Fast computation of the Nth term of an algebraic series in positive characteristic

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Functional Equations
in
Limoges, March 29th 2016

Motivation

- Ubiquity of algebraic functions (combinatorics, number theory, algebraic geometry)
- Confluence of several domains:
  - functional equations
  - automatic sequences
  - complexity theory
- *One of the most difficult questions in modular computations is the complexity of computations mod p for a large prime p of coefficients in the expansion of an algebraic function.*

D. Chudnovsky & G. Chudnovsky, 1990

Computer Algebra in the Service of

Mathematical Physics and Number Theory

Problem and main result

Input:

- field $\mathbb{K} = \mathbb{F}_p$
- $E(x, y) \in \mathbb{K}[x, y]$, with $E(0, 0) = 0$, $E_y(0, 0) \neq 0$
- $N \in \mathbb{N}_{\geq 0}$

Output:

- the Nth coefficient of the unique solution $f(x) \in \mathbb{K}[[x]]$ of $E(x, f(x)) = 0$, $f(0) = 0$

Main result:

- arithmetic complexity linear in $\log N$ and almost linear in p

State of the art

First N coefficients

Method	char. 0	char. p
Undetermined coefficients	$O(N^d)^\dagger$	✓
Fixed point iteration	$\widetilde{O}(N^2)^\P$	✓¶
Newton iteration	$\widetilde{O}(N)^\P$	✓¶
Linear recurrence	$O(N)$	✓*

† $d = \deg_y E(x, y)$ ¶ FFT used all along * with p-adic computations

Kung + Traub, 1976
All algebraic functions can be computed fast

Chudnovsky + Chudnovsky, 1986
On expansion of algebraic functions in power and Puiseux series, I

State of the art

Only Nth term (with precomputation)

Method	char. 0	char. p
Binary powering ($d = 1$ only)	$O(\log_2 N) + O(1)$	✓
Baby steps – Giant steps	$\widetilde{O}(\sqrt{N}) + O(1)$	✓*
Divide and Conquer	✘	$O(\log_p N) \times \widetilde{O}(p^{3d})$

Rational series by binary powering

Miller + Brown, 1966,
An algorithm for evaluation of remote terms in a linear recurrence sequence

Fiduccia, 1985
An efficient formula for linear recurrences

State of the art

Only Nth term (with precomputation)

Method	char. 0	char. p
Rational series	$O(\log_2 N) + O(1)$	✓
Baby steps – Giant steps	$\widetilde{O}(\sqrt{N}) + O(1)$	✓*
Divide and Conquer	✘	$O(\log_p N) \times \widetilde{O}(p^{3d})$

Baby steps – Giant steps after precomputing
algebraic equation $\longrightarrow$ differential equation $\longrightarrow$ linear recurrence

Chudnovsky + Chudnovsky, 1988
Approximations and complex multiplication according to Ramanujan

State of the art

Only Nth term (with precomputation)

Method	char. 0	char. p
Rational series	$O(\log_2 N) + O(1)$	✓
Baby steps – Giant steps	$\widetilde{O}(\sqrt{N}) + O(1)$	✓*
Divide and Conquer	✘	$O(\log_p N) \times \widetilde{O}(p^{3d})$

Divide and Conquer after precomputing
algebraic equation $\longrightarrow$ Mahler equation $\longrightarrow$ DAC recurrence

Christol + Kamae
+ Mendès France + Rauzy, 1980,
Suites algébriques, automates et substitutions

