An analytic pointview on the Mock Theta functions of Ramanujan

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Plan and key-words

- Motivation: Dedekind eta-function, integer partitions, Ramanujan-Hardy formula, rank and co-rank, Appell-Lerch of the third order, mock theta functions of Ramanujan.
- Asymptotics found from modular formula, definition of theta-type function, Eulerian series and Ramanujan theta-functions.
- Singular $q$-difference equation, heat kernel (Gaussian) and Jacobi theta function, Stokes phenomenon, modular-like relation.
- Continued fractions, asymptotic behavior of Appell-Lerch series, some technical remarks
- Conclusion: Stokes phenomenon implies modularity.
Number of partitions and formula of Hardy-Ramanujan

- A partition of a positive integer \( n \), also called an integer partition, is a way of writing \( n \) as a sum of positive integers. The number of partitions of \( n \) is denoted by \( p(n) \).

- Example: \( 4 = 4 \times 1 (= 1 + 1 + 1 + 1) = 2 \times 1 + 1 \times 2 = 1 \times 1 + 1 \times 3 = 2 \times 2 = 1 \times 4 \), so \( p(4) = 5 \).

- Since Euler, one knows that \( \sum_{n \geq 0} p(n)q^n = \frac{1}{(q; q)_\infty} \), where and in the following:

\[
(a; q)_\infty = \prod_{n \geq 0} (1 - aq^n), \quad |q| < 1.
\]

- An asymptotic expression for \( p(n) \) is given by Hardy and Ramanujan:

\[
p(n) \sim \frac{1}{4n \sqrt{3}} \exp \left( \pi \sqrt{\frac{2n}{3}} \right) \quad \text{as} \ n \to \infty.
\]

Rademacher has completed this asymptotic formula into an exact formula.

- A key point consists of the fact that the Dedekind \( \eta \)-function

\[
\eta(\tau) := q^{1/24}(q; q)_\infty
\]

is modular, where \( q = e^{2\pi i \tau}, \tau \in \mathcal{H} \).
Rank of a partition and Appell-Lerch series of order 3

- Following Dyson, the rank of a partition is its largest part minus the number of its parts.
- Thanks to Garvan and Andrews, one knows that

\[
R(a; q) := \sum_{n \geq 0} \sum_{\lambda \in \mathcal{P}_n} a^{\text{rank}(\lambda)} q^n = \sum_{n \geq 0} \frac{q^{n^2}}{\prod_{m=1}^{n} (1 - aq^m)(1 - q^m/a)}.
\]

- Given an integer \( m \), knowing the number of partitions of \( n \) with rank congruent to \( r \) modulo \( m \) for all \( r \in \mathbb{Z}/m\mathbb{Z} \) and all \( n \) is equivalent to knowing the specialization of \( R(a; q) \) to all \( m \)-th roots of unity \( a = e^{2\pi ik/m} \).
- Gordon and McIntosh give that

\[
R(a; q) = \frac{1 - a}{(q; q)_{\infty}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(3n^2+n)/2}}{1 - q^n a},
\]

where the last series is an Appell-Lerch series of order 3.
Specialization at $a = -1$ for $R(a; q)$

- Putting $a = -1$ into the expression

$$R(a; q) = \sum_{n \geq 0} \frac{q^{n^2}}{\prod_{m=1}^{n}(1 - aq^m)(1 - q^m/a)}.$$  

yields

$$R(-1; q) = 1 + \sum_{n \geq 1} \frac{q^{n^2}}{\prod_{m=1}^{n}(1 + q^m)^2}.$$  

- Ramanujan claimed that

$$\alpha_n \sim \frac{(-1)^{n+1}}{\sqrt{n}} \exp \left( \pi \sqrt{\frac{n}{6}} \right) \text{ as } n \to \infty$$

where $\alpha_n$ denotes the coefficient of $q^n$ in the power series expansion $R(-1; q) = 1 + \sum_{n \geq 1} \alpha_n q^n$.

- This asymptotic relation has first been completed into an exact formula as conjecture by Andrews-Dragonette and proved later by K. Ono and K. Bringmann : by using Cauchy + Unit circle method.
Ramanujan mock-theta functions of order 3

- In *The final problem* (1935), G. N. Watson discussed the last letter of Ramanujan to Hardy in which a list of $q$-series had been introduced:

\[
f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1 + q)^2(1 + q^2)^2 \cdots (1 + q^n)^2},
\]

\[
\phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1 + q^2)(1 + q^4) \cdots (1 + q^{2n})},
\]

- These series are named by Ramanujan *mock theta functions*.

- Some of these functions (NOT ALL) can be directly expressed in terms of $R(a; q)$:

\[
f(q) = R(-1; q), \quad \phi(q) = R(i; q), \quad ...
\]
What might mean mock-theta?

Following the lastest letter of Ramanujan to Hardy, one may guess that a mock theta function means a function $f$ of the complex variable $q$, defined by a $q$-series of a particular type (Ramanujan calls this the Eulerian form), which converges for $|q| < 1$ and satisfies the following conditions:

1. infinitely many roots of unity are exponential singularities,
2. for every root $\zeta$ of unity, there is a theta function $T_\zeta(q)$ such that the difference $f(q) - T_\zeta(q)$ is bounded as $q \to \zeta$ radially,
3. $f$ is not the sum of two functions, one of which is a theta function and the other a function which is bounded radially toward all roots of unity.

