Convergence rate of strong approximations of compound random maps

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Contents

1 The problem: approximation of $F(\Theta)$

1.1 Statement, objectives

✓ $(\Omega, \mathcal{F}, \mathbb{P})$: probability space

✓ $(\mathcal{E}, |.|)$: separable Banach space

✓ a $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable mapping $(\omega, x) \in (\Omega, \mathbb{R}^d) \mapsto F(\omega, x) \in \mathcal{E}$, continuous in x for a.e. ω

✓ $\Theta : \Omega \mapsto \mathbb{R}^d$ be a $\mathcal{F}$-random variable.

Our aim:

✓ F^N and Θ^N are some approximations of F and Θ

✓ control in $\mathbf{L}_p$ the error $\omega \in \Omega \mapsto F^N(\omega, \Theta^N(\omega)) - F(\omega, \Theta(\omega))$

✓ non asymptotic error estimates

✓ Strong approximation rates: crucial for Multi-Level Monte Carlo methods **[Heinrich '01, Giles '08, Glynn-Rhee '12]**

Θ and F may be dependent.

1.2 Application to unbiased simulation of $\mathbb{E}(f(F(\Theta)))$

Theorem (adaptation of Glynn-Rhee '12). Assume

✓ $F(\theta) \in \mathbb{R}^q$ and f Lipschitz bounded;

✓ the computational cost for simulating $F^N(\Theta^N)$ is $O(N)$ as $N \to +\infty$;

✓ strong approximation at order $r > 0$ in any L_p

$$\left\| F^N(\Theta^N) - F(\Theta) \right\|_{\mathbf{L}_p} \leq c_p N^{-r}, \quad \forall N \geq 1, \forall p \geq 1;$$

✓ $N_l := 2^l$ and $L \overset{\mathrm{d}}{=} \mathcal{G}(2^{-\gamma})$ (for $\gamma > 1$), independent from $(F^N(\Theta^N))_{N \geq 1}$.

Then, $\mathbf{Z} := \mathbf{f}(\mathbf{F}^{\mathbf{N_0}}(\boldsymbol{\Theta}^{\mathbf{N_0}})) + \sum_{\mathbf{l}=1}^{\mathbf{L}} \frac{\mathbf{f}(\mathbf{F}^{\mathbf{N_l}}(\boldsymbol{\Theta}^{\mathbf{N_l}})) - \mathbf{f}(\mathbf{F}^{\mathbf{N_{l-1}}}(\boldsymbol{\Theta}^{\mathbf{N_{l-1}}}))}{\mathbb{P}(\mathbf{L} \geq \mathbf{l})}$ is such that

✓ **No bias**: $\mathbb{E}(Z) = \mathbb{E}(f(F(\Theta)))$.

✓ The **average computational cost** for simulating Z is **finite**.

✓ Z **has a finite L_p-moment** ($p > 1$) provided $r > \gamma(1 - 1/p)$

➠ **MC confidence Intervals** given by von Barh-Essen inequality, rate $M^{1-1/p}$.

1.3 Examples of $F(\theta)$ with processes at random times

Typically, $(F(t))_{t\geq 0}$ for the process and $\Theta \geq 0$ for the random time.

1.3.1 When Θ is stopping time

Examples.

✓ SKOROKHOD EMBEDDING PROBLEM: given a distribution μ on $\mathbb{R}$, find a stopping time τ s.t.

$$\mu \overset{\mathbf{d}}{=} \mathbf{M}_\tau$$

where M is a given scalar martingale.

[Skorokhod '61, . . . , Azema-Yor '79, . . . , overview by Obloj '04]

- ✓ EXIT TIME POSITION FROM A DOMAIN D: approximation of

$$\mathbf{X}_\tau$$

where $\tau = \inf\{t \geq 0 : X_t \notin D\}$.