From algebraic equation to Mahler equation

Algebraic equation $y = 2x + 5xy + 4xy^2 + xy^3$

$p = 3$

$$
\begin{array}{c|c}
 & \begin{array}{cccc} y & y^3 & y^9 & y^{27} \end{array} \\
\begin{array}{c} 1 \\ y \\ y^2 \end{array} &
\left[\begin{array}{cccc}
0 & 1 & \frac{1+2x^2+x^3}{x^3} & \frac{1+2x^2+x^3+2x^5+x^6+x^8+x^9+2x^{11}+x^{12}}{x^{12}} \\
1 & \frac{1+x}{x} & \frac{1+x+2x^2+x^3+x^4}{x^4} & \frac{1+x+2x^2+x^3+x^4+2x^5+2x^6+2x^7+x^8+x^9+x^{10}+2x^{11}+x^{12}+x^{13}}{x^{13}} \\
0 & 2 & 2\,\frac{1+x+2x^2+x^3}{x^3} & \frac{2+2x+x^2+2x^3+2x^4+x^5+x^6+x^7+2x^8+2x^9+2x^{10}+x^{11}+2x^{12}}{x^{12}}
\end{array}\right]
\end{array}
$$

Mahler equation

$$
\begin{aligned}
&\left(2x^2 + 2x^3 + x^4\right) f(x) \\
&\qquad + \left(1 + x^2 + 2x^3 + 2x^4 + x^5 + 2x^6\right) f(x^3) \\
&\qquad\qquad + \left(2 + 2x^3 + 2x^5 + x^6 + 2x^9\right) f(x^9) + x^9 f(x^{27}) = 0
\end{aligned}
$$

Frobenius $f(x)^p = f(x^p)$

From algebraic equation to Mahler equation

Algebraic equation $y = 2x + 5xy + 4xy^2 + xy^3$
Mahler equation

$$\begin{aligned}&\left(2x^2 + 2x^3 + x^4\right) f(x)\\ &\qquad + \left(1 + x^2 + 2x^3 + 2x^4 + x^5 + 2x^6\right) f(x^3)\\ &\qquad\quad + \left(2 + 2x^3 + 2x^5 + x^6 + 2x^9\right) f(x^9) + x^9 f(x^{27}) = 0\end{aligned}$$

Divide-and-conquer recurrence

$$\begin{aligned}&2f_{n-2} + 2f_{n-3} + f_{n-4}\\ &\qquad + f_{\frac{n}{3}} + f_{\frac{n-2}{3}} + 2f_{\frac{n-3}{3}} + 2f_{\frac{n-4}{3}} + f_{\frac{n-5}{3}} + 2f_{\frac{n-6}{3}}\\ &\qquad\quad + 2f_{\frac{n}{9}} + 2f_{\frac{n-3}{9}} + 2f_{\frac{n-5}{9}} + f_{\frac{n-6}{9}} + 2f_{\frac{n-9}{9}} + f_{\frac{n-9}{27}} = 0\end{aligned}$$

$f_x = 0$ if $x \notin \mathbb{N}_{\geq 0}$

From algebraic equation to Mahler equation

$$+5x^{131}+2x^{130}+4x^{129}+x^{128}+2x^{127}+5x^{126}+3x^{125}+2x^{124}+4x^{123}+x^{122}+x^{121}+6x^{120}+6x^{119}$$

$$+2x^{118}+2x^{117}+5x^{116}+3x^{115}+6x^{114}+4x^{113}+2x^{112}+6x^{111}+4x^{110}+6x^{109}+5x^{108}+6x^{107}$$

$$+5x^{106}+6x^{105}+4x^{104}+3x^{103}+4x^{102}+x^{101}+2x^{100}+5x^{99}+3x^{98}+4x^{97}+5x^{71}+4x^{69}+2x^{68}+2x^{67}$$

$$+4x^{66}+5x^{65}+3x^{64}+x^{62}+2x^{57}+3x^{55}+3x^{54}+3x^{53}+6x^{52}+4x^{51}+5x^{50}+4x^{48}+2x^{46}+4x^{45}$$

$$+3x^{44}+x^{43}+6x^{42}+5x^{41}+2x^{40}+5x^{39}+3x^{38}+6x^{37}+3x^{36}+2x^{35}+x^{34}+4x^{33}+3x^{32}+6x^{31}+5x^{30}$$

$$+3x^{29}+4x^{28}+x^{27}+3x^{26}+6x^{25}+5x^{24}+5x^{23}+5x^{22}+2x^{21}+4x^{20}+x^{18}+2x^{17}+5x^{16}+5x^{15}+3x^{14}$$

