlying the actions
- Exploit the hidden lower-dimensional structure of the network for inference/prediction

Dimension d is large:

- Need a simple parametric model on μ and φ
- For inference: we want a **tractable** and **scalable** optimization problem
- We want to encode some prior assumptions by penalizing the likelihood

Simple parametrization:

- Constant baselines $\mu_j(\cdot) \equiv \mu_j$
- Take

$$\varphi_{j,k}(t) = a_{j,k} e^{-\alpha_{j,k} t}$$

- $a_{j,k}$ = level of interaction between nodes j and k
- $\alpha_{j,k}$ = lifetime of instantaneous excitation of node j by node k

The matrix

$$\boldsymbol{A} = [a_{j,k}]_{1 \leq j,k \leq d}$$

is understood has a **weighted adjacency matrix** of mutual excitement of th nodes $\{1, \ldots, d\}$

- $\boldsymbol{A}$ is non-symmetric: oriented graph

We end up with intensities

$$\lambda_{j,\theta}(t) = \mu_j + \int_{(0,t)} \sum_{k=1}^{d} a_{j,k} e^{-\alpha_{j,k}(t-s)} dN_k(s)$$

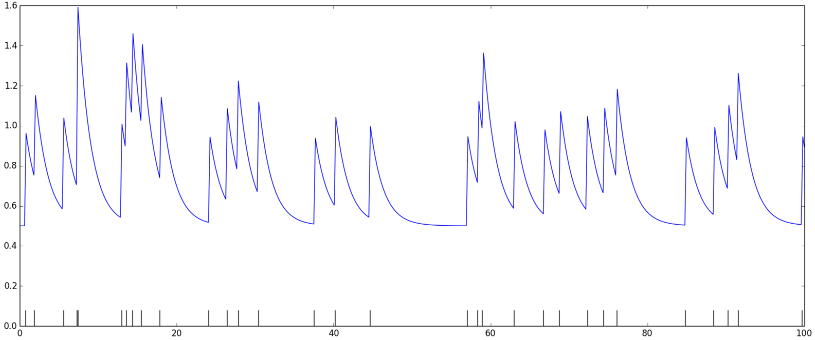
for $j \in \{1, \ldots, d\}$ where

$$\theta = [\mu, \boldsymbol{A}, \boldsymbol{\alpha}]$$

with

- baselines $\mu = [\mu_1, \ldots, \mu_d]^\top \in \mathbb{R}_+^d$
- interactions $\boldsymbol{A} = [a_{j,k}]_{1 \le j,k \le d} \in \mathbb{R}_+^{d \times d}$
- decays $\boldsymbol{\alpha} = [\alpha_{j,k}]_{1 \le j,k \le d} \in \mathbb{R}_+^{d \times d}$

For $d = 1$, intensity λ_θ looks like this:



Goodness-of-fit $= -$ log-likelihood is given by:

$$-\ell_T(\theta) = \sum_{j=1}^{d} \Big\{ \int_0^T (\lambda_{j,\theta}(t) - 1)dt - \int_0^T \log \lambda_{j,\theta}(t) dN_j(t) \Big\}$$

with

$$\lambda_{j,\theta}(t) = \mu_j + \sum_{k=1}^{d} a_{j,k} \int_{(0,t)} \exp\big(- \alpha_{j,k}(t-s)\big) dN_k(s)$$

where $\theta = [\boldsymbol{\mu}, \boldsymbol{A}, \boldsymbol{\alpha}]$ with $\boldsymbol{\mu} = [\mu_j]$, $\boldsymbol{A} = [a_{j,k}]$, $\boldsymbol{\alpha} = [\alpha_{j,k}]$

Prior assumptions

- Some users are basically inactive and react only if stimulated:

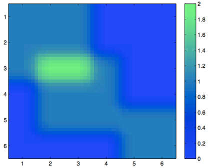
$$\boldsymbol{\mu} \text{ is sparse}$$

- Everybody does not interact with everybody:

$$\boldsymbol{A} \text{ is sparse}$$

- Interactions have community structure, possibly overlapping, a small number of factors explain interactions:

$$\boldsymbol{A} \text{ is low-rank}$$



- Decays $\boldsymbol{\alpha}$ are not sparse, but $\alpha_{j,k}$ should be regularized proportionaly to $a_{j,k}$

Standard convex relaxations [Tibshirani (01), ..., Srebro et al. (05), Bach (08), Candès & Recht (08), ...]

- Convex relaxation of $\|\boldsymbol{A}\|_0 = \sum_{j,k} \mathbf{1}_{\boldsymbol{A}_{j,k}>0}$ is ℓ_1-norm:

$$\|\boldsymbol{A}\|_1 = \sum_{j,k} |\boldsymbol{A}_{j,k}|$$

- Convex relaxation of rank is trace-norm:

$$\|A\|_* = \sum_j \sigma_j(A) = \|\sigma(A)\|_1$$

where $\sigma_1(A) \geq \cdots \geq \sigma_d(A)$ singular values of $\boldsymbol{A}$

So, we use the following penalizations

- Use ℓ_1 penalization on $\boldsymbol{\mu}$
- Use ℓ_1 penalization on $\boldsymbol{A}$
- Use trace-norm penalization on $\boldsymbol{A}$
- Use ℓ_2^2 penalization on $\boldsymbol{\alpha}$, weighted by $\boldsymbol{A}$

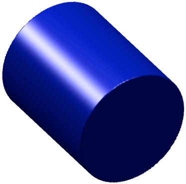
[but other choices might be interesting...]

