A particle system for singular non-linear McKean-Vlasov stochastic differential equations.

Numerical schemes for SDEs and SPDEs (Lille)

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June 1-3, 2016

Introduction

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V_t = V_0 + \int_0^t b(V_s)ds + \sigma W_t - M_t \left(V^{th} - V^r\right) \\
M_t = \sum_{k \geq 1} \mathbb{1}_{[0,t]}(\tau_k) \\
\tau_{k+1} = \inf\{t \geq \tau_k,\ V_{t-} \geq V^{th}\} \qquad\qquad (\tau_0 = 0)
\end{cases}
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where

- $V_0 < V^{th}$ almost surely.
- $V^r < V^{th}$
- $(W_t)_{t \geq 0}$ is a 1 dim. Brownian motion
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A singular McKean-Vlasov equation

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Stochastic process

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$$e'(t) = \frac{d}{dt} \mathbb{E}(M_t)$$

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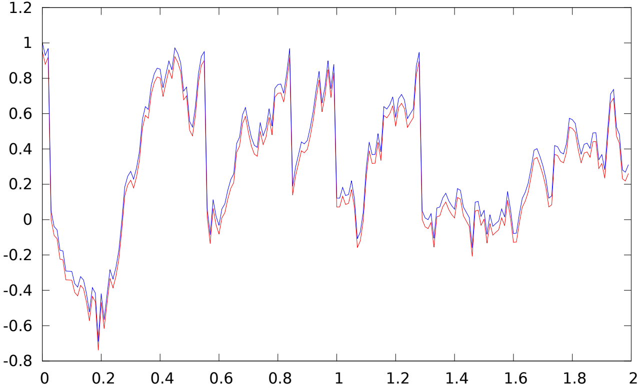

An existence result with continuous e

"Notations"

- $V^{th} = 1 \qquad V^r = 0 \qquad \sigma = 1$

Theorem

Let $\epsilon > 0$ *and* $V_0 \leq 1 - \epsilon$.
$\exists \alpha_0 \in (0, 1]$ *such that for any* $0 \leq \alpha \leq \alpha_0$ *and any* $T > 0$, *there exists a unique solution*

$$\begin{cases} V_t = V_0 + \int_0^t b(V_s) ds + W_t - M_t + \alpha \mathbb{E}(M_t) \\ M_t = \sum_{k \geq 1} \mathbb{1}_{[0,t]}(\tau_k) \qquad \tau_{k+1} = \inf\{t \geq \tau_k, \; V_{t-} \geq 1\} \end{cases}$$

with $e \in \mathcal{C}^1[0, T]$.

A non-existence result with continuous e

Blow up phenomenon

For any fixed α_0, there exist initial conditions V_0 such that any solution blows up in finite time.
[Cáceres, Carrillo, and Perthame. *J. Math. Neurosci.* (2011)]

Discontinuity of $t \mapsto e(t)$

When a blow up occurs at time t_1, a macroscopic proportion of neurons spikes and $e(t_1) = \mathbb{E}(M_{t_1}) \neq e(t_1-)$

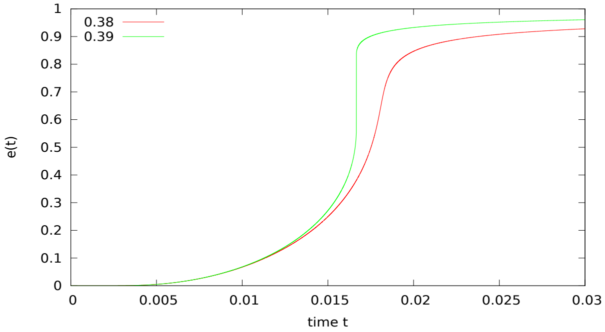

Figure: Plot of $t \mapsto e(t)$ for $v_0 = 0.8$, $b(v) \equiv 0$, $\alpha = 0.38$ and $\alpha = 0.39$.

More general solutions including discontinuous e

$$\begin{cases} V_t = V_0 + \int_0^t b(V_s)ds + W_t - M_t + \alpha\mathbb{E}(M_t) \\ M_t = \sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k) \\ \tau_{k+1} = \inf\{t \geq \tau_k,\ V_{t-} + \alpha\Delta e(t) \geq 1\} \end{cases}$$

Proposition

Assume that the pair $(V_t, M_t)_{t\geq 0}$ of càdlàg processes satisfies

1. $(M_t)_{t\geq 0}$ has integrable marginal distributions;
2. for all $t \geq 0$, $\mathbb{P}(\Delta M_t \leq 1) = 1$;
3. $\mathbb{P}$-almost surely, the previous equations hold true.