$$+4x^{13}+6x^{12}+2x^{11}+4x^{10}+2x^{9}+x^{8}+2x^{7}+5x^{6}+5x^{4}+3x^{3}+4x^{2}+1\Big)f(x^{7})$$

$$+\Big(6x^{165}+5x^{164}+2x^{163}+2x^{162}+4x^{161}+3x^{160}+2x^{155}+3x^{153}+2x^{151}+4x^{150}+3x^{149}+6x^{147}$$

$$+x^{144}+2x^{143}+5x^{142}+4x^{141}+3x^{140}+6x^{139}+x^{137}+2x^{136}+5x^{135}+x^{134}+3x^{133}+5x^{132}+2x^{130}$$

$$+4x^{129}+3x^{128}+4x^{127}+6x^{126}+6x^{125}+6x^{123}+5x^{122}+2x^{121}+2x^{120}+4x^{119}+3x^{118}+5x^{116}$$

$$+3x^{115}+4x^{114}+4x^{113}+x^{112}+6x^{111}+4x^{106}+6x^{104}+4x^{102}+x^{101}+6x^{100}+5x^{98}+2x^{71}+3x^{69}$$

$$+x^{67}+2x^{66}+5x^{65}+5x^{64}+3x^{63}+4x^{62}+5x^{57}+4x^{55}+5x^{53}+3x^{52}+4x^{51}+x^{49}+5x^{46}+3x^{45}+4x^{44}$$

$$+6x^{43}+x^{42}+2x^{41}+5x^{39}+3x^{38}+4x^{37}+5x^{36}+x^{35}+4x^{34}+3x^{32}+6x^{31}+x^{30}+6x^{29}+2x^{28}+2x^{27}$$

$$+2x^{25}+4x^{24}+3x^{23}+6x^{21}+6x^{18}+5x^{17}+2x^{16}+2x^{15}+4x^{14}+3x^{13}+2x^{8}+3x^{6}+2x^{4}+4x^{3}+3x^{2}+6\Big)f(x^{49})$$

$$+\Big(x^{165}+2x^{164}+5x^{163}+5x^{162}+3x^{161}+4x^{160}+5x^{155}+4x^{153}+5x^{151}+3x^{150}+4x^{149}+x^{147}\Big)f(x^{343})=0$$

$O(\log_p N) \times p^{O(d)}$

Sections

Section operators

$$S_r \sum_{n\geq 0} f_n x^n = \sum_{k\geq 0} f_{pk+r} x^k, \quad 0 \leq r < p$$

Lemma

Let f be in $\mathbb{K}[[x]]$ and let $N = (N_\ell \cdots N_1 N_0)_p$ be the radix p expansion of N. Then $f_N = (S_{N_\ell} \cdots S_{N_1} S_{N_0} f)(0)$.

Sections

$$
\begin{aligned}
N &= 100000 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad p = 7 \\
&= 5 \times 16807 + 6 \times 2401 + 4 \times 343 + 3 \times 49 + 5 \times 7 + 5 \times 1 \\
&= (5,6,4,3,5,5)_7
\end{aligned}
$$

$$
\begin{aligned}
f(x) &= f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 + f_5 x^5 + \cdots \\
S_5 f(x) &= f_5 + f_{12} x + f_{19} x^2 + f_{26} x^3 + f_{33} x^4 + f_{40} x^5 + \cdots \\
S_5 S_5 f(x) &= f_{40} + f_{89} x + f_{138} x^2 + f_{187} x^3 + f_{236} x^4 + f_{285} x^5 + \cdots \\
S_3 S_5 S_5 f(x) &= f_{187} + f_{530} x + f_{873} x^2 + f_{1216} x^3 + f_{1559} x^4 + f_{1902} x^5 + \cdots \\
S_4 S_3 S_5 S_5 f(x) &= f_{1559} + f_{3960} x + f_{6361} x^2 + f_{8762} x^3 + f_{11163} x^4 + \cdots \\
S_6 S_4 S_3 S_5 S_5 f(x) &= f_{15965} + f_{32772} x + f_{49579} x^2 + f_{66386} x^3 + f_{83193} x^4 + \cdots \\
S_5 S_6 S_4 S_3 S_5 S_5 f(x) &= f_{100000} + f_{217649} x + f_{335298} x^2 + f_{452947} x^3 + \cdots \\
S_5 S_6 S_4 S_3 S_5 S_5 f(0) &= f_{100000}
\end{aligned}
$$