NB1: to induce **sparsity AND low-rank** on $\boldsymbol{A}$, we use the mixed penalization

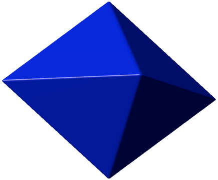
$$\boldsymbol{A} \mapsto w_* \|\boldsymbol{A}\|_* + w_1 \|\boldsymbol{A}\|_1$$

NB2: recent work by Richard et al (2013): better way to induce sparsity and low-rank than the sum

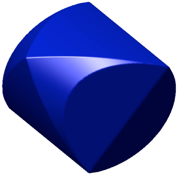
Sparse and low-rank matrices



$\{\boldsymbol{A} : \|A\|_* \leq 1\}$



$\{\boldsymbol{A} : \|\boldsymbol{A}\|_1 \leq 1\}$



$\{\boldsymbol{A} : \|\boldsymbol{A}\|_1 + \|\boldsymbol{A}\|_* \leq 1\}$

The balls are computed on the set of 2×2 symmetric matrices, which is identified with $\mathbb{R}^3$.

Penalized maximum likelihood

Finally, consider

$$\hat{\theta} \in \underset{\theta=(\boldsymbol{\mu},\boldsymbol{A},\boldsymbol{\alpha})}{\operatorname{argmin}} \Big\{ -\frac{1}{T}\ell_T(\theta) + \tau\|\boldsymbol{\mu}\|_1 + \gamma_1\|\boldsymbol{A}\|_1 + \gamma_*\|\boldsymbol{A}\|_* + \frac{\kappa}{2}\|\boldsymbol{A}\odot\boldsymbol{\alpha}\|_F^2 \Big\}$$

where we recall

$$-\frac{1}{T}\ell_T(\theta) = \frac{1}{T}\sum_{j=1}^d \Big\{ \int_0^T \lambda_{j,\theta}(t)dt - \int_0^T \log\lambda_{j,\theta}(t)dN_j(t) \Big\}$$

with

$$\lambda_{j,\theta}(t) = \mu_j + \sum_{k=1}^d a_{j,k}\int_{(0,t)} \exp\big(-\alpha_{j,k}(t-s)\big)dN_k(s)$$

Penalized maximum likelihood: a problem

Problem: $\theta \mapsto -\ell_T(\theta)$ not convex! Indeed

$$(a, \alpha) \mapsto a h_\alpha(t)$$

never convex when $\alpha \mapsto h_\alpha(t)$ is convex

We **want** convexity for:

- Convergence to a global optimum
- Plethora of optimization algorithm
- If smooth (Lispchitz gradient): optimal first-order techniques [first order=mandatory for large scale problems]

Generic in the chosen penalization [if proximal operator easy to compute]

Penalized maximum likelihood

A solution: the **perspective function** trick:

- If $\alpha \mapsto h_\alpha(t)$ is convex, then

$$(a, \alpha) \mapsto a h_{\alpha/a}(t)$$

  is **convex!**

- Reparametrization $\beta_{j,k} = a_{j,k}\alpha_{j,k}$, leading to

$$\lambda_{j,\theta}(t) = \mu_j + \sum_{k=1}^{d} a_{j,k} \int_{(0,t)} \exp\Big(- \frac{\beta_{j,k}}{a_{j,k}}(t-s)\Big) dN_k(s)$$

  with $\theta = [\boldsymbol{\mu}, \boldsymbol{A}, \boldsymbol{\beta}]$ for $\boldsymbol{\beta} = [\beta_{j,k}]_{1 \leq j,k \leq d}$

- With this reparametrization

$$\theta \mapsto \lambda_{j,\theta}(t)$$

  is **convex**!

The reparametrization $\boldsymbol{\beta} = \boldsymbol{A} \odot \boldsymbol{\alpha}$ leads to

$$\hat{\theta} \in \operatorname*{argmin}_{\theta=(\boldsymbol{\mu},\boldsymbol{A},\boldsymbol{\beta})} \Big\{ -\frac{1}{T}\ell_T(\theta) + \tau\|\boldsymbol{\mu}\|_1 + \gamma_1\|\boldsymbol{A}\|_1 + \gamma_*\|\boldsymbol{A}\|_* + \frac{\kappa}{2}\|\boldsymbol{\beta}\|_F^2 \Big\} \tag{1}$$

where

$$-\frac{1}{T}\ell_T(\theta) = \frac{1}{T}\sum_{j=1}^{d}\Big\{ \int_0^T \lambda_{j,\theta}(t)dt - \int_0^T \log \lambda_{j,\theta}(t)dN_j(t)\Big\}$$

with

$$\lambda_{j,\theta}(t) = \mu_j + \sum_{k=1}^{d} a_{j,k}\int_{(0,t)} \exp\Big(-\frac{\beta_{j,k}}{a_{j,k}}(t-s)\Big)dN_k(s)$$

