Exact simulation of Brownian diffusions with drift admitting jumps

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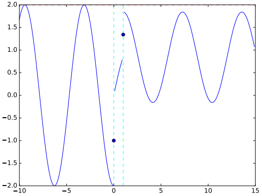
Real valued Brownian diffusion with discontinuous drift

Let $b \in \mathcal{C}^1_b(\mathbb{R} \setminus \{0, z\})$,

$$\boxed{dX_t = b(X_t)\,dt + dW_t, \quad t \in [0,T] \quad \longrightarrow \quad \mathbb{P}_b \in \mathscr{P}(\mathcal{C}([0,T];\mathbb{R}))}$$

Example:



Intensity of the jumps:

$$\theta_1 = \frac{b(0^+) - b(0^-)}{2}, \ \theta_2 = \frac{b(z^+) - b(z^-)}{2}.$$

Value at the discontinuities:

$$b(0) := \frac{b(0^+) + b(0^-)}{2}, \ b(z) := \frac{b(z^+) + b(z^-)}{2}$$

Aim (Exact simulation of $\mathbb{P}_b$)

One can sample directly from $\mathbb{P}_b$ without approximating b by continuous drifts.

Exact simulation, retrospective rejection sampling scheme

For $b \in \mathcal{C}^1_b(\mathbb{R})$ [Beskos, Roberts, Papaspiliopoulos (2008)];
The exact simulation scheme consists in

- $\mathfrak{Q} \in \mathscr{P}(\mathcal{C}([0, T]; \mathbb{R}))$ easy to sample from,
- a functional of the path of the form $\int_0^T \varphi(X_t)dt$ with $0 \leq \varphi \leq M$,

such that

$$\boxed{\mathbb{P}_b(dX) \propto e^{-\int_0^T \varphi(X_t)dt}\,\mathfrak{Q}(dX)}$$

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Proposition (Retrospective rejection sampling)

1. *Simulate a Poisson Point Process on the rectangle* $[0,T] \times [0,M]$ *with intensity* $\lambda = 1$, *and obtain* $\{(t_i, y_i),\ i = 1, \ldots, N\}$
2. *sample the marginals* $(X_{t_i})_{i \leq N}$ *under the measure* $\mathfrak{Q}$. *Obtain* $(x_i)_{i \leq N}$
3. *accept if* $y_i \geq \varphi(x_i)$ *for all* i, *otherwise reject and repeat.*

Output: the finite-dimensional marginals of $\mathbb{P}_b$:

$$\{X_{t_1}, X_{t_2}, \ldots, X_{t_N}, X_T\,;\, 0 \leq t_1 < t_2 < \ldots < t_N \leq T\}.$$

Exact algorithm if the drift b has two (or more) jumps

- It is easy to find $\mathfrak{Q}$, but not to sample from it.

$$\mathfrak{Q}(dX) \propto e^{B(X_T)-B(X_0)-\theta_1 L^0_T(X)-\theta_2 L^z_T(X)}\,\mathbb{P}(dX), \quad B(x) := \int_0^x b(u)du.$$

- The function $\varphi := \frac{1}{2}(b^2 + b') + c$ a.e., c constant such that $\varphi \geq 0$.

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- The function $\varphi := \frac{1}{2}(b^2 + b') + c$ a.e., c constant such that $\varphi \geq 0$.

Suppose $\theta_2 = 0$. $\mathfrak{Q}$ is BM conditioned on $(X_T, L^0_T(X)) \sim h(y, \ell)dy\, d\ell$

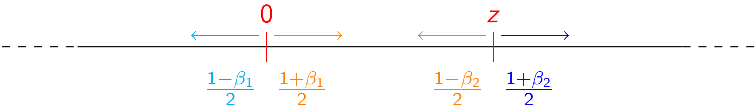
$$h(y, \ell)dy\, d\ell \propto \exp\left(B(y) - B(x_0) - \theta_1\, \ell\right)\mathbb{P}(X_T \in dy, L^0_T \in d\ell).$$

Remark (Two approaches)

- *Obtain the rejection sampling as the limit of rejection sampling schemes for one-skew BM with drift b. [Étoré and Martinez (2013)]*
- *Sample jointly Brownian motion and its local times [Papaspiliopoulos, Roberts and Taylor (2015)]*