**Question** – theta functions = ?

Following Andrews and Hickerson, one can define $\theta$-products by using Jacobi’s theta functions and one can guess that a theta is the sum of a finitely many theta-products.
What might mean theta for Ramanujan?

- One finds that four functions play a particular role in his theory:
  \((q; q)_{\infty}, \quad (-q; q)_{\infty}, \quad (q; q^2)_{\infty}, \quad (-q; q^2)_{\infty}\).

- Since 
  \((a^2; q^2)_{\infty} = (a, -a; q)_{\infty}\)
  and 
  \((a; q)_{\infty} = (a, aq; q^2)_{\infty}\),
  one finds that
  \((-q; q)_{\infty} = \frac{(q^2; q^2)_{\infty}}{(q; q)_{\infty}}, \quad (q; q^2)_{\infty} = \frac{(q; q)_{\infty}}{(q^2; q^2)_{\infty}}\)

and that

\((-q; q^2)_{\infty} = \frac{(q^2; q^4)_{\infty}}{(q; q^2)_{\infty}} = \frac{(q^2; q^2)_{\infty}}{(q; q)_{\infty} (q^4; q^4)_{\infty}}\).

Thus, all the four above functions can be expressed in terms of (quotients of) 
\((q^k; q^k)_{\infty}\), where \(k\) may be 1, 2 or 4.

- Remember that 
  \(q^{1/24} (q; q)_{\infty}\) is the Dedekind eta-modular function, so
  Ramanujan’s theta functions might merely be related to modular functions.
A brief revisit of Jacobi’s Theta functions (I)

Let \( z \in \mathbb{C}^* \), \( x = e(z) = e^{2\pi i z} \), \( \tau \in \mathcal{H} \), \( q = e(\tau) = e^{2\pi i \tau} \), and let \( \theta(x; q) = \theta(z \mid \tau) \) with

\[
\theta(z \mid \tau) = \sum_{n \in \mathbb{Z}} q^{n(n-1)/2} x^n = (q, -x, -\frac{q}{x}; q)_\infty.
\]

The classical \( \theta \)-modular formula says that

\[
\theta(z + \frac{1}{2} \mid \tau) = A \sqrt{m} \sqrt{i} \sqrt{\hat{\tau}} e\left(-\frac{1}{2} \hat{\tau} \left(z + \frac{1}{2m} - \frac{1}{m^2 \hat{\tau}}\right)^2\right) \theta\left(\frac{z}{m \hat{\tau}} + \frac{1}{2} \mid \tau_m\right),
\]

where \( A = A(m, p, \alpha, \beta) \) denotes some unity root,

\[
\left(\begin{array}{cc}
\alpha & \beta \\
-m & p
\end{array}\right) \in SL(2, \mathbb{Z}), \quad m \in \mathbb{Z}_{>0}
\]

and where

\[
\hat{\tau} = \tau - \frac{p}{m}, \quad \tau_m = -\frac{\alpha}{m} - \frac{1}{m^2 \hat{\tau}}.
\]
A brief revisit of Jacobi’s Theta functions (II)

As $\tau \to \frac{p}{m}$, it follows that $\hat{\tau} \to 0$ and $\tau_m \to i\infty$ in $\mathcal{H}$. This implies that if $q_m = e(\tau_m)$, then $q_m \to 0$ exponentially.

Lemma

Given $z_0, z_1 \in \mathbb{R}$, if $z = z_0 + z_1 \tau$ and $\tau \to \frac{p}{m}$, then

$$\theta(z + \frac{1}{2} | \tau) = \frac{A'}{\sqrt{m}} \sqrt{\frac{i}{\hat{\tau}}} \exp(\frac{\lambda_-}{\hat{\tau}} + \lambda_+ \hat{\tau}) \theta(\frac{z}{m\hat{\tau}} + \frac{1}{2} | \tau_m),$$

with $\hat{\tau} = \tau - \frac{p}{m} \to 0$,

1. $|A'| = 1$, $\lambda_- = -\frac{1}{2} (z_0 + \frac{z_1 p}{m} + \frac{1}{2m})^2$, $\lambda_+ = -\frac{1}{2} (z_1 - \frac{1}{2})^2$,
2. $\theta(\frac{z}{m\hat{\tau}} + \frac{1}{2} | \tau_m) = C q^\delta_m (1 + O(q^\kappa_m))$, where $\delta \in \mathbb{R}$ and $\kappa > 0$. 

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This revisit about Jacobi’s theta functions shows that:

- All modular function $f(\tau)$ has a similar asymptotic behavior at every root of unity $e(p/m) = e^{2\pi ip/m}$:

$$f(\tau) = \frac{A'}{\sqrt{m}} \sqrt{i \frac{i}{\hat{\tau}}} e\left(\frac{\lambda_-}{\hat{\tau}} + \lambda_+ \hat{\tau}\right) \left(1 + O(e^{-\kappa/|\hat{\tau}|})\right),$$

with $\hat{\tau} = \tau - \frac{p}{m} \to 0$.