Convex optimization – numerical aspects

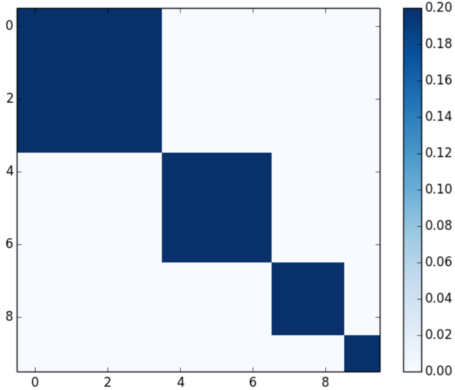
- Can be solved using optimal first-order routines
- Gradient of $-\ell_T(\theta)$ using a recursion formula [Ogata (1988)]

  $\rightarrow$ When carefully done complexity of one gradient is $O(nd)$ (instead of $O(n^2 d)$ for the naive approach), where $n =$ number of events (very large)

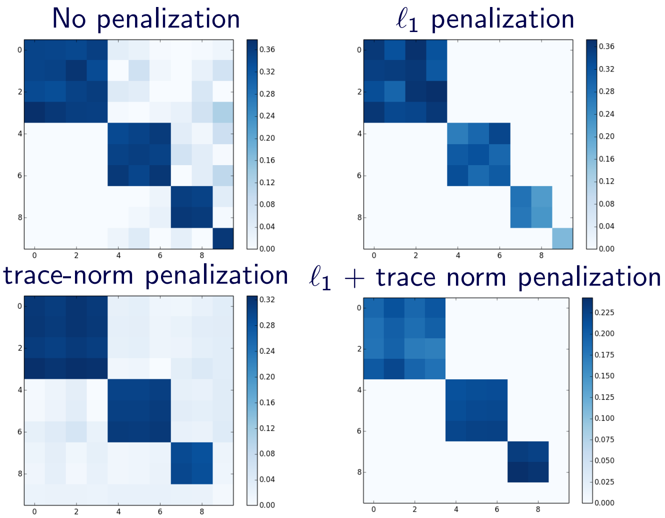
  $\rightarrow$ We have scalable / parallelized code for this: the gradient on each node $j \in \{1, \ldots, d\}$ can be computed in a **distributed** fashion
- Computation bootleneck is the heavy use of exp and log [accelerated using some ugly hacking]
- Proximal of trace norm requires many truncated SVD: we use the default's Lanczos's implementation of Python, fast enough when using incremental truncation

Numerical experiment

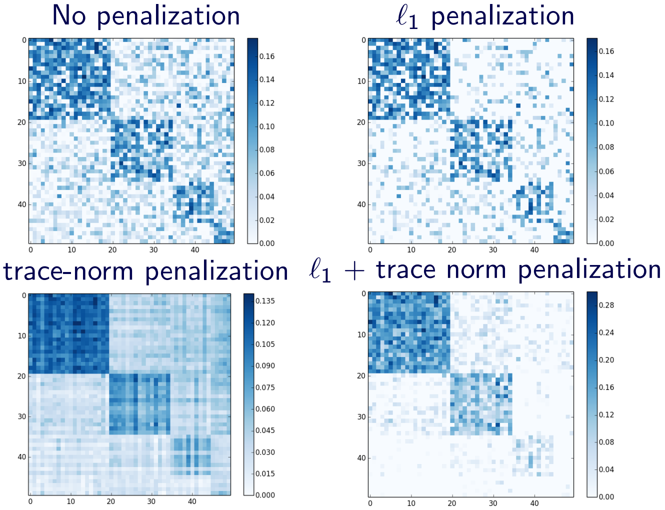
Toy example: take matrix $\boldsymbol{A}$ as



Numerical experiment: dimension 10, 210 parameters



Numerical experiment: dimension 100, 20100 parameters



A simplified framework: fix a set $\{h_{j,k} : 1 \le j, k \le d\}$ and intensities

$$\lambda_{j,\theta}(t) = \mu_j + \int_{(0,t)} \sum_{k=1}^{d} a_{j,k} h_{j,k}(t-s) dN_k(s),$$

where $\theta = [\mu, \boldsymbol{A}]$ with $\mu = [\mu_1, \ldots, \mu_d]^\top$ and $\boldsymbol{A} = [a_{j,k}]_{1 \le j,k \le d}$

Instead of $-\log$ likelihood, consider least squares

$$R_T(\theta) = \|\lambda_\theta\|_T^2 - \frac{2}{T} \sum_{j=1}^{d} \int_{[0,T]} \lambda_{j,\theta}(t) dN_j(t)$$

where $\|\lambda_\theta\|_T^2 = \langle \lambda_\theta, \lambda_\theta \rangle_T$ with

$$\langle \lambda_\theta, \lambda_{\theta'} \rangle_T = \frac{1}{T} \sum_{j=1}^{d} \int_{[0,T]} \lambda_{j,\theta}(t) \lambda_{j,\theta'}(t) dt.$$