Then, for any time $t \geq 0$, the jump $\Delta e(t)$ satisfies

$$\Delta e(t) = \mathbb{P}\big(V_{t-} + \alpha \Delta e(t) \geq 1\big).$$

Proof

- For fixed $t \geq 0$, $M_t \neq M_{t-}$ iff $V_{t-} + \alpha \Delta e(t) \geq 1$
- $\Delta e(t) = \mathbb{P}(\Delta M_t = 1) = \mathbb{P}(V_{t-} + \alpha \Delta e(t) \geq 1)$.

Definition of "physical" solutions

A pair $(V_t, M_t)_{t\geq 0}$ of càdlàg adapted processes such that

1. $(M_t)_{t\geq 0}$ has integrable marginal distributions;
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3. $\mathbb{P}$-almost surely, hold true;

$$\begin{cases} V_t = V_0 + \int_0^t b(V_s)ds + W_t - M_t + \alpha\mathbb{E}(M_t) \\ M_t = \sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k) \\ \tau_{k+1} = \inf\{t \geq \tau_k,\ V_{t-} + \alpha\Delta e(t) \geq 1\} \end{cases}$$

4. the discontinuity points of $e(t) = \mathbb{E}(M_t)$ satisfy

$$\begin{aligned} \Delta e(t) &= \sup\left\{\eta \geq 0 : \forall \eta' \leq \eta, \mathbb{P}\left(V_{t-} + \alpha\eta' \geq 1\right) \geq \eta'\right\} \\ &= \inf\left\{\eta \geq 0 : \mathbb{P}\left(V_{t-} + \alpha\eta \geq 1\right) < \eta\right\}. \end{aligned}$$

A first approximation by a particle system

$$\begin{cases} V_t^{i,N} = V_0^{i,N} + \int_0^t b\big(V_s^{i,N}\big)\,ds + \dfrac{\alpha}{N}\sum_{j=1}^N M_t^{j,N} + W_t^i - M_t^{i,N} \\ V_0^{i,N} \overset{d}{=} V_0 \text{ independent and identically distributed,} \end{cases}$$

$$M_t^{i,N} := \sum_{k\geq 1} \mathbb{1}_{[0,t]}(\tau_k^{i,N}),$$

where $\tau_0^{i,N} = 0$ and

$$\tau_k^{i,N} := \inf\left\{ t > \tau_{k-1}^{i,N} : V_{t-}^{i,N} + \frac{\alpha}{N}\sum_{j=1}^N \big(M_t^{j,N} - M_{t-}^{j,N}\big) \geq 1 \right\}, \quad k \geq 1,$$

Again: non uniqueness!

A precise definition of the cascade of spikes

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Remark

We say that neurons in Γ_{k+1} spikes *after* neurons in $\Gamma_0 \cup \cdots \cup \Gamma_k$.

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Property

- M_t can be expressed in terms of $(Z_s)_{0\leq s\leq t}$

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Dynamics of Z

$$Z_t = Z_0 + \int_0^t b(Z_s - M_s)ds + \alpha\mathbb{E}(M_t) + W_t.$$

Convergence of the particles system

- Let $\mu_N = \frac{1}{N}\sum_i \delta_{Z^{i,N}}$
- μ_N is a random variable with values in $\mathcal{P}(\hat{\mathcal{D}}([0,T],\mathbb{R}))$
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Theorem