[Recent L_p-approximations by Bouchard-Geiss-G' 16]

- ✓ DAMBIS, DUBBINS-SCHWARZ THEOREM AND OTHER TIME-CHANGE: representation of continuous local martingales by time-changed Brownian motion

$$\mathbf{M_t} = \beta_{\langle \mathbf{M} \rangle_\mathbf{t}}.$$

- ✓ BASIC CONTROLS WHEN Θ AND Θ^N ARE STOPPING TIMES: from BDG inequalities when M martingale

$$\|\mathbf{M}_{\boldsymbol{\Theta}^\mathbf{N}} - \mathbf{M}_{\boldsymbol{\Theta}}\|_{\mathbf{L_p}} \leq c_p \left\| \sqrt{\int_{\Theta \wedge \Theta^N}^{\Theta \vee \Theta^N} \mathrm{d}\langle M \rangle_t} \right\|_{\mathbf{L}_p} \leq \mathbf{c_p} \left| \frac{\mathrm{d}\langle \mathbf{M} \rangle_\mathbf{t}}{\mathrm{d}\mathbf{t}} \right|_\infty \|\boldsymbol{\Theta}^\mathbf{N} - \boldsymbol{\Theta}\|_{\mathbf{L_{p/2}}}^{\frac{1}{2}}.$$

1.3.2 When Θ is not a stopping time

✓ PROCESS EVALUATED AT ITS MAXIMUM $\Theta = \inf\{t \in [0,T] : X_t = \max_{s\in[0,T]} X_s\}$

$$\mathbf{X}_{\boldsymbol{\Theta}} \approx ?$$

- Discrete time approximation?
- When X is a scalar BM **[Asmussen, Glynn, Pitman AAP'95]**:

$$\Theta^N = \inf\{t_i = iT/N \in [0,T] : X_{t_i} = \max_{t_j \le T} X_{t_j}\}$$

and

$$\sqrt{N}(X_\Theta - X_{\Theta^N}) \overset{weakly}{\Longrightarrow} \sqrt{T} \min_{i\in\mathbb{Z}} R(U+i)$$

where R is a two-sided BES(3) process and $U \overset{d}{=} \mathcal{U}(0,1)$.
Specifically relies on path-decomposition of BM at its maximum.

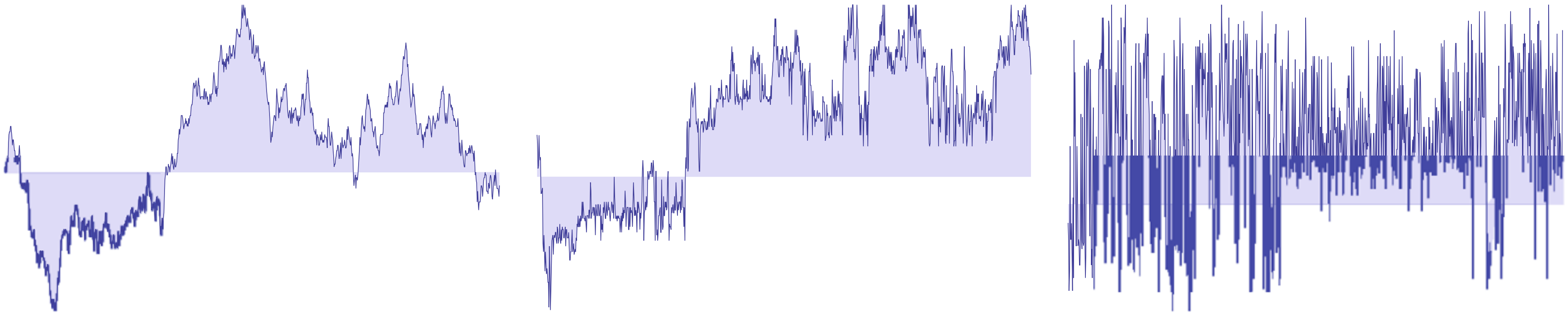
✓ Process at Brownian time: Iterated Brownian motion

Introduced by **[Burdzy '93]** as realization of BM on a random fractal. The second-order iterated Brownian motion is defined by

$$\mathbf{Z_t = B_{W_t}}$$

where

- $(B_t)_{t\in\mathbb{R}}$: two-sided BM
- $(W_t)_{t\geq 0}$: usual BM, ind. of B



Self-similar, stationary increments, α-Holder continuous (with $\alpha < 1/4$).