Least-squares goodness-of-fit

$$R_T(\theta) = \|\lambda_\theta\|_T^2 - \frac{2}{T}\sum_{j=1}^{d}\int_{[0,T]} \lambda_{j,\theta}(t) dN_j(t)$$

is natural : if N has ground truth intensity λ^*:

$$\mathbb{E}[R_T(\theta)] = \mathbb{E}\|\lambda_\theta\|_T^2 - 2\mathbb{E}\langle \lambda_\theta, \lambda^* \rangle_T = \mathbb{E}\|\lambda_\theta - \lambda^*\|_T^2 - \|\lambda^*\|_T$$

where we used "signal + noise" decomposition (Doob-Meyer):

$$dN_j(t) = \lambda^*(t)dt + dM_j(t)$$

where M_j martingale

Introduce

$$\hat{\theta} \in \operatorname*{argmin}_{\theta \in \mathbb{R}_+^d \times \mathbb{R}_+^{d \times d}} \left\{ R_T(\theta) + \mathrm{pen}(\theta) \right\},$$

with

$$\mathrm{pen}(\theta) = \|\mu\|_{1,\hat{w}} + \|\boldsymbol{A}\|_{1,\hat{\boldsymbol{W}}} + \hat{w}_* \|\boldsymbol{A}\|_*$$

- Penalization tuned by data-driven weights $\hat{w}$, $\hat{\boldsymbol{W}}$ and $\hat{w}_*$
- Comes from sharp controls of the noise terms, using new probabilistic tools

Towards a statistical guarantee: first order condition can be written as: for any θ

$$\|\lambda_{\hat{\theta}} - \lambda^*\|_T^2 + \|\lambda_{\hat{\theta}} - \lambda_\theta\|_T^2 - \|\lambda_\theta - \lambda^*\|_T^2 \leq -\langle \theta_\partial, \hat{\theta} - \theta \rangle + \frac{2}{T}\langle \hat{\mu} - \mu, \bar{M}_T \rangle + \frac{2}{T}\langle \hat{\boldsymbol{A}} - \boldsymbol{A}, \boldsymbol{Z}_T \rangle,$$

for $\theta_\partial \in \partial\, \mathrm{pen}(\theta)$ and we use $\frac{2}{T}\langle \hat{\boldsymbol{A}} - \boldsymbol{A}, \boldsymbol{Z}_T \rangle \leq \frac{2}{T}\|\hat{\boldsymbol{A}} - \boldsymbol{A}\|_* \|\boldsymbol{Z}_T\|_{\mathrm{op}}$

$\bar{M}_T = [\int_0^T dM_1(t) \ \cdots \ \int_0^T dM_d(t)]^\top$ and $\boldsymbol{Z}_t$ matrix martingale with entries

$$(\boldsymbol{Z}_t)_{j,k} = \int_0^t \int_{(0,s)} h_{j,k}(s-u) dN_k(u) dM_j(s), \qquad (2)$$

or

$$\boldsymbol{Z}_t = \int_0^t \mathrm{diag}[dM_s] \boldsymbol{H}_s,$$

with $\boldsymbol{H}_t$ predictable process with entries

$$(\boldsymbol{H}_t)_{j,j'} = \int_{(0,t)} h_{j,j'}(t-s) dN_{j'}(s)$$

Noise term is a matrix-martingale in continuous time:

$$\frac{1}{T} \boldsymbol{Z}_T$$

wee need to control $\frac{1}{T} \|\boldsymbol{Z}_T\|_{\mathrm{op}}$

A consequence of our new concentration inequalities:

$$\mathbb{P}\Bigg[\frac{\|\boldsymbol{Z}_t\|_{\mathrm{op}}}{t} \geq \sqrt{\frac{2v(x+\log(2d))}{t}} + \frac{b(x+\log(2d))}{3t}, \\ b_t \leq b, \quad \lambda_{\max}(\boldsymbol{V}_t) \leq v\Bigg] \leq e^{-x},$$

for any $v, x, b > 0$, where

$$\boldsymbol{V}_t = \frac{1}{t}\int_0^t \|\boldsymbol{H}_s\|_{2,\infty}^2 \begin{bmatrix} \mathrm{diag}[\lambda_s^*] & \mathbf{0} \\ \mathbf{0} & \boldsymbol{H}_s^\top \mathrm{diag}[\boldsymbol{H}_s \boldsymbol{H}_s^\top]^{-1} \mathrm{diag}[\lambda_s^*] \boldsymbol{H}_s \end{bmatrix} ds$$

and $b_t = \sup_{s\in[0,t]} \|\boldsymbol{H}_s\|_{2,\infty}$ ($\|\cdot\|_{2,\infty}$ = maximum ℓ_2 row norm)

Useless for statistical learning! Event $\lambda_{\max}(\boldsymbol{V}_t) \leq v$ is annoying and $\boldsymbol{V}_t$ is **not observable** (depends on λ^*)!