- *The family $(\Pi_N)_{N\geq 1}$ is tight in $\mathcal{P}(\mathcal{P}(\hat{\mathcal{D}}([0,T],\mathbb{R})))$*
- *Let Π_∞ be a weak limit. For Π_∞-a.e. measure $\mu \in \mathcal{P}(\hat{\mathcal{D}}([0,T],\mathbb{R}))$, the canonical process $(z_t)_{t\in[0,T]}$ on $\hat{\mathcal{D}}([0,T],\mathbb{R})$ generates, under μ, a physical solution, i.e.*

$$(z_t - z_0 - \int_0^t b(z_s - m_s)ds - \alpha\langle \mu, m_t\rangle)_{t\in[0,T)}$$

is a Brownian motion, where $m_t = \lfloor (\sup_{0\leq s\leq t} z_s)_+ \rfloor$

Propagation of chaos

Theorem

- *Assume there exists a unique physical solution.*
- *Denote by J the (at most countable) set of discontinuity points of $t \mapsto \mathbb{E}(M_t)$.*
- *For any $k \geq 1$ and $S \in \mathbb{R}^+ \setminus J$*

$$\left((Z_s^{1,N}, M_s^{1,N}), \ldots, (Z_s^{k,N}, M_s^{k,N})\right)_{s\in[0,S]} \Rightarrow \mathbb{P}^{\otimes k}_{(Z_s, M_s)_{s\in[0,S]}}.$$

Another approximation by a delayed interaction

$$V_t^\delta = V_0 + \int_0^t b(V_s^\delta)ds + \alpha e_\delta(t) + W_t - M_t^\delta, \quad t \geq 0.$$

where

$$e_\delta(t) := \begin{cases} 0 & \text{if } t \leq \delta \\ \mathbb{E}(M_{t-\delta}^\delta) & \text{if } t > \delta. \end{cases}$$

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Proposition

Let $T > 0$ and $\alpha \in (0,1)$.
There exists a unique càdlàg process $(V_t^\delta, M_t^\delta)_{t\in[0,T]}$, solution of the delayed system.
The resulting map e_δ is continuously differentiable.

Convergence of the delayed system

Theorem

Consider the family of solutions $((V_t^\delta)_{t\in[0,T]})_{\delta\in(0,1)}$ *to the delayed equation and the associated* $((Z_t^\delta)_{t\in[0,T]})_{\delta\in(0,1)}$.
Define by μ^δ *the law of* $(Z_t^\delta)_{t\in[0,T]}$ *on* $\hat{\mathcal{D}}([0,T],\mathbb{R})$. *Then, the family* $(\mu^\delta)_{\delta\in(0,1)}$ *is tight in* $\mathcal{P}(\hat{\mathcal{D}}([0,T],\mathbb{R}))$.
Under any weak limit μ *as* δ *tends to* 0*, the canonical process* $(z_t)_{t\in[0,T]}$ *on* $\hat{\mathcal{D}}([0,T],\mathbb{R})$ *generates a physical solution.*

A more precise result

Theorem

- *Assume there exists a unique physical solution.*
- *Denote by J the (at most countable) set of discontinuity points of $t \mapsto \mathbb{E}(M_t)$.*
- *For any $S \in \mathbb{R}^+ \setminus J$*

$$(Z_s^\delta, M_s^\delta)_{s\in[0,S]} \Rightarrow \mathbb{P}_{(Z_s,M_s)_{s\in[0,S]}}$$

Proofs

Theorem

Let $\epsilon > 0$ *and* $V_0 \leq 1 - \epsilon$.
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with $e \in \mathcal{C}^1[0, T]$.

Proof of the existence result

- Step 1 Look for a solution in small time T_0 via a fixed point argument
- Step 2 Prove a bound on the density of the solution at time T_0, i.e. a bound on the density of V_{T_0}.
- Step 3 Prove that the equation has a solution in small time when V_0 has a density that behaves as the density of V_{T_0} as in the previous step.
- Step 4 Show that an iteration of this procedure gives the existence and uniqueness of a solution up to any finite time.

Step 1: Solution in small time

- Let $e(t)$ be a deterministic non-decreasing continuous function on $[0, T]$.
- We introduce the space

$$\mathcal{L}(T, C_T, K, \theta) := \big\{ e \in \mathcal{C}[0, T] : \forall s \le t \quad 0 = e(0) \le e(s) \le e(t), \\ e(t) \le C_T(\sqrt{t} + K)e^{\theta t} \big\},$$

- for $e \in \mathcal{L}(T, C_T, K, \theta)$, we consider the linear equation

$$\begin{cases} V_t^e = V_0 + \int_0^t b(V_s^e)ds + \alpha e(t) + W_t - M_t^e \\ M_t^e = \sum_{k \ge 1} \mathbb{1}_{[0,t]}(\tau_k^e) \\ \tau_k^e = \inf\{t > \tau_{k-1}^e : V_{t-}^e = 1\} \end{cases}$$

Step 1: Solution in small time

We define

$$\Gamma(e)(t) := \mathbb{E}(M_t^e).$$

Lemma

For any $T > 0$, there exists constants C_T and θ depending only on T, α and C_b such that $\mathcal{L}(T, C_T, \mathbb{E}|V_0|, \theta)$ is stable by Γ.