Connexion with PDE of Bi-Laplacian form (or fractional-time derivatives):
If we set $u(t,x) = \mathbb{E}(f(Z_t)|Z_0 = x)$, then it solves

$$\partial_t u(t,x) = \frac{\Delta f(x)}{\sqrt{8\pi t}} + \frac{1}{8}\Delta^2 u(t,x), \quad t > 0, x \in \mathbb{R}^d,$$
$$u(0,x) = f(x), \quad x \in \mathbb{R}^d.$$

[Allouba-Zheng (AoP '01); Deblassie (AAP '04); Baeumer, Meerschaert, Nane (TAMS '09)]

✓ Brownian-time SDE

If B is replaced by a SDE X, then $Z_t = X^x_{|W_t|}$ is a Brownian-time SDE (with similar PDE).

Recent interest in approximating hitting times and positions of Z, and therefore strong approximations.

✓ Fractional Brownian motion in Brownian time

[Zeineddine '13; Nourdin and Zeineddine '14] Defined by

$$\mathbf{Z_t} = \mathbf{B^{(H)}_{W_t}}$$

where

- $(B_t^{(H)})_{t\in\mathbb{R}}$: Brownian motion with Hurst parameter $H \in (0,1)$, i.e. centered, Gaussian, continuous process with covariance function

$$\mathbb{E}(B_t^{(H)} B_s^{(H)}) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right).$$

- $(W_t)_{t\geq 0}$: usual BM, ind. of B^H

Stationary increments, self-similar with order $H/2$, α-Holder continuous with $\alpha < H/2$.

✓ Non-semimartingale model and stopping times

$$\mathbf{B^{(H)}_{\Theta}} \approx ?$$

(previous BDG arguments do not apply anymore).

1.4 Examples of $F(\theta)$ with composition of random functions

✓ COMPOUND SDEs AND SPDEs

2 Stochastic Differential Equations driven by a scalar BM $(W_t)_{t\geq 0}$:

$$\mathrm{d}X_t(x) = \mu(t, X_t(x))\mathrm{d}t + \sigma(t, X_t(x))\mathrm{d}W_t, \quad X_0(x) = x,$$
$$\mathrm{d}Y_t(y) = b(t, Y_t(y))\mathrm{d}t + \gamma(t, Y_t(y))\mathrm{d}W_t, \quad Y_0(y) = y.$$

Under regularity assumptions, the Itô-Ventzel formula yields that $U(t,y) := X_t(Y_t(y))$ **solves a SPDE**

$$\begin{aligned}\mathrm{d}U(t,y) = &\Bigg(\partial_y U(t,y)\frac{b(t,Y_t(y))}{\partial_y Y(t,y)} + \frac{1}{2}\Big(\partial_y^2 U(t,y) - \partial_y U(t,y)\frac{\partial_y^2 Y_t(y)}{\partial_y Y_t(y)}\Big)\frac{\gamma^2(t,Y_t(y)}{(\partial_y Y_t(y))^2}\\ &+ \mu(t,U(t,y)) + \partial_x\sigma(t,U(t,y))\gamma(t,Y_t(y))\Bigg)\mathrm{d}t\\ &+ \Bigg(\partial_y U(t,y)\frac{\gamma(t,Y_t(y))}{\partial_y Y(t,y)} + \sigma(t,U(t,y))\Bigg)\mathrm{d}W_t.\end{aligned}$$

Approximation of $X_t(Y_t(y))$?

✓ PROGRESSIVE STOCHASTIC UTILITIES

[Musiela, Zariphopoulou '10; El Karoui, Mrad '13]. Assume $x \mapsto U(\omega, t, x)$ is a classic utility function (strictly increasing, concave) s.t.

$$\mathbb{E}\left(\mathbf{U}(\mathbf{t}, \mathbf{X}_{\mathbf{t}}^{\pi}) \mid \mathcal{F}_{\mathbf{s}}\right) \leq \mathbf{U}(\mathbf{s}, \mathbf{X}_{\mathbf{s}}^{\pi}), \quad \forall \mathbf{s} \leq \mathbf{t},$$

for any admissible portolio X^{π}.