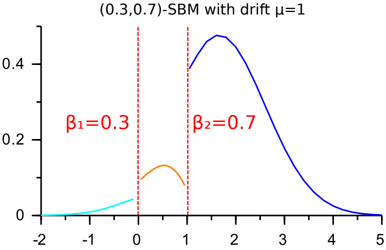
Heuristics on the (β_1, β_2)-skew BM

Let $\beta_1, \beta_2 \in [-1, 1]$



$$\begin{cases} dX_t = dW_t + \mu\, dt + \beta_1\, dL_t^0(X) + \beta_2\, dL_t^z(X); \\ L_t^0(X) = \int_0^t \mathbb{I}_{\{X_s=0\}}\, dL_s^0(X), \quad L_t^z(X) = \int_0^t \mathbb{I}_{\{X_s=z\}}\, dL_s^z(X) \end{cases}$$

Example:
$y \mapsto p_1^{(0.3,0.7)}(1, 0.5, y)$
is the density of X_1 where $(X_t)_t$ is the $(0.3, 0.7)$-skew BM with drift $\mu = 1$ starting at $X_0 = 0.5$.



The transition density of the (β_1, β_2)-skew BM

Proposition (without drift)

$$p^{(\beta_1,\beta_2)}(t,x,y) = p(t,x,y) \cdot v^{(\beta_1,\beta_2)}(t,x,y)$$

where z is the distance between the barriers,

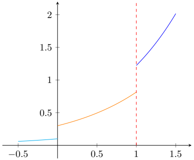
$$v^{(\beta_1,\beta_2)}(t,x,y) = \sum_{k=0}^{\infty} (-\beta_1\beta_2)^k \sum_{j=1}^{4} c_j(y) e^{-\frac{\left(a_j(x,y)+2\,z\,k\right)^2}{2t}} e^{-|x-y|\frac{a_j(x,y)+2\,z\,k}{t}}$$

$$\begin{cases} c_1(y) \equiv 1 \\ c_2(y) = \left(2\mathbb{1}_{[0,+\infty)}(y) - 1\right)\beta_1 \\ c_3(y) = \left(2\mathbb{1}_{[z,+\infty)}(y) - 1\right)\beta_2 \\ c_4(y) = \left(1 - 2\mathbb{1}_{[0,z)}(y)\right)\beta_1\beta_2 \end{cases} \qquad \begin{cases} a_1(x,y) \equiv 0 \\ a_2(x,y) = |y| + |x| - |y-x| \\ a_3(x,y) = |y-z| + |y-z| - |y-x| \\ a_4(x,y) = 2\left(z - \max(x,y,0)\right)^+ + 2\left(\min(x,y,z)\right)^+ \end{cases}$$

Representation of the transition density

Let $k(x) = \begin{cases} \frac{1}{4}(1-\beta_1)(1-\beta_2)e^{2\mu x} & x < 0, \\ \frac{1}{4}(1+\beta_1)(1-\beta_2)e^{2\mu x} & 0 \leq x < z, \\ \frac{1}{4}(1+\beta_1)(1+\beta_2)e^{2\mu x} & x \geq z. \end{cases}$

$\beta_1 = 0.2, \ \beta_2 = 0.5, \ \mu = 0.5$



Lemma

The infinitesimal generator $\mathcal{L}$ is self adjoint in $L^2(k(x)dx)$, and

$$\boxed{p_\mu^{(\beta_1,\beta_2)}(t,x,y) = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t} G_\mu^{(\beta_1,\beta_2)}(x,y;\lambda)d\lambda,}$$

where Γ is a complex contour of $\sigma(\mathcal{L}) \subseteq (-\infty, 0]$, and $G_\mu^{(\beta_1,\beta_2)}$ is the Green's function of the resolvent $(\lambda - \mathcal{L})^{-1}$.

If $\mu = 0$, this is known as the Titchmarsh-Kodaira-Yoshida method.