Diagonal

$$F(x,y) = \frac{y(1-5xy-xy^2-3xy^3)}{1-2x-5xy-4xy^2-xy^3} = \qquad\qquad p = 7$$

y	$+2xy$	$+4x^2y$	$+x^3y$	$+2x^4y$	$+4x^5y$	$+x^6y$	$+2x^7y$	$+4x^8y$	$+x^9y$	$+2x^{10}y$
		$+3x^2y^2$	$+5x^3y^2$	$+x^4y^2$	$+5x^5y^2$	$+2x^6y^2$	$+2x^7y^2$		$+6x^9y^2$	$+3x^{10}y^2$
	$+3xy^3$		$+3x^3y^3$		$+6x^5y^3$	$+4x^6y^3$	$+x^7y^3$	$+6x^8y^3$		$+6x^{10}y^3$
	$+5xy^4$	$+6x^2y^4$		$+x^4y^4$	$+x^5y^4$		$+x^7y^4$	$+3x^8y^4$	$+5x^9y^4$	
		$+2x^2y^5$	$+2x^3y^5$		$+6x^5y^5$	$+4x^6y^5$	$+5x^7y^5$		$+4x^9y^5$	$+4x^{10}y^5$
		$+2x^2y^6$	$+3x^3y^6$	$+5x^4y^6$		$+5x^6y^6$	$+6x^7y^6$		$+4x^9y^6$	$+6x^{10}y^6$
		$+5x^2y^7$	$+6x^3y^7$		$+5x^5y^7$		$+6x^7y^7$		$+3x^9y^7$	$+5x^{10}y^7$
				$+2x^4y^8$	$+3x^5y^8$	$+2x^6y^8$		$+3x^8y^8$	$+6x^9y^8$	$+5x^{10}y^8$
			$+x^3y^9$	$+x^4y^9$	$+3x^5y^9$	$+6x^6y^9$	$+6x^7y^9$		$+x^9y^9$	$+6x^{10}y^9$
			$+5x^3y^{10}$		$+6x^5y^{10}$	$+2x^6y^{10}$		$+x^8y^{10}$		$+4x^{10}y^{10}$
				$+x^4y^{11}$	$+x^5y^{11}$	$+5x^6y^{11}$	$+4x^7y^{11}$	$+4x^8y^{11}$	$+2x^9y^{11}$	
					$+6x^5y^{12}$	$+2x^6y^{12}$	$+x^7y^{12}$		$+3x^9y^{12}$	$+3x^{10}y^{12}$
				$+5x^4y^{13}$		$+5x^6y^{13}$			$+3x^9y^{13}$	$+x^{10}y^{13}$
					$+5x^5y^{14}$		$+3x^7y^{14}$		$+4x^9y^{14}$	$+2x^{10}y^{14}$
					$+6x^5y^{15}$	$+x^6y^{15}$	$+3x^7y^{15}$	$+x^8y^{15}$	$+2x^9y^{15}$	$+4x^{10}y^{15}$
					$+5x^5y^{16}$	$+4x^6y^{16}$	$+4x^7y^{16}$		$+5x^9y^{16}$	$+4x^{10}y^{16}$

$$f(x) = DF(x) = 2x + 3x^2 + 3x^3 + x^4 + 6x^5 + 5x^6 + 6x^7 + 3x^8 + x^9 + 4x^{10} + x^{14} + 5x^{15} + \cdots$$

Diagonal

Theorem (Furstenberg's theorem)