Theorem [Something better]. For any $x > 0$, we have

$$\frac{\|\boldsymbol{Z}_t\|_{\mathrm{op}}}{t} \leq 8\sqrt{\frac{(x + \log d + \hat{\ell}_{x,t})\lambda_{\max}(\hat{\boldsymbol{V}}_t)}{t}} + \frac{(x + \log d + \hat{\ell}_{x,t})(10.34 + 2.65 b_t)}{t}$$

with a probability larger than $1 - 84.9e^{-x}$, where

$$\hat{\boldsymbol{V}}_t = \frac{1}{t}\int_0^t \|\boldsymbol{H}_s\|_{2,\infty}^2 \begin{bmatrix} \mathrm{diag}[dN_s] & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{H}_s^\top \mathrm{diag}[\boldsymbol{H}_s\boldsymbol{H}_s^\top]^{-1} \mathrm{diag}[dN_s]\boldsymbol{H}_s \end{bmatrix} ds$$

and small ugly term:

$$\hat{\ell}_{x,t} = 4 \log\log\left(\frac{2\lambda_{\max}(\hat{\boldsymbol{V}}_t) + 2(4 + b_t^2/3)x}{x} \vee e\right) + 2\log\log\left(b_t^2 \vee e\right).$$

This is a non-commutative deviation inequality with **observable** variance

- These concentration inequalities leads to a data-driven tuning of penalization
- Solves the **"scaling" problem** in this context $\approx$ **features scaling** in supervised learning

Controls on $\|\boldsymbol{Z}_T\|_\infty = \max_{j,k} |A_{j,k}|$ and $\|\boldsymbol{Z}_T\|_{\mathrm{op}}$ leads to the following tuning of the penalizations

For ℓ_1 penalization of μ: $\|\mu\|_{1,\hat{w}} = \sum_{j=1}^{d} \hat{w}_j |\mu_j|$ with

$$\hat{w}_j = 6\sqrt{2}\sqrt{\frac{(x + \log d + \hat{\ell}_{x,j,T})N_j([0,T])/T}{T}} + 27.93\frac{x + \log d + \hat{\ell}_{x,j,T}}{T}$$

where $N_j([0,T]) = \int_0^T dN_j(t)$, namely

$$\hat{w}_j \approx c\sqrt{\frac{N_j([0,T])/T}{T}}$$

- Each coordinate j of μ is penalized (roughly) by $N_j([0,T)]/T$: estimated average intensity of events of node j

For ℓ_1 penalization of $\boldsymbol{A}$: $\|\boldsymbol{A}\|_{1,\hat{\boldsymbol{W}}} = \sum_{1 \le j,k \le d} \hat{\boldsymbol{W}}_{j,k} |\boldsymbol{A}_{j,k}|$ with

$$\hat{\boldsymbol{W}}_{j,k} = 4\sqrt{2}\sqrt{\frac{(x + 2\log d + \hat{\ell}_{x,j,k,T})\hat{\boldsymbol{V}}_{j,k}(T)}{T}} + 18.62\frac{(x + 2\log d + \hat{\ell}_{x,j,k,T})\boldsymbol{B}_{j,k}(T)}{T}$$

where

$$\boldsymbol{B}_{j,k}(t) = \sup_{s \in [0,t]} \int_{(0,t)} h_{j,k}(t-s) dN_k(s)$$

$$\hat{\boldsymbol{V}}_{j,k}(t) = \frac{1}{t} \int_0^t \Big(\int_{(0,s)} h_{j,k}(s-u) dN_k(u) \Big)^2 dN_j(s)$$

namely

$$\hat{\boldsymbol{W}}_{j,k} \approx c\sqrt{\frac{\hat{\boldsymbol{V}}_{j,k}(T)}{T}}$$

$\hat{\boldsymbol{V}}_{j,k}(t)$ estimates the variance of self-excitements between nodes j and k

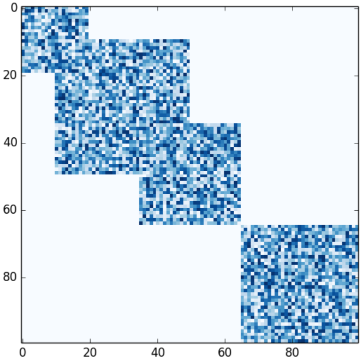
For trace-norm penalization of $\boldsymbol{A}$: $\hat{w}_* \|\boldsymbol{A}\|_*$ with

$$\hat{w}_* = 8\sqrt{\frac{(x + \log d + \hat{\ell}_{x,T})\lambda_{\max}(\hat{\boldsymbol{V}}_T)}{T}} + \frac{2(x + \log d + \hat{\ell}_{x,T})(10.34 + 2.65 b_t)}{T}$$