The Green's function for the (β_1, β_2)-SBM with drift

Let $y \in \mathbb{R}$ and $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, the Green's function solves

$$\begin{cases} (\lambda - \mathcal{L})G(x) = \delta_{\{y\}}(x), & G \in \mathcal{C}^2(\mathbb{R} \setminus \{0, z\}) \cap \mathcal{C}(\mathbb{R}) \\ k(0^+)u'(0^+) = k(0^-)u'(0^-), & k(z^+)u'(z^+) = k(z^-)u'(z^-). \end{cases}$$

Let the change of variable: $\lambda \overset{\phi}{\mapsto} \sqrt{2\lambda + \mu^2} =: w \in \{v \in \mathbb{C}; \Re(v) > 0\}$

Lemma (The Green's function)

$$G_{\mu}^{(\beta_1,\beta_2)}(x, y; w) = \frac{e^{\mu(y-x)}}{w} \frac{\sum_{j=1}^{4} c_j(\mu, y; w) e^{-w(a_j(x,y)+|x-y|)}}{\beta_1\beta_2 e^{-2wz}(w^2 - \mu^2) + (w + \beta_1\mu)(w + \beta_2\mu)}.$$

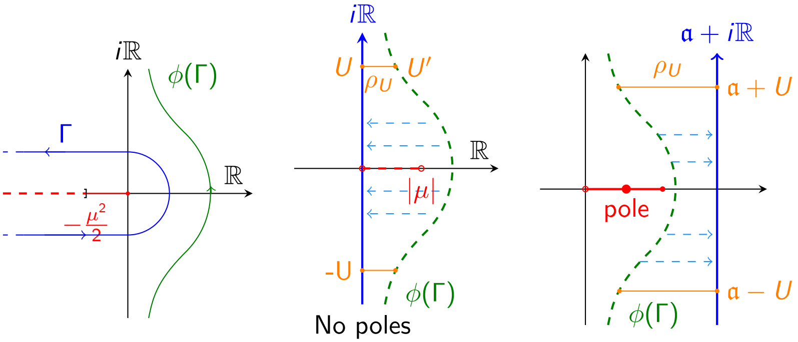
$c_j(\mu, y; w) = w^2 c_{j,0}(y) + w\mu c_{j,1}(y) + \mu^2 c_{j,2}(y)$, *where*

$$\begin{cases} c_{1,0} = 1, \\ c_{2,0} = \left(2\mathbb{1}_{[0,+\infty)}(y) - 1\right)\beta_1 \\ c_{3,0} = \left(2\mathbb{1}_{[z,+\infty)}(y) - 1\right)\beta_2 \\ c_{4,0} = \left(1 - 2\mathbb{1}_{[0,z)}(y)\right)\beta_1\beta_2 \end{cases} \quad \begin{cases} c_{1,1} = \beta_1 + \beta_2 \\ c_{2,1} = -\beta_1 - c_{4,0} \\ c_{3,1} = -\beta_2 + c_{4,0} \\ c_{4,1} = 0 \end{cases} \quad \begin{cases} c_{1,2} = \beta_1\beta_2 \\ c_{2,2} = \beta_1 c_{3,0} \\ c_{3,2} = -\beta_2 c_{2,0} \\ c_{4,2} = -c_{4,0}. \end{cases}$$

Ideas of the proof

$$p_{\mu}^{(\beta_1,\beta_2)}(t,x,y) = \frac{1}{2\pi i}\int_{\Gamma} e^{\frac{1}{2}\phi(\lambda)^2 t} e^{-\frac{1}{2}\mu^2 t} G_{\mu}^{(\beta_1,\beta_2)}(x,y;\phi(\lambda))d\lambda,$$

and we use techniques of complex analysis:



The transition density for the (β_1, β_2)-SBM with drift

Proposition

$$p_{\mu}^{(\beta_1,\beta_2)}(t,x,y) = p_{\mu}(t,x,y)v_{\mu}^{(\beta_1,\beta_2)}(t,x,y)$$

where the function $v_{\mu}^{(\beta_1,\beta_2)}$ is given by a series of Fourier transforms. They involve two functions

$$\begin{cases} \mathcal{J}_0(\omega, \beta_i\mu\sqrt{t}) := \sqrt{2\pi}e^{-\frac{\omega^2}{2}}e^{\frac{1}{2}(\omega+\beta_i\mu\sqrt{t})^2}\Phi^c(\omega+\beta_i\mu\sqrt{t}) \text{ with } i \in \{1,2\} \\ \mathcal{J}_1(\omega) := -e^{-\frac{\omega^2}{2}}, \end{cases}$$

evaluated in $\omega \in \left\{\omega_{j,k} := \frac{a_j(x,y)+2zk+|y-x|}{\sqrt{t}},\ j=1,2,3,4,\ k \in \mathbb{N}\right\}$.

The explicit representation of transition density...