➠ The solution $(t, x) \mapsto U(t, x)$ solves a second-order fully nonlinear SPDE that can be solved by **composition of stochastic flows.**

✓ FORWARD AND BACKWARD EULER SCHEMES **[Cohort, Mrad, G', 2016]**

Scalar diffusion on $[0,1]$: $\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t$.

Forward Euler scheme at times $t_i = i/N$:

$$\overrightarrow{X}^{\mathbf{N}}_{\mathbf{0}} = \mathbf{X_0}, \quad \overrightarrow{X}^{\mathbf{N}}_{\mathbf{t_{i+1}}} = \overrightarrow{X}^{\mathbf{N}}_{\mathbf{t_i}} + \mu(\overrightarrow{X}^{\mathbf{N}}_{\mathbf{t_i}})\frac{\mathbf{1}}{\mathbf{N}} + \sigma(\overrightarrow{X}^{\mathbf{N}}_{\mathbf{t_i}})\mathbf{\Delta W_{t_i}}.$$

Backward Euler scheme **[Kunita '97]**:

$$\overleftarrow{X}^{\mathbf{N}}_{\mathbf{1}} = \overrightarrow{X}^{\mathbf{N}}_{\mathbf{1}},$$

$$\overleftarrow{X}^{\mathbf{N}}_{\mathbf{t_i}} = \overleftarrow{X}^{\mathbf{N}}_{\mathbf{t_{i+1}}} - \mu(\overleftarrow{X}^{\mathbf{N}}_{\mathbf{t_{i+1}}})\frac{\mathbf{1}}{\mathbf{N}} - \sigma(\overleftarrow{X}^{\mathbf{N}}_{\mathbf{t_{i+1}}})\mathbf{\Delta W_{t_i}} + \sigma'_{\mathbf{x}}(\overleftarrow{X}^{\mathbf{N}}_{\mathbf{t_{i+1}}})\sigma(\overleftarrow{X}^{\mathbf{N}}_{\mathbf{t_{i+1}}})\frac{\mathbf{1}}{\mathbf{N}}.$$

- **Convergence rate for $\overleftarrow{X}^{N}_{t_i} - X^{N}_{t_i}$?**
  ➠ Composition of two (dependent) Euler schemes
- Allows to save memory by reversing the Random Number Generator and resimulating the (almost) same paths.
- Useful for Regression based Monte Carlo algorithms (American options, BSDEs).

2 Main approximation result for $F(\Theta)$

2.1 General assumptions

(H1) For any $p > 0$, $\exists\ \alpha_p^{(\mathbf{H1})} \geq 0$ s.t.

$$\sup_{\lambda \geq 1} \lambda^{-\alpha_p^{(\mathbf{H1})}} \left\| \sup_{|x| \leq \lambda} |F(\cdot, x)| \right\|_{\mathbf{L}_p} < +\infty.$$

(H2) $\exists\ \kappa \in (0, 1]$ such that $\forall\ p > 0$, $\exists\ \alpha_p^{(\mathbf{H2})} \geq 0$ s.t.

$$\sup_{\lambda \geq 1} \lambda^{-\alpha_p^{(\mathbf{H2})}} \left\| \sup_{x \neq y, |x| \leq \lambda, |y| \leq \lambda} \frac{|F(\cdot, y) - F(\cdot, x)|}{|y - x|^{\kappa}} \right\|_{\mathbf{L}_p} < +\infty.$$

(H3) For any $p > 0$, $\exists\ \alpha_p^{(\mathbf{H3})} \geq 0$ and a non-negative sequence $(\varepsilon_p^{N,(\mathbf{H3})})_{N \geq 1}$ s.t.

$$\sup_{\lambda \geq 1} \lambda^{-\alpha_p^{(\mathbf{H3})}} \left\| \sup_{|x| \leq \lambda} |F^N(\cdot, x) - F(\cdot, x)| \right\|_{\mathbf{L}_p} \leq \varepsilon_{\mathbf{p}}^{\mathbf{N},(\mathbf{H3})}, \quad \forall N \geq 1.$$

(H4) For any $p > 0$, there exists a non-negative sequence $(\varepsilon_p^{N,(\mathbf{H4b})})_{N\geq 1}$ s.t.