Every algebraic series is the diagonal of a bivariate rational function.

$$E(x, f(x)) = 0, \quad f(0) = 0, \quad E_y(0,0) \neq 0$$

$$f(x) = D\frac{a}{b} \qquad \text{with} \quad a(x,y) = yE_y(xy, y), \quad b(x,y) = E(xy,y)/y$$

Furstenberg, 1967, *Algebraic functions over finite fields*

Actions

univariate sections S_r: action on formal series

$$S_r f = S_r D \frac{a}{b}$$

diagonal and commutation rule

$$S_r D = D S_r$$

bivariate sections S_r: action on bivariate rational functions

$$S_r D \frac{a}{b} = D S_r \frac{a}{b}$$

Frobenius

pseudo-sections T_r: action on bivariate polynomials

$$D S_r \frac{a}{b} = D \frac{S_r a b^{p-1}}{b} = D \frac{T_r a}{b}$$

Christol, 1979, *Ensembles presque périodiques k-reconnaissables*

Actions

univariate sections S_r: action on formal series

$$f_N = (S_{N_\ell} \cdots S_{N_1} S_{N_0} f)(0)$$

$$\downarrow$$

bivariate sections S_r: action on bivariate rational functions

$$\downarrow$$

pseudo-sections T_r: action on bivariate polynomials

$$f_N = \frac{(T_{N_\ell} \cdots T_{N_1} T_{N_0} a)(0,0)}{b(0,0)}$$

Finite dimensional stable subspace

$$DS_r \frac{a}{b} = D \frac{S_r a b^{p-1}}{b} = D \frac{T_r a}{b}$$

$$d_x = \max(\deg_x a, \deg_x b), \qquad d_y = \max(\deg_y a, \deg_y b)$$

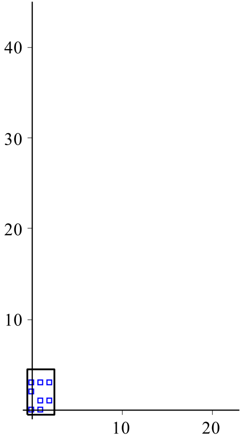
$$\begin{array}{rcl} \mathbb{F}_p[x,y]_{d_x,d_y} \longrightarrow & \mathbb{F}_p[x,y]_{pd_x,pd_y} \longrightarrow & \mathbb{F}_p[x,y]_{d_x,d_y} \\ v \longmapsto & vb^{p-1} = vB \longmapsto & S_r vB = T_r v \end{array}$$

$\mathbb{F}_p[x,y]_{d_x,d_y}$ left stable by T_r, $0 \le r < p$

Finite dimensional stable subspace

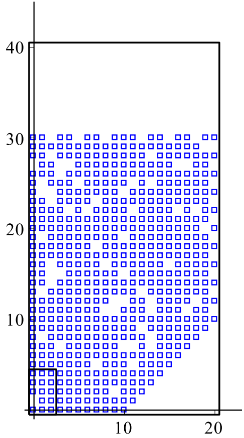
$$\begin{array}{rcl} \mathbb{F}_p[x,y]_{d_x,d_y} \longrightarrow & \mathbb{F}_p[x,y]_{pd_x,pd_y} \longrightarrow & \mathbb{F}_p[x,y]_{d_x,d_y} \\ v \longmapsto & vb^{p-1} = vB \longmapsto & S_r vB = T_r v \end{array}$$

$p = 11$
$E = (1+x)(x-y)+x^2y^2+(1+x)y^3+y^4$
$a = -y-xy^2+3y^3+4y^4+3xy^4+2x^2y^4$
$b = -1+x-xy+y^2+x^2y+y^3+xy^3+x^2y^3$
$d_x = 2$, $d_y = 4$
small box $[0,d_x]\times[0,d_y]$
b