Let $\mathfrak{a} \geq 0, -2\beta_1\mu, -2\beta_2\mu$. The function $v_\mu^{(\beta_1,\beta_2)}$, which does not depend on $\mathfrak{a}$, admits the following series representation

$$v_\mu^{(\beta_1,\beta_2)}(t,x,y) = \sum_{k=0}^{\infty}(-\beta_1\beta_2\mu^2 t)^k \sum_{j=1}^{4} F_{j,k}(\omega_{j,k},\mathfrak{a}),$$

$$F_{j,k} := \begin{cases} \sum_{n=0}^{k} \binom{2k-n}{k} \frac{\mathfrak{C}_{j,k}(\mathfrak{a})}{n!(\beta_1\mu\sqrt{t}-\beta_2\mu\sqrt{t})^{2k+1-n}} \mathscr{F}_{k+h-s,m,n}(\omega_{j,k},\mathfrak{a}), & \text{if } \beta_1 \neq \beta_2; \\ (-1)^{k+1} \frac{\mathfrak{C}_{j,k}(\mathfrak{a})}{(2k+1)!} \mathscr{G}_{k+h-s,m,2k+1}(\omega_{j,k},\mathfrak{a}\sqrt{t},\beta_1\mu\sqrt{t}), & \text{if } \beta_1 = \beta_2; \end{cases}$$

where

$$\mathfrak{C}_{j,k}(\mathfrak{a}) := e^{\frac{1}{2}\omega_{1,0}^2} \sum_{m=0}^{k}\sum_{s=0}^{k-m}\sum_{h=0}^{2} \binom{k-m}{s}\binom{k}{m} \frac{(-2\mathfrak{a}\sqrt{t})^{k-m-s}(\mu^2-\mathfrak{a}^2)^s C_{j,2-h}(y)}{\mu^{2k}t^{k-s}}$$

$$C_{j,0} := c_{j,0}, \quad C_{j,1} := \mu c_{j,1} + 2c_{j,0}\mathfrak{a}, \quad C_{j,2} := c_{j,2}\mu^2 + c_{j,1}\mu\mathfrak{a} + c_{j,0}\mathfrak{a}^2.$$

The transition density... completion

$$\mathscr{F}_{K,m,n}(\omega, \mathfrak{a}) := \mathscr{G}_{K,m,n}(\omega, \mathfrak{a}\sqrt{t}, \beta_2\mu\sqrt{t}) - (-1)^n \mathscr{G}_{K,m,n}(\omega, \mathfrak{a}\sqrt{t}, \beta_1\mu\sqrt{t}).$$

$$\mathscr{G}_{K,m,n}(\omega, \mathfrak{a}, \tau) := \sum_{\ell=0}^{\lfloor \frac{K+m}{2} \rfloor} \frac{(-1)^{\ell+K}}{2^\ell} \frac{(K+m)!}{\ell!(K+m-2\ell)!} \mathcal{S}_{K+m-2\ell,n}(\omega, \mathfrak{a}, \tau)$$

$$\mathcal{S}_{L,n}(\omega, \mathfrak{a}, \tau) = \sum_{n'=0}^{n} \sum_{L'=0}^{L} \binom{n}{n'} \binom{L}{L'} (\omega+\tau)^{n-n'} (\mathfrak{a}+\tau)^{L-L'} \mathcal{J}_{n'+L'}(\omega, \tau),$$

$$\mathcal{J}_q := \begin{cases} \mathcal{J}_0(\omega,\tau) = \sqrt{2\pi} e^{-\frac{1}{2}\omega^2} e^{\frac{1}{2}(\omega+\tau)^2} \Phi^c(\omega+\tau) & q=0, \\ \mathcal{J}_1(\omega) = -e^{-\frac{1}{2}\omega^2} & q=1, \\ \mathcal{J}_0(\omega,\tau)(q-1)!! - \mathcal{J}_1(\omega) \sum_{k=0}^{\frac{q}{2}-1} (\omega+\tau)^{q-2k-1} \frac{(q-1)!!}{(q-2k-1)!!} & q \geq 2 \text{ even}, \\ \mathcal{J}_1(\omega) \sum_{k=0}^{\frac{q-1}{2}} (\omega+\tau)^{(q-1-2k)} 2^k \frac{(\frac{q-1}{2})!}{(\frac{q-1}{2}-k)!} & q \geq 3 \text{ odd} \end{cases},$$

where $(2n+1)!! = (2n+1)\cdot(2n-1)\cdot \ldots \cdot 3 \cdot 1,\ n \in \mathbb{N}$.