$$\sup_{N\geq 1}\left[\|\Theta\|_{\mathbf{L}_p} \vee \left\|\Theta^N\right\|_{\mathbf{L}_p}\right] < +\infty,$$

$$\left\|\Theta^N - \Theta\right\|_{\mathbf{L}_p} \leq \varepsilon_{\mathbf{p}}^{\mathbf{N},(\mathbf{H4b})}, \quad \forall N \geq 1.$$

Theorem (general result). Assume (**H1-H2-H3-H4**). Then for any $p > 0$ and any $p_2 > p$, there is a constant c independent on N such that

$$\left\|\mathbf{F}^{\mathbf{N}}(\boldsymbol{\Theta}^{\mathbf{N}}) - \mathbf{F}(\boldsymbol{\Theta})\right\|_{\mathbf{L_p}} \leq \mathbf{c}\left(\varepsilon_{\mathbf{2p}}^{\mathbf{N},(\mathbf{H3})} + [\varepsilon_{\kappa\mathbf{p_2}}^{\mathbf{N},(\mathbf{H4b})}]^{\kappa}\right).$$

Corollary (rule of thumb). If

- ✓ $F_N - F$ "=" $O(N^{-\gamma_F})$ in any L_p,
- ✓ $\Theta^N - \Theta = O(N^{-\gamma_\theta})$ in any L_p,

the order of L_p-convergence of $F^N(\Theta^N) - F(\Theta)$ is $\gamma_{\mathbf{F}} \wedge (\kappa\gamma_\theta)$.

2.2 Simplified assumptions

How to get uniform estimates from local ones?

Theorem. Let $p > d$. Assume that

- ✓ G is $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable mapping $(\omega, x) \in (\Omega, \mathbb{R}^d) \mapsto G(\omega, x) \in \mathcal{E}$,
- ✓ G is continuous in x for a.e. ω,
- ✓ $G(x)$ is in $\mathbf{L}_p$ for any x,
- ✓ there exist constants $C^{(G)} \geq 0$, $\beta^{(G)} \in (d/p, 1]$ and $\tau^{(G)} \geq 0$ s.t.

$$\|\mathbf{G}(\mathbf{x}) - \mathbf{G}(\mathbf{y})\|_{\mathbf{L_p}} \leq \mathbf{C}^{(\mathbf{G})} |\mathbf{x} - \mathbf{y}|^{\beta^{(\mathbf{G})}} (\mathbf{1} + |\mathbf{x}| + |\mathbf{y}|)^{\tau^{(\mathbf{G})}}, \quad \forall (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{\mathbf{d}} \times \mathbb{R}^{\mathbf{d}}. \quad (1)$$

Then, for any $\beta \in (0, \beta^{(G)} - d/p)$, we have

$$\sup_{\lambda \geq \mathbf{1}} \lambda^{-\tau^{(\mathbf{G})} - \beta^{(\mathbf{G})} + \beta} \left\| \sup_{\mathbf{x} \neq \mathbf{y}, |\mathbf{x}| \leq \lambda, |\mathbf{y}| \leq \lambda} \frac{|\mathbf{G}(\mathbf{y}) - \mathbf{G}(\mathbf{x})|}{|\mathbf{y} - \mathbf{x}|^{\beta}} \right\|_{\mathbf{L_p}} < +\infty,$$

$$\sup_{\lambda \geq \mathbf{1}} \lambda^{-\tau^{(\mathbf{G})} - \beta^{(\mathbf{G})}} \left\| \sup_{|\mathbf{x}| \leq \lambda} |\mathbf{G}(\mathbf{x})| \right\|_{\mathbf{L_p}} < +\infty.$$