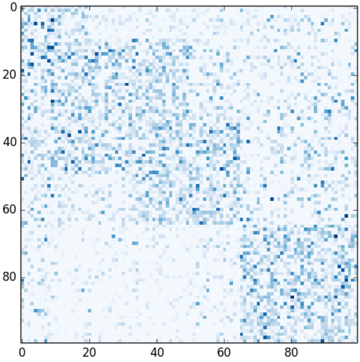
namely

$$\hat{w}_* \approx \sqrt{\frac{\lambda_{\max}(\hat{\boldsymbol{V}}_T)}{T}}$$

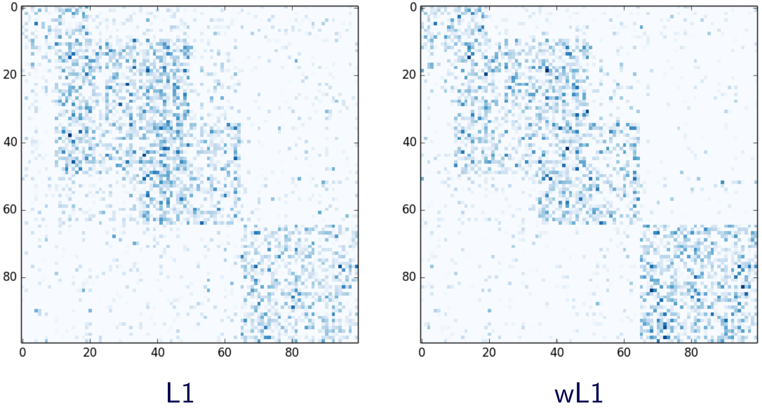
- Data-driven weights that comes from "empirical" Bernstein's inequalities, entrywise and for operator norm of $\boldsymbol{Z}_T$
- $\hat{\boldsymbol{V}}_{j,k}(t)$ and $\lambda_{\max}(\hat{\boldsymbol{V}}_t)$ are estimations (based on optional variation) of the variance terms from Bernstein's inequality
- $\boldsymbol{B}_{j,k}(t)$ and b_t are L^∞ terms (sub-exponential actually) from these Bernstein's inequalities
- Leads to a data-driven scaling of penalization: deals correctly with the inhomogeneity of information over nodes



Truth



No pen


0
20
40
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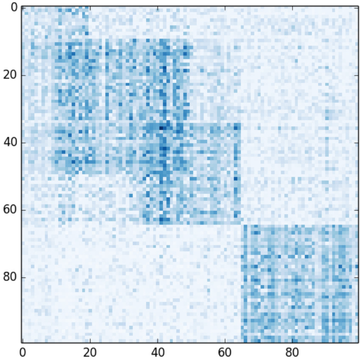
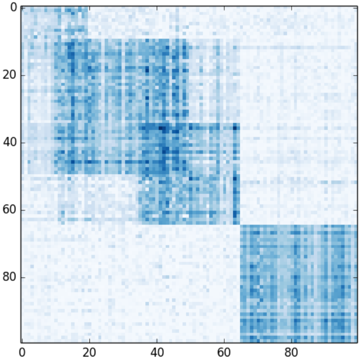
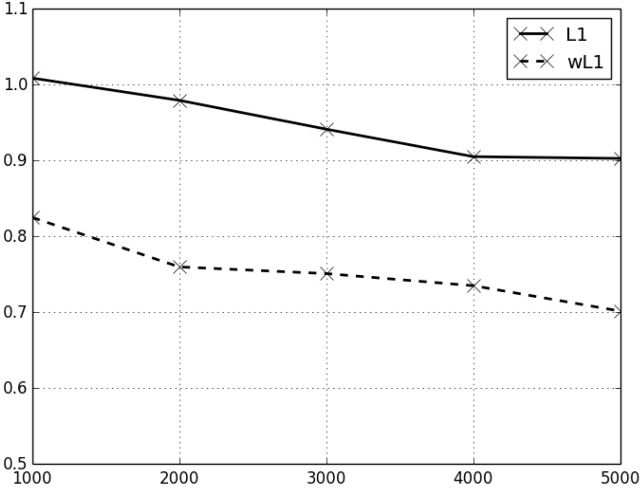
L1
wL1



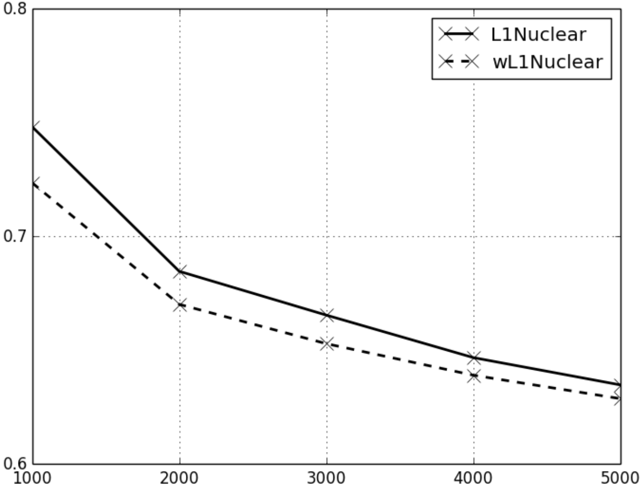
Nuclear



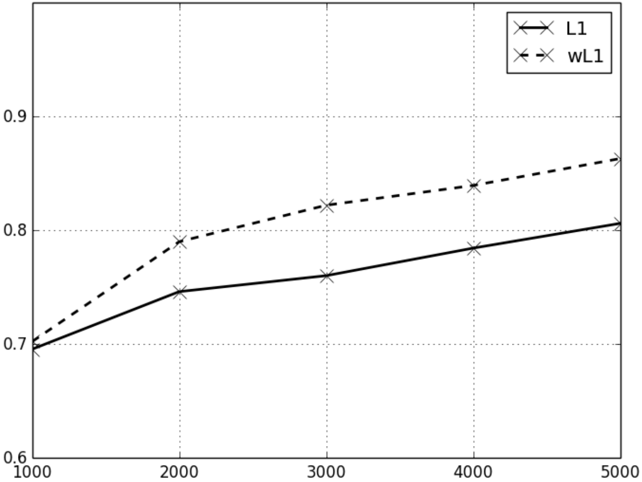
wNuclear



Error for L1 and wL1



Error for L1Nuclear and wL1Nuclear



AUC (support selection) for L1Nuclear and wL1Nuclear

A sharp oracle inequality

- Recall $\langle \lambda_1, \lambda_2 \rangle_T = \frac{1}{T} \sum_{j=1}^{d} \int_0^T \lambda_{1,j}(t) \lambda_{2,j}(t) dt$ and $\|\lambda\|_T^2 = \langle \lambda, \lambda \rangle_T$
- Assume RE in our setting (Restricted Eigenvalues), which is a standard assumption to obtain fast rates for the Lasso (and other convex-relaxation based procedures)