Sample the finite-dimensional distributions of $\mathfrak{Q}$. Propagation of the retrospective rejection sampling

Let $\mathbb{P}_b^{(\beta_1,\beta_2)}$ the distribution of the $(\beta_1, \beta_2)-$skew BM with drift b.

Lemma

Choosing $\beta_1 = \frac{1}{n}$, $\beta_2(n) = \frac{\theta_2}{\theta_1 n + b(0) - b(z)}$ *and* $\mu(n) = \theta_1 n + b(0)$

$$\mathbb{P}_b^{(\beta_1,\beta_2)}(dX) \propto e^{-\int_0^T \varphi(X_t)dt}\, \mathfrak{Q}^{(\beta_1,\beta_2)}(dX)$$

$$\mathfrak{Q}^{(\beta_1,\beta_2)}(dX) \propto \exp\left(B(X_T) - B(X_0) - \mu(n)(X_T - X_0)\right) \mathbb{P}_{\mu(n)}^{(\beta_1,\beta_2)}(dX)$$

$$\begin{array}{ccc}
\mathbb{P}_b^{(\beta_1,\beta_2)} & \beta_i(n) \to 0 & \mathbb{P}_b \\
\updownarrow & & \updownarrow \\
\mathbb{P}_{\mu(n)}^{(\beta_1,\beta_2)} & & \mathbb{P} \\
\updownarrow & & \updownarrow \\
\mathfrak{Q}^{(\beta_1,\beta_2)} & \beta_i(n) \to 0 & \mathfrak{Q}
\end{array}$$

Sampling from $\mathfrak{Q}^{(\beta_1,\beta_2)}$

The finite-dimensional distributions of $\mathfrak{Q}^{(\beta_1,\beta_2)}$ converge to the finite-dimensional distributions of $\mathfrak{Q}$, and are given by the product

$$h^{(\beta_1,\beta_2)}(y) \prod_{i=0}^{N-1} q_\mu^{(\beta_1,\beta_2)}(t_i, t_{i+1}, T, y_i, y_{i+1}, y) dy_1 \dots dy_N dy,$$

where $q_\mu^{(\beta_1,\beta_2)}$ are the bridges of the skew BM with drift and

$$h^{(\beta_1,\beta_2)}(y) \propto e^{B(y)-B(x_0)}\; p(T, x_0, y)\; v_\mu^{(\beta_1,\beta_2)}(T, x_0, y)$$

$$\frac{q_\mu^{(\beta_1,\beta_2)}(t', t, T, x', y, x)}{q(t', t, T, x', y, x)} = \frac{v_\mu^{(\beta_1,\beta_2)}(T-t, y, x)\; v_\mu^{(\beta_1,\beta_2)}(t-t', x', y)}{v_\mu^{(\beta_1,\beta_2)}(T-t', x', x)}$$

Lemma

There exists a pointwise limit $v^{(\theta_1,\theta_2)}(t,x,y)$ of $v_\mu^{(\beta_1,\beta_2)}(t,x,y)$ for $\beta_i \to 0,\ \beta_i\mu \to \theta_i,\ i = 1,2$ where θ_i are the half jumps.

Sampling from $\mathfrak{Q}$

The finite-dimensional distributions of $\mathfrak{Q}^{(\beta_1,\beta_2)}$ converge to the finite-dimensional distributions of $\mathfrak{Q}$ given by the product

$$h^{(\theta_1,\theta_2)}(y) \prod_{i=0}^{N-1} q^{(\theta_1,\theta_2)}(t_i, t_{i+1}, T, y_i, y_{i+1}, y) dy_1 \dots dy_N dy,$$

where $q^{(\theta_1,\theta_2)}$ are the limit of the bridges of the skew BM with drift and

$$h^{(\theta_1,\theta_2)}(y) \propto e^{B(y)-B(x_0)}\ p(T, x_0, y)\ v^{(\theta_1,\theta_2)}(T, x_0, y)$$

$$\frac{q^{(\theta_1,\theta_2)}(t', t, T, x', y, x)}{q(t', t, T, x', y, x)} = \frac{v^{(\theta_1,\theta_2)}(T-t, y, x)\ v^{(\theta_1,\theta_2)}(t-t', x', y)}{v^{(\theta_1,\theta_2)}(T-t', x', x)}$$

Lemma

There exists a pointwise limit $v^{(\theta_1,\theta_2)}(t, x, y)$ of $v_\mu^{(\beta_1,\beta_2)}(t, x, y)$ for $\beta_i \to 0,\ \beta_i\mu \to \theta_i,\ i = 1, 2$ where θ_i are the half jumps.