3 Applications of the main convergence result

3.1 Strong approximation of compound SDEs

- ✓ Standard probability space $(\Omega, \mathbb{F}, \mathbb{P})$ supporting two q-dimensional standard BM $W = (W^1, \dots, W^q)$ and $B = (B^1, \dots, B^q)$ on $[0, T]$.
- ✓ W and B may be dependent.
- ✓ Two $\mathbb{R}^d$-valued stochastic processes X and Y, solutions of the SDEs (with Lipschitz coefficients)

$$\mathrm{d}X_t(x) = \mu(t, X_t(x))\mathrm{d}t + \sum_{i=1}^{q} \sigma_i(t, X_t(x))\mathrm{d}W_t^i, \quad X_0(x) = x,$$

$$\mathrm{d}Y_t(y) = b(t, Y_t(y))\mathrm{d}t + \sum_{i=1}^{q} \gamma_i(t, Y_t(y))\mathrm{d}B_t^i, \quad Y_0(y) = y,$$

- ✓ Denote by $X_T^N(x)$ (resp. $Y_T^N(y)$) the related Euler scheme with time step T/N.

Aim: approximation of $X_t(Y_s(y))$ **by** $X_t^N(Y_s^N(y))$.

Working assumptions for X

(HP1) The coefficients μ and σ are Lipschitz continuous in space uniformly in time: $\exists C^X$ s.t. $\forall t \in [0,T]$ and $x, y \in \mathbb{R}^d$

$$\begin{cases} |\mu(t,x)-\mu(t,y)| \le C^X|x-y|, & |\mu(t,0)| \le C^X, \\ |\sigma(t,x)-\sigma(t,y)| \le C^X|x-y|, & |\sigma(t,0]| \le C^X. \end{cases} \quad (\textbf{HP1})$$

(HP2) μ and σ are continuously space-differentiable functions such that their derivatives are δ-Hölder for a certain exponent $\delta \in (0,1]$:

$$\begin{cases} |\nabla_x\mu(t,x)-\nabla_x\mu(t,y)| \le C^{X,\nabla}|x-y|^\delta, & |\nabla_x\mu(t,x)| \le C^{X,\nabla}, \\ |\nabla_x\sigma(t,x)-\nabla_x\sigma(t,y)| \le C^{X,\nabla}|x-y|^\delta, & |\nabla_x\sigma(t,x)| \le C^{X,\nabla}. \end{cases} \quad (\textbf{HP2})$$

(HP3) μ and σ are α^X-Hölder continuous in time, locally in space:

$$|\mu(t,x)-\mu(s,x)| + |\sigma(t,x)-\sigma(s,x)| \le C^X(1+|x|)|t-s|^{\alpha^{\mathbf{X}}}. \quad (\textbf{HP3})$$

(HP4) μ and σ are continuously space-differentiable functions such that their derivatives are α^X-Hölder continuous in time, locally in space:

$$|\nabla_x \mu(t,x) - \nabla_x \mu(s,x)| + |\nabla_x \sigma(t,x) - \nabla_x \sigma(s,x)| \leq C^{X,\nabla}(1+|x|)|t-s|^{\alpha^{\mathbf{X}}}. \tag{\textbf{HP4}}$$

Working assumptions for Y: (**HP1-HP3**) are satisfied for b and γ (instead of μ and σ) with a Holder-exponent α^Y (instead of α^X).

Theorem. The compound Euler scheme $X^N_.(Y^N_.)$ converges to $X_.(Y_.)$ in any $\mathbf{L}_p$-norm, at the order $\beta := \min(\alpha^{\mathbf{X}}, \alpha^{\mathbf{Y}}, \frac{1}{2})$ w.r.t. N:

$$\sup_{\mathbf{s},\mathbf{t}\in[\mathbf{0},\mathbf{T}]} \left\| \mathbf{X}^{\mathbf{N}}_{\mathbf{t}}(\mathbf{Y}^{\mathbf{N}}_{\mathbf{s}}) - \mathbf{X}_{\mathbf{t}}(\mathbf{Y}_{\mathbf{s}}) \right\|_{\mathbf{L_p}} = \mathbf{O}(\mathbf{N}^{-\beta}).$$

Sketch of proof

Application of the general result with extra intermediate results.

Proposition. Assume **HP1**. For any $p > 0$, $\exists C_{p,(2)}, C_{p,(3)}$ s.t.

$$\|X_t(x)\|_{\mathbf{L}_p} \le C_{p,(2)}(1 + |x|), \tag{2}$$

$$\|X_t(x) - X_t(y)\|_{\mathbf{L}_p} \le C_{p,(3)}|x - y|, \tag{3}$$

for any $(t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$.