Definition

$$\mathcal{H}(A,T) = \left\{ e \in \mathcal{L}\left(T, C_T, \mathbb{E}|V_0|, \theta\right) \cap \mathcal{C}^1[0,T] : \sup_{0\leq t\leq T} |e'(t)| \leq A \right\}$$

equipped with the norm $\|e\|_{\mathcal{H}(A,T)} = \|e\|_{\infty,T} + \|e'\|_{\infty,T}$,

A contraction

Theorem

Suppose $\mathbb{P}(V_0 \in dv_0) \leq \beta(1 - v_0)dv_0$ *with* $\beta \geq 1$, *or* $V_0 = 0$. *Then there exists* $A_1 \geq 1$ *and* $T_1 > 0$ *s.t.*

$$\Gamma : \mathcal{H}(A_1, T_1) \to \mathcal{H}(A_1, T_1).$$

Moreover, for all $e_1, e_2 \in \mathcal{H}(A_1, T_1)$

$$\|\Gamma(e_1) - \Gamma(e_2)\|_{\mathcal{H}(A_1,T_1)} \leq \frac{1}{2} \|e_1 - e_2\|_{\mathcal{H}(A_1,T_1)}.$$

Hence there exists a unique fixed point of the function $\Gamma : \mathcal{H}(A_1, T_1) \to \mathcal{H}(A_1, T_1)$.

Remark

The solution of the linear equation with the fixed point e is solution of the non-linear equation up until time T_1.

Step 2: An a priori bound on the firing rate

Theorem

For a given $\epsilon \in (0,1)$, there exists a positive constant $\alpha_0 \in (0,1]$, only depending upon ϵ, the drift coefficient b, such that, for any $\alpha \in (0,\alpha_0)$ and any positive time $T > 0$, there exist a constant $\mathcal{M}_T$, only depending on T, ϵ, and b, such that, for any initial condition $V_0 = v_0 \leq 1 - \epsilon$, any solution of the non linear equation satisfies

$$\forall t \in [0,T], \quad e'(t) = \frac{d}{dt}\mathbb{E}(M_t) \leq \mathcal{M}_T.$$

Step 2: A bound on the density of the solution

Lemma

Given an initial condition $V_0 = v_0 \leq 1 - \epsilon$, with $\epsilon \in (0, 1)$, and a solution $(V_t)_{0 \leq t \leq T}$ on some interval $[0, T]$, the random variable V_t has a density on $(-\infty, 1]$, for any $t \in (0, T]$.
Moreover, we have an explicit uniform bound of the density of V in a neighbourhood of 1.

Step 3 and 4

Step 3 was proved during Step 1

Step 4
- We construct a solution up to time T_1.
- If $T_1 < T$, we know that

$$\mathbb{P}(V_{T_1} \in dv) \leq C_{T_1}(1 - v)dv \leq C_T(1 - v)dv$$

- We construct a solution for $t \in [T_1, T_1 + T_2]$ where T_2 depends on C_T.
- If $T_1 + T_2 < T$, we know that

$$\mathbb{P}(V_{T_1+T_2} \in dv) \leq C_{T_1}(1 - v)dv \leq C_T(1 - v)dv$$

- We construct a solution for $t \in [T_1 + T_2, T_1 + 2T_2]$.

Recall on the M1 Skorohod topology

- $\hat{\mathcal{D}}([0,T],\mathbb{R})$: càdlàg functions on $[0,T]$, left continuous at T
- for $f \in \hat{\mathcal{D}}([0,T],\mathbb{R})$, $\mathcal{G}_f$ is the completed graph of f

$$\mathcal{G}_f := \{(x,t) \in \mathbb{R} \times [0,T] : x \in [f(t-), f(t)]\},$$

- an order on $\mathcal{G}_f$

$$(x_1,t_1) \leq (x_2,t_2) \text{ iff } \begin{cases} t_1 < t_2 \\ \text{or} \\ t_1 = t_2 \text{ and } |f(t_1-) - x_1| \leq |f(t_1-) - x_2| \end{cases}$$

- a parametric representation of $\mathcal{G}_f$ is a pair of continuous functions (u,r)

$$(u(t), r(t)) \in \mathcal{G}_f,$$

- $\mathcal{R}_f$ the set of all parametric representations of $\mathcal{G}_f$.