Finite dimensional stable subspace

$$\begin{array}{rcccl} \mathbb{F}_p[x,y]_{d_x,d_y} & \longrightarrow & \mathbb{F}_p[x,y]_{pd_x,pd_y} & \longrightarrow & \mathbb{F}_p[x,y]_{d_x,d_y} \\ v & \longmapsto & vb^{p-1} = vB & \longmapsto & S_r vB = T_r v \end{array}$$



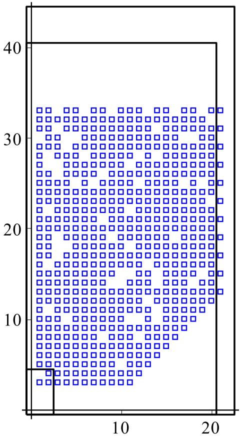
homothety with ratio $p-1$

almost large box $[0,(p-1)d_x] \times [0,(p-1)d_y]$

$B = b^{p-1}$

Finite dimensional stable subspace

$$\begin{array}{rcccl} \mathbb{F}_p[x,y]_{d_x,d_y} & \longrightarrow & \mathbb{F}_p[x,y]_{pd_x,pd_y} & \longrightarrow & \mathbb{F}_p[x,y]_{d_x,d_y} \\ v & \longmapsto & vb^{p-1} = vB & \longmapsto & S_r vB = T_r v \end{array}$$



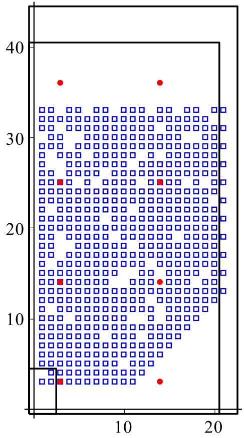
$v = xy^3$ in the canonical basis

translation by $(1,3)$

large box $[0, pd_x] \times [0, pd_y]$ xy^3B

Finite dimensional stable subspace

$$\begin{array}{rcccc} \mathbb{F}_p[x,y]_{d_x,d_y} & \longrightarrow & \mathbb{F}_p[x,y]_{pd_x,pd_y} & \longrightarrow & \mathbb{F}_p[x,y]_{d_x,d_y} \\ v & \longmapsto & vb^{p-1} = vB & \longmapsto & S_r vB = T_r v \end{array}$$

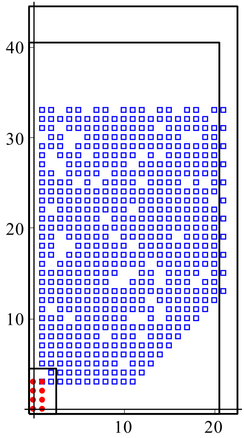


filter $r = 3$

xy^3B

Finite dimensional stable subspace

$$\begin{array}{rcccl} \mathbb{F}_p[x,y]_{d_x,d_y} & \longrightarrow & \mathbb{F}_p[x,y]_{pd_x,pd_y} & \longrightarrow & \mathbb{F}_p[x,y]_{d_x,d_y} \\ v & \longmapsto & vb^{p-1} = vB & \longmapsto & S_r vB = T_r v \end{array}$$



contraction

small box $[0,d_x]\times[0,d_y]$

$T_3 xy^3 = S_3 xy^3 B$

Precomputation of $A_0, A_1, \ldots, A_{p-1}$

All information is in $B = b^{p-1}$.

$$\text{matrix } A_r\colon\ x^n y^m \xrightarrow[\text{translation}]{\hspace{8em}} x^n y^m B \xrightarrow[\text{selection}]{\hspace{8em}} S_r x^n y^m B$$

No computation in $\mathbb{K} = \mathbb{F}_p$, except raising b to the power $p-1$

Cost: $\widetilde{O}(p^2)$ (binary powering, Kronecker substitution, FFT)