Corollary (From local to uniform estimates). Assume **HP1**. For any $p > 0$ and any $\beta \in (0, 1)$, $\exists C_{p,(4)}, C_{p,(5)}$ s.t.

$$\sup_{t\in[0,T]} \left\| \sup_{|x|\le\lambda} |X_t(x)| \right\|_{\mathbf{L}_p} \le C_{p,(4)}\lambda, \quad \forall\lambda \ge 1, \tag{4}$$

$$\sup_{t\in[0,T]} \left\| \sup_{x\ne y,|x|\le\lambda,|y|\le\lambda} \frac{|X_t(x) - X_t(y)|}{|y - x|^\beta} \right\|_{\mathbf{L}_p} \le C_{p,(5)}\lambda^{1-\beta}, \quad \forall\lambda \ge 1. \tag{5}$$

These estimates are also valid for X^N.

Proposition. Assume **HP1** and **HP2**. For any $p > 0$, $\exists C_{p,(6)}$ s.t.

$$\|\nabla X_t(x) - \nabla X_t(y)\|_{\mathbf{L}_p} \leq C_{p,(6)}|x - y|^\delta \tag{6}$$

for any $(t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$.

Corollary (From local to uniform estimates). Assume **HP1** and **HP2**. For any $p > 0$ and any $\beta \in (0, \delta)$, $\exists\ C_{p,(7)}, C_{p,(8)}, C_{p,(9)}$ s.t.

$$\sup_{t\in[0,T]} \left\| \sup_{|x|\leq\lambda} |\nabla X_t(x)| \right\|_{\mathbf{L}_p} \leq C_{p,(7)}\lambda^\delta, \quad \forall\lambda \geq 1, \tag{7}$$

$$\sup_{t\in[0,T]} \left\| \sup_{x\neq y,|x|\leq\lambda,|y|\leq\lambda} \frac{|\nabla X_t(x) - \nabla X_t(y)|}{|y - x|^\beta} \right\|_{\mathbf{L}_p} \leq C_{p,(8)}\lambda^{\delta-\beta}, \quad \forall\lambda \geq 1, \tag{8}$$

$$\sup_{t\in[0,T]} \left\| \sup_{x\neq y,|x|\leq\lambda,|y|\leq\lambda} \frac{|X_t(x) - X_t(y)|}{|y - x|} \right\|_{\mathbf{L}_p} \leq C_{p,(9)}\lambda^\delta, \quad \forall\lambda \geq 1. \tag{9}$$

EULER SCHEME ERRORS, UNIFORMLY IN THE STARTING POINT

Theorem. Assume **HP1**, **HP2**, **HP3**, **HP4** and set $\beta^X = \min(\alpha^X, \frac{1}{2})$. Then, for any $p > 0$, $\exists C_{p,(10)}$ s.t.

$$\sup_{t\in[0,T]} \left\| \sup_{|x|\leq\lambda} |X_t(x) - X_t^N(x)| \right\|_{\mathbf{L}_p} \leq \frac{C_{p,(10)}}{N^{\beta^X}} \lambda^2, \quad \forall \lambda \geq 1. \tag{10}$$

The rest of the proof of $\sup_{s,t\in[0,T]} \left\| X_t^N(Y_s^N) - X_t(Y_s) \right\|_{\mathbf{L}_p} = O(N^{-\beta})$ follows by applying the main theorem. □

3.2 Strong approximation of stochastic processes

3.2.1 The case of semimartingales at random times

Let $(M_t)_{t\geq 0}$ be a $\mathbb{R}^d$-valued continuous martingale, which componentwise bracket is of the form $\langle M^{(i)}\rangle_t = \int_0^t m_s^{(i)} \mathrm{d}s$ for a progressively measurable process $m^{(i)}$ bounded by $C^{(M)}$.

Proposition. The measurable mapping $(\omega, t) \mapsto M_t(\omega)$ satisfies (**H1-H2**) for any $\kappa \in (0, \frac{1}{2})$.