Theorem. We have

$$\|\lambda_{\hat{\theta}} - \lambda^*\|_T^2 \leq \inf_{\theta} \Big\{ \|\lambda_\theta - \lambda^*\|_T^2 + \kappa(\theta)^2 \Big(\frac{5}{4} \|(\hat{w})_{\mathrm{supp}(\mu)}\|_2^2 + \frac{9}{8} \|(\hat{\boldsymbol{W}})_{\mathrm{supp}(\boldsymbol{A})}\|_F^2 + \frac{9}{8} \hat{w}_*^2 \operatorname{rank}(\boldsymbol{A}) \Big) \Big\}$$

with a probability larger than $1 - 146 e^{-x}$.

- Leading constant 1

Roughly, $\hat{\theta}$ achieves an optimal tradeoff between approximation and complexity given by

$$\frac{\|\mu\|_0(x + \log d)}{T} \max_j N_j([0, T])/T + \frac{\|\boldsymbol{A}\|_0(x + 2\log d)}{T} \max_{j,k} \hat{\boldsymbol{V}}_{j,k}(T) + \frac{\operatorname{rank}(A)(x + \log d)}{T} \lambda_{\max}(\hat{\boldsymbol{V}}_T)$$

- Complexity measured both by sparsity and rank
- Convergence has shape $(\log d)/T$, where $T =$ length of the observation interval
- These terms are balanced by the empirical variance terms

Main tool: new concentration inequalities for matrix martingales in continuous time

Introduce

$$\boldsymbol{Z}_t = \int_0^t \boldsymbol{A}_s(\boldsymbol{C}_s \odot d\boldsymbol{M}_s)\boldsymbol{B}_s,$$

where $\{\boldsymbol{A}_t\}$, $\{\boldsymbol{C}_t\}$ and $\{\boldsymbol{B}_t\}$ predictable and where $\{\boldsymbol{M}_t\}_{t\geq 0}$ is a "white" matrix martingale, in the sense that $[\mathrm{vec}\boldsymbol{M}]_t$ is diagonal

NB: entries of $\boldsymbol{Z}_t$ are given by

$$(\boldsymbol{Z}_t)_{i,j} = \sum_{k=1}^{p}\sum_{l=1}^{q}\int_0^t (\boldsymbol{A}_s)_{i,k}(\boldsymbol{C}_s)_{k,l}(\boldsymbol{B}_s)_{l,j}(d\boldsymbol{M}_s)_{k,l}.$$

- $\langle \boldsymbol{M} \rangle_t$ = entrywise predictable quadratic variation, so that

  $$\boldsymbol{M}_t^{\odot 2} - \langle \boldsymbol{M} \rangle_t$$

  martingale
- vectorization operator $\mathrm{vec} : \mathbb{R}^{p \times q} \to \mathbb{R}^{pq}$ stacks vertically the columns of $\boldsymbol{X}$
- $\langle \mathrm{vec} \boldsymbol{M} \rangle_t$ is the $pq \times pq$ matrix with entries that are all pairwise quadratic covariations, so that

  $$\mathrm{vec}(\boldsymbol{M}_t)\mathrm{vec}(\boldsymbol{M}_t)^\top - \langle \mathrm{vec} \boldsymbol{M} \rangle_t$$

  is a martingale.

- $\boldsymbol{M}_t = \boldsymbol{M}_t^c + \boldsymbol{M}_t^d$, where $\boldsymbol{M}_t^c$ is a continuous martingale and $\boldsymbol{M}_t^d$ is a purely discountinuous martingale. Its (entrywise) quadratic variation is defined as

$$[\boldsymbol{M}]_t = \langle \boldsymbol{M}^c \rangle_t + \sum_{0 \le s \le t} (\Delta \boldsymbol{M}_t)^2, \tag{3}$$

and its quadratic covariation by

$$[\mathrm{vec}\boldsymbol{M}]_t = \langle \mathrm{vec}\boldsymbol{M}^c \rangle_t + \sum_{0 \le s \le t} \mathrm{vec}(\Delta \boldsymbol{M}_s)\mathrm{vec}(\Delta \boldsymbol{M}_s)^\top.$$

We say that $\boldsymbol{M}$ is *purely discontinuous* if the process $\langle \mathrm{vec}\boldsymbol{M}^c \rangle_t$ is identically the zero matrix.