Generalized rejection sampling method

Rejection sampling method
Instrumental density: $g(x)dx$ of a random variable Y
Goal density: $h(x)dx$ of a random variable X

$$\text{assume } \exists M > 0 \text{ such that } f(x) := \frac{1}{M}\frac{h(x)}{g(x)} \leq 1 \text{ for all } x \in \mathbb{R};$$

then $X \overset{(d)}{=} (Y|U < f(Y))$ is an *exact* simulation if f can be evaluated.

Generalization

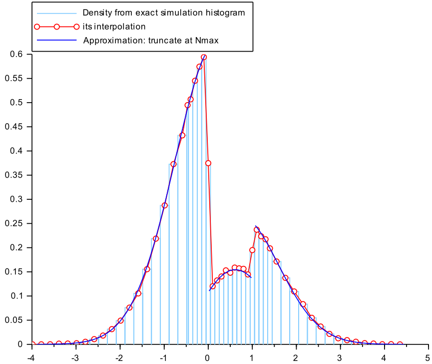
If there exists a sequence of functions $(f_n)_n$
converging pointwise to f *at a decreasing rate* $(\delta_n)_n$*.*
Moreover for each $x \in \mathbb{R}$ $f_n(x), \delta_n(x)$ *can be evaluated,*
then $X \overset{(d)}{=} (Y| \; \exists n : \; U < f_n(Y) - \delta_n(Y))$.

Application of the generalized RS method at the 2-SBM

Lemma (Bound, uniform in x and y)

For each term of the series $v_\mu^{(\beta_1,\beta_2)}(t,x,y)$,

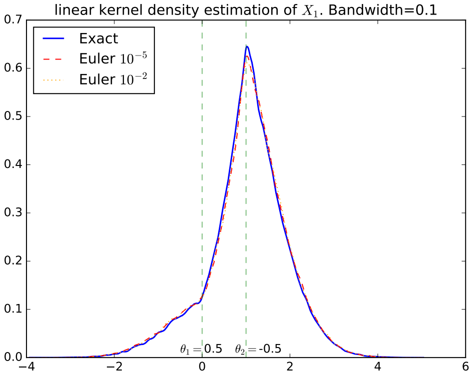
$$\sup_{x,y\in\mathbb{R}} \left| \left(v_\mu^{(\beta_1,\beta_2)} \right)_k (t,x,y) \right| \leq C_{\beta_1,\beta_2}\, e^{-\frac{2z^2}{t}k}$$

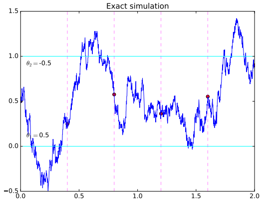


$y \mapsto p^{(-0.7,0.3)}(1, 0.5, y)$ with drift $\mu = 0$:
comparison between
-the truncation at the 10^{th} term
-the histogram for 50000 *exact* simulations through GRS.

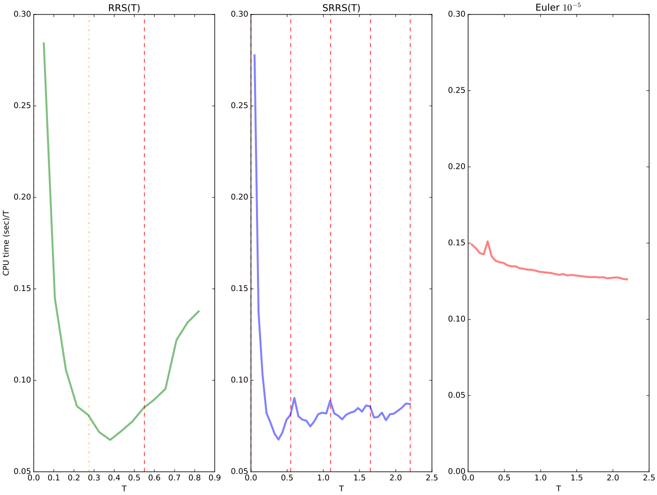
Exact simulation of the BM with drift $b = \mathbb{1}_{(0,1)}$