Proof. Follows from local estimates $\|M_t - M_s\|_{\mathbf{L}_p} \leq C_p^{(M)} |t-s|^{1/2}$.

Theorem. Let θ^N and θ be random times with finite moments at any order, uniformly bounded w.r.t. N. For any $p > 0$, any $\kappa \in (0, 1/2)$, $\exists\ c_{p,\kappa}$ s.t.

$$\|\mathbf{M}_{\theta^{\mathbf{N}}} - \mathbf{M}_{\theta}\|_{\mathbf{L_p}} \leq \mathbf{c}_{\mathbf{p},\kappa} \left\|\theta^{\mathbf{N}} - \theta\right\|_{\mathbf{L}_{\mathbf{p}/\mathbf{2}}}^{\kappa}.$$

Remind that in the case of stopping times, we can take $\kappa = \frac{1}{2}$.

Corollary (approximation of the maximum). Set

$$\Theta = \inf\{t \in [0,T] : M_t = \max_{s \leq T} M_s\}$$

and

$$\Theta^N = \inf\{t_i = iT/N \in [0,T] : M_{t_i} = \max_{t_j \leq T} M_{t_j}\} :$$

we have

$$\|\mathbf{M}_{\theta^{\mathbf{N}}} - \mathbf{M}_{\theta}\|_{\mathbf{L_p}} = \mathbf{O}(\mathbf{N}^{-\kappa})$$

for any $\kappa < \frac{1}{2}$.

3.2.2 The case of non-semimartingales at random times

Example (Fractional Brownian motion).

✓ $(B_t^H)_{t\in\mathbb{R}}$: Brownian motion with Hurst parameter $H \in (0,1)$.

Centered, Gaussian, continuous process with covariance function

$$\mathbb{E}(B_t^H B_s^H) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right).$$

✓ Mandelbrot and van Ness representation (1968):

$$B_t^{(H)} = \frac{1}{\Gamma(H+\frac{1}{2})}\int_{-\infty}^{t}\left[(t-s)^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}\right]\mathrm{d}B_s$$

where B is a standard two-sided BM.

✓ Approximation of $B_\Theta^{(H)}$ for a random time $\Theta \in [0,T]$?

By using Randow Walks for B, **[Szabados (SPA, '01)]** provides an explicit continuous approximation of $(B_t^{(H),N})_{0\leq t\leq T}$ (with $N = 2^{2m}$ points) and sharp deviation probability in sup-norm.

Theorem. Let $H \in (0,1)$. Then $(B_t^{(H),N})_{0\leq t\leq T}$ converges uniformly to a fBM $(B_t^{(H)})_{0\leq t\leq T}$ and

$$\left\| \sup_{\mathbf{0\leq t\leq T}} |\mathbf{B_t^{(H),N}} - \mathbf{B_t^{(H)}}| \right\|_{\mathbf{L_p}} = \mathbf{O}(\mathbf{N}^{-\min(\mathbf{H},\frac{1}{2})}\log(\mathbf{N})).$$

Since $\left\| B_t^{(H)} - B_s^{(H)} \right\|_{\mathbf{L}_p} \leq c|s-t|^H$ (for $s, t \in [0,T]$), our main result gives

Theorem. For any random times Θ, Θ^N taking values in $[0,T]$, we have

$$\left\| \mathbf{B}^{(\mathbf{H}),\mathbf{N}}_{\boldsymbol{\Theta}^{\mathbf{N}}} - \mathbf{B}^{(\mathbf{H})}_{\boldsymbol{\Theta}} \right\|_{\mathbf{L_p}} \leq \mathbf{c} \left(\mathbf{N}^{-\min(\mathbf{H},\frac{1}{2})}\log(\mathbf{N}) + \left\| \boldsymbol{\Theta}^{\mathbf{N}} - \boldsymbol{\Theta} \right\|_{\mathbf{L_p}}^{\mathbf{H}} \right).$$

Includes Brownian times.

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Home

News

Summer school

Research projects

Seminars

Registration

Participants

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During the first week, a classical summer school is proposed. It consists of several lectures given by leading scientists and related to the topics of the research projects. The remaining 5 weeks are dedicated to working on the rechearch projects, possibly after a morning seminar.