Concentration for purely discountinuous matrix martingale:

- $\boldsymbol{M}_t$ is purely discountinuous and we have

$$\langle \boldsymbol{M} \rangle_t = \int_0^t \boldsymbol{\lambda}_s ds$$

  for a non-negative and predictable intensity process $\{\boldsymbol{\lambda}_t\}_{t \geq 0}$.
- Standard moment assumptions (subexponential tails)

Introduce

$$\boldsymbol{V}_t = \int_0^t \|\boldsymbol{A}_s\|_{\infty,2}^2 \|\boldsymbol{B}_s\|_{2,\infty}^2 \boldsymbol{W}_s ds$$

where

$$\boldsymbol{W}_t = \begin{bmatrix} \boldsymbol{W}_t^1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{W}_t^2 \end{bmatrix}, \tag{4}$$

$$\boldsymbol{W}_t^1 = \boldsymbol{A}_t \operatorname{diag}[\boldsymbol{A}_t^\top \boldsymbol{A}_t]^{-1} \operatorname{diag}\left[(\boldsymbol{C}_t^{\odot 2} \odot \boldsymbol{\lambda}_t)\mathbb{1}\right] \boldsymbol{A}_t^\top$$
$$\boldsymbol{W}_t^2 = \boldsymbol{B}_t^\top \operatorname{diag}[\boldsymbol{B}_t \boldsymbol{B}_t^\top]^{-1} \operatorname{diag}\left[(\boldsymbol{C}_t^{\odot 2} \odot \boldsymbol{\lambda}_t)^\top \mathbb{1}\right] \boldsymbol{B}_t$$

Introduce also

$$b_t = \sup_{s \in [0,t]} \|\boldsymbol{A}_s\|_{\infty,2} \|\boldsymbol{B}_s\|_{2,\infty} \|\boldsymbol{C}_s\|_{\infty}.$$

Theorem.

$$\mathbb{P}\Bigg[\|\boldsymbol{Z}_t\|_{\mathrm{op}} \geq \sqrt{2v(x+\log(m+n))} + \frac{b(x+\log(m+n))}{3}, \\ b_t \leq b, \quad \lambda_{\max}(\boldsymbol{V}_t) \leq v \Bigg] \leq e^{-x},$$

- First result of this type for matrix-martingale in continuous time

Corollary. $\{\boldsymbol{N}_t\}$ a $p \times q$ matrix, each $(\boldsymbol{N}_t)_{i,j}$ is an independent inhomogeneous Poisson processes with intensity $(\boldsymbol{\lambda}_t)_{i,j}$. Consider the martingale $\boldsymbol{M}_t = \boldsymbol{N}_t - \boldsymbol{\Lambda}_t$, where $\boldsymbol{\Lambda}_t = \int_0^t \boldsymbol{\lambda}_s ds$ and let $\{\boldsymbol{C}_t\}$ be deterministic and bounded. We have

$$\Big\| \int_0^t \boldsymbol{C}_s \odot d(\boldsymbol{N}_t - \boldsymbol{\Lambda}_t) \Big\|_{\mathrm{op}} \leq \sqrt{2\Big(\Big\| \int_0^t \boldsymbol{C}_s^{\odot 2} \odot \boldsymbol{\lambda}_s ds \Big\|_{1,\infty} \vee \Big\| \int_0^t \boldsymbol{C}_s^{\odot 2} \odot \boldsymbol{\lambda}_s ds \Big\|_{\infty,1}\Big)(x + \log(p+q))} + \frac{\sup_{s \in [0,t]} \|\boldsymbol{C}_s\|_\infty (x + \log(p+q))}{3}$$

holds with a probability larger than $1 - e^{-x}$.

Corollary. Even more particualar: $\boldsymbol{N}$ random matrix where $\boldsymbol{N}_{i,j}$ are independent Poisson variables with intensity $\boldsymbol{\lambda}_{i,j}$. We have

$$\|\boldsymbol{N} - \boldsymbol{\lambda}\|_{\mathrm{op}} \leq \sqrt{2(\|\boldsymbol{\lambda}\|_{1,\infty} \vee \|\boldsymbol{\lambda}\|_{\infty,1})(x + \log(p+q))} + \frac{x + \log(p+q)}{3}.$$

- Up to our knowledge, not previously stated in literature
- NB: In the Gaussian case: variance depends on maximum ℓ_2 norm of rows and columns (cf. Tropp (2011))

- We have as well a non-commutative Hoeffding's inequality when $\boldsymbol{M}_t$ has continuous paths, with a similar variance term
- Tools from stochastic calculus, use of the dilation operator and some classical matrix inequalities about the trace exponential and the SDP order.

A difficult proposition: a control of the quadratic variation of the pure jump process

$$\boldsymbol{U}_t^u = \sum_{0 \le s \le t} \Big(e^{u \Delta \mathscr{S}(\boldsymbol{Z}_s)} - u \Delta \mathscr{S}(\boldsymbol{Z}_s) - \boldsymbol{I} \Big)$$

given by

$$\langle \boldsymbol{U}^\xi \rangle_t \preceq \int_0^t \frac{\varphi\big(\xi \|\boldsymbol{A}_s\|_{\infty,2} \|\boldsymbol{B}_s\|_{2,\infty} \|\boldsymbol{C}_s\|_\infty\big)}{\|\boldsymbol{C}_s\|_\infty^2} \boldsymbol{W}_s ds,$$

where $\varphi(x) = e^x - x - 1$.

Conclusion

- Theoretical study of learning algorithms for “time-oriented” models need new probabilistic results
- In our case new concentration results for matrix martingales in continuous time
- Solves the scaling problem of penalizations

Perpectives:

- Experiments on Twitter, BlogoSphere and High-frequency Finance (ongoing)
- Superposition of Hawkes for viral diffusion of contents
- Better solvers using stochastic gradient based algorithms