The solution of $dX_t = dW_t + \mathbb{1}_{(0,1)}(X_t)dt,\ \ t \in [0, T]$,
with initial condition $X_0 = 0.5$,





CPU times


RRS(T)
SRRS(T)
Euler 10^{-5}
CPU time (sec)/T
T
T
T
0.30
0.25
0.20
0.15
0.10
0.05
0.00
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.5
2.0
2.5

Sampling from the finite-dimensional distributions of $\mathfrak{Q}$

$$\frac{h^{(\theta_1,\theta_2)}(y)}{p_0(\frac{T}{1-\delta}, x_0, y)} = \underbrace{\frac{C_{\theta,x_0,T}}{\sqrt{1-\delta}}\; e^{M_B}\, \bar{v}(T)}_{C^{\mathcal{H}}} \underbrace{\frac{e^{-\frac{(y-x_0)^2}{2T}\delta}\; e^{B(y)-B(x_0)}}{e^{M_B}} \; \frac{v^{(\theta_1,\theta_2)}(T, x_0, y)}{\bar{v}(T)}}_{f_\delta^{\mathcal{H}}(y)},$$

where $C_{\theta,x_0,T}$ is the normalizing constant for the density $h^{(\theta_1,\theta_2)}$ and $M_B \leq \frac{\|b\|_\infty^2 T}{2\delta}$ is an upper bound for $B(y) - B(x) - \frac{(y-x)^2\delta}{2T}$. An appropriate choice of δ, for each specific case, can make the bound sharper.

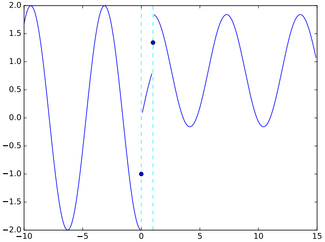
$$\frac{q^{(\theta_1,\theta_2)}(t, T, x_1, x_2, x)}{q(t, T, x_1, x_2, x)} = \underbrace{\frac{\bar{v}(t)\;\bar{v}(T-t)}{v^{(\theta_1,\theta_2)}(T, x_1, x_2)}}_{C^{\mathcal{B}}} \underbrace{\frac{v^{(\theta_1,\theta_2)}(t, x_1, x)}{\bar{v}(t)} \; \frac{v^{(\theta_1,\theta_2)}(T-t, x, x_2)}{\bar{v}(T-t)}}_{f^{\mathcal{B}}_{x_1,x_2}(x)}.$$

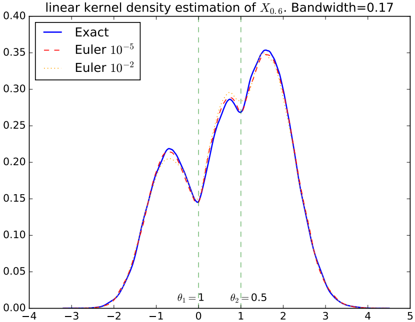
Another example

The solution of $dX_t = dW_t + b(X_t)dt,\ t \in [0, T]$; with initial condition $X_0 = 0.5$ for a piecewise continuous drift

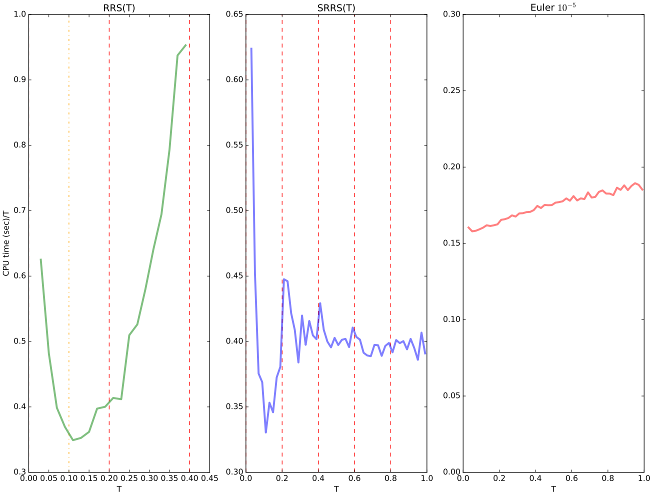
$$b(x) = \begin{cases} -2\cos(x) & x < 0 \\ \sin(x) & 0 < x < 1 \\ \cos(x-1) + sin(1) & x > 1 \end{cases},$$

The drift b





CPU times



Acknowledgements

Thank you all for your attention!

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