Theorem

The Nth coefficient can be computed in time

$$\widetilde{O}(p^2) + O(\log_p N).$$

Only a small part of $B = b^{p-1}$ is enough

Task: Computation of A_0, A_1, …, A_{p-1}

row index $i = (k, \ell)$
column index $j = (n, m)$

$$B = \sum_{\alpha,\beta} c_{\alpha,\beta} x^\alpha y^\beta$$

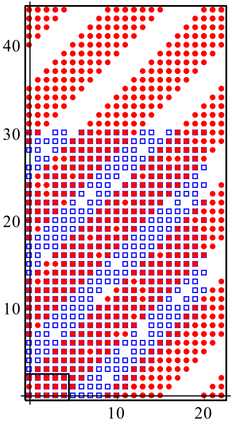
$$x^n y^m \xrightarrow[\text{translation}]{} x^n y^m B \xrightarrow[\text{selection}]{} S_r x^n y^m B$$

$$x^n y^m \longrightarrow \sum_{\alpha,\beta} c_{\alpha,\beta} x^{n+\alpha} y^{m+\beta} \longrightarrow \sum_{\substack{\alpha,\beta \\ (C)}} c_{\alpha,\beta} x^k y^\ell$$

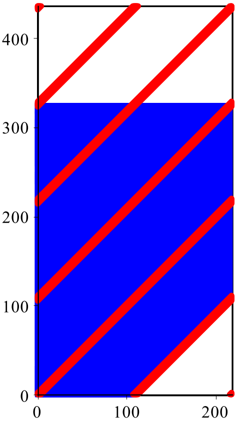
$$(C) \begin{cases} n + \alpha = pk + r \\ m + \beta = p\ell + r \end{cases} \implies \beta - \alpha = p(\ell - k) + n - m$$

Only a small part of $B = b^{p-1}$ is enough

$p = 11$



$p = 109$



Improved precomputation

$$B(x/t,t) = \sum_{\alpha,\beta} c_{\alpha,\beta} x^{\alpha} t^{\beta-\alpha} = \sum_{\delta} \pi_{\delta}(x) t^{\delta}$$

$$B(x/t,t) = b(x/t,t)^{p-1} = \frac{b(x/t,t)^p}{b(x/t,t)} = \frac{b(x^p/t^p, t^p)}{b(x/t,t)}$$

$$\frac{1}{b(x/t,t)} = \sum_{u} c_u(x) t^u \qquad b(x^p/t^p, t^p) = \textstyle\sum_v b_v(x^p) t^{pv}$$

rational series

for free

$$\pi_{\delta}(x) = \sum_{u+pv=\delta} c_u(x) b_v(x^p)$$

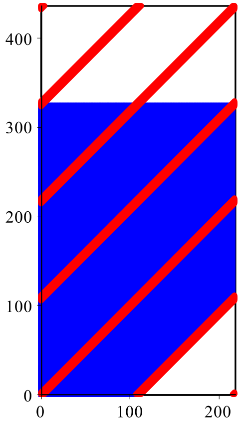
Improved precomputation

$\delta = \beta - \alpha =$ intercept of the strips with the ordinates axis

big leaps from one strip to the next $\widetilde{O}(p)$ (Kronecker substitution, FFT, Newton iteration)

Fiduccia, 1985

An efficient formula for linear recurrences



Main result

Theorem

Let E be in $\mathbb{F}_p[x,y]_{h,d}$ satisfy $E(0,0)=0$ and $E_y(0,0)\neq 0$, and let $f\in\mathbb{F}_p[[x]]$ be its unique root with $f(0)=0$. One can compute the coefficient f_N of f in

$$h^2(d+h)^2\log_p N+\widetilde{O}(h(d+h)^5 p)$$

operations in $\mathbb{F}_p$.

Timings

Maple implementation; timings on Intel Core i5, 2.8 GHz, 3MB.
With $p = 9001$, $N = 10^{10^k}$, $k = 1..6$,

$$ComputingTime(N) \simeq 0.0011 \cdot \log_p(N) - 1.3563$$

With $p = NextPrime(2^k)$, $k = 1..12$,

$$PrecomputingTime(p) \simeq 0.00037 \cdot p \log(p) + 0.04835$$

Documents at url
`http://specfun.inria.fr/dumas/Research/AlgModp/`

Theorem

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Thanks for your attention!