- The M1 distance between f_1 and f_2 in $\hat{\mathcal{D}}([0,T],\mathbb{R})$

$$d_{M_1}(f_1,f_2) := \inf_{\substack{(u_j,r_j)\in\mathcal{R}_{f_j}\\ j=1,2}} \left\{\|u_1-u_2\|_\infty \vee \|r_1-r_2\|_\infty\right\}.$$

- Define for $f \in \hat{\mathcal{D}}([0,T],\mathbb{R})$, $t\in[0,T]$ and $\delta>0$,

$$w_T(f,t,\delta) := \sup_{0\vee(t-\delta)\le t_1<t_2<t_3\le T\wedge(t+\delta)} \left\| f(t_2) - [f(t_1),f(t_3)] \right\|$$

where

$$\left\| f(t_2) - [f(t_1),f(t_3)] \right\| = \inf_{\theta\in[0,1]} \left| \theta f(t_1) + (1-\theta) f(t_3) - f(t_2) \right|$$

i.e. the distance between $f(t_2)$ and the set $[f(t_1),f(t_3)]$

- The M1 distance between f_1 and f_2 in $\hat{\mathcal{D}}([0,T],\mathbb{R})$

$$d_{M_1}(f_1,f_2) := \inf_{\substack{(u_j,r_j)\in\mathcal{R}_{f_j}\\ j=1,2}} \{\|u_1-u_2\|_\infty \vee \|r_1-r_2\|_\infty\}.$$

- Define for $f\in\hat{\mathcal{D}}([0,T],\mathbb{R})$, $t\in[0,T]$ and $\delta>0$,

$$w_T(f,t,\delta) := \sup_{0\vee(t-\delta)\le t_1<t_2<t_3\le T\wedge(t+\delta)} \left\| f(t_2) - [f(t_1), f(t_3)] \right\|$$

- If f is monotone, then $w_T(f,t,\delta)=0$.

Theorem

A sequence of functions $(f_n)_{n\geq 1} \subset \hat{\mathcal{D}}([0,T],\mathbb{R})$ *converges to some* $f \in \hat{\mathcal{D}}([0,T],\mathbb{R})$ *in the M1 topology if and only if* $f_n(t) \to f(t)$ *for* t *in a dense subset of* $[0,T]$ *that includes* 0 *and* T*, and*

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \sup_{t\in[0,T]} w_T(f_n, t, \delta) = 0.$$

Remark

If f_n is monotone for each n, then $f_n \to f$ in M1 iff $f_n(t) \to f(t)$ for all t in a dense subset of $[0,T]$ including 0 and T.

Theorem

Assume $f_n \to f$.
Then for all points $t \in [0, T]$ *at which* f *is continuous*

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \sup_{s \in [0 \vee (t-\delta), T \wedge (t+\delta)]} |f_n(s) - f(s)| = 0.$$

- *In particular, if* f *is continuous,* $(f_n)_{n \geq 1}$ *conv. uniformly to* f.

Theorem

A subset A of $\hat{\mathcal{D}}([0,T],\mathbb{R})$ *has compact closure in the M1 topology if and only if* $\sup_{f\in A}\|f\|<\infty$ *and*

$$\lim_{\delta\to 0}\sup_{f\in A}\Big\{\Big(\sup_{t\in[0,T]} w_T(f,t,\delta)\Big)\vee v_T(f,0,\delta)\vee v_T(f,T,\delta)\Big\}=0$$

where $v_T(f,t,\delta):=\sup_{0\vee(t-\delta)\le t_1\le t_2\le T\wedge(t+\delta)}\big|f(t_1)-f(t_2)\big|.$

Properties

- $\hat{\mathcal{D}}$ endowed with M1 is Polish
- the Borel σ-field coincides with the σ-field generated by the evaluation mappings
- the law of a process over $\hat{\mathcal{D}}$ is characterized by its finite-dimensional distribution.

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