- Moreover, the factor $1 + O(e^{-\kappa/|\hat{\tau}|})$ can be expressed as a germ of analytic function of the NEW (modular) variable $e(-1/\hat{\tau})$ at $0 \in \mathbb{C}$.
- Modularity $\implies$ Exponential scale for the asymptotics: transseries!
Radial limit behavior at each root of unity

For simplify, a function $f$ will be called **theta-type** at a given root $\zeta = e^{2\pi i \frac{p}{m}}$ of unity, and one writes $f \in \mathcal{T}_\zeta$, if there exists a quadruplet $(\nu, \lambda, I, \gamma)$, composed of a couple $(\nu, \lambda) \in \mathbb{Q} \times \mathbb{R}$, a strictly increasing and unbounded sequence $I$, and a $\mathbb{C}^*$-valued map $\gamma$ defined on $I$, such that the following relation holds for any $N \in \mathbb{Z}_{\geq 0}$ as $\tau \xrightarrow{a. v.} r$:

$$f(q) = \left(\frac{i}{\hat{\tau}}\right)^\nu q^{\lambda} \left( \sum_{k \in I \cap (-\infty, N]} \gamma(k) q_m^k + o(q_m^N) \right).$$

In the above, $\hat{\tau} = \tau - \frac{p}{m}$ and $q_m$ denotes the modular variable related to $\zeta$:

$$q_m = e^{-2\pi i \frac{1}{m^2 \hat{\tau}}} e^{2\pi i \frac{p'}{m}}$$

with $pp' = -1 \pmod{m}$. 
When Euler products is theta-type?

The infinite product \((x; q)_\infty = (1 - x)(1 - xq)(1 - xq^2)\ldots\) is called Euler product.

**Theorem**

The following conditions are equivalent for any Eulerian product \((x; q)_\infty\) while \(x = e^{2\pi i a} q^b\) with \(a, b \in \mathbb{R}, q = e^{2\pi i \tau}, \tau \in \mathcal{H}\).

1. \((x; q)_\infty \in \mathbb{F}_1,
2. \((x; q)_\infty \in \mathbb{F}_\zeta\) for some root of unity,
3. \((x; q)_\infty \in \mathbb{F}_\zeta\) for every root of unity,
4. \(x \in \{q, -q, \sqrt{q}, -\sqrt{q}\}\).
Strategy for the proof

- Let \( x_1 = e^{2\pi i \frac{a+b\tau}{\tau}} \) and \( q_1 = e^{-2\pi i \frac{1}{\tau}} \). Comparing \( (x; q) \infty \) with \( (x_1; q_1) \infty \) yields that
  \[
  (x; q) \infty = e^{T(x; q)} (x_1; q_1) \infty ,
  \]
  where \( T \) can be expressed with dilogarithm plus a Laplace integral.

- \( (x; q) \infty \in \mathcal{I}_1 \) means that the above Laplace integral defines a polynomial so we have NO divergence as \( q \to 1 \).

- Continued fractions \( \rightarrow \) other roots of unity from 1.

Counterexample – Knowing \( (q; q) \infty \in \mathcal{I}_1 \), one can see that \( (q^2; q) \infty \) does not belong to \( \mathcal{I}_1 \).
Indeed, \( (q^2; q) \infty = \frac{(q; q) \infty}{1-q} \) and \( \frac{1}{1-q} \notin \mathcal{I}_1 \) ! For
\[
\frac{1}{1-e^{2\pi i \tau}} \neq C \left( \frac{i}{\tau} \right)^\nu e\left( \frac{\lambda-}{\tau} + \lambda_+\tau \right) \left( 1 + O(e^{-\kappa/|\tau|}) \right). \]

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The above-introduced **theta-type** property can be extended as follows $(0 = e(i\infty))$:

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**Definition**

Let $q = e(\tau)$, $\tau \in \mathcal{H}$, $\zeta \in \mathbb{U} \cup \{0\}$, where $\mathbb{U}$ is the set of the roots of unity.

1. One says that $f$ is of **almost theta-type** as $q \to \zeta$ and one writes $f \in \tilde{T}_\zeta$, if there exists $f_\zeta \in T_\zeta$ such that $f - f_\zeta = O(1)$ as $q \to \zeta$.

2. One says that $f$ is a **theta-type function** and one writes $f \in T$, if $f \in T_\zeta$ for all $\zeta \in \mathbb{U} \cup \{0\}$.

3. One says that $f$ is a **false theta-type function** and one writes $f \in \tilde{F}$, if $f \notin T$ and there exists $T \in T$ such that $f(q) - T(q) = O(1)$ for all $\zeta \in \mathbb{U} \cup \{0\}$.

4. One says that $f$ is a **mock theta-type function** and one writes $f \in \tilde{M}$, if $f \in \tilde{T}_\zeta$ for all $\zeta = \mathbb{U} \cup \{0\}$ and, moreover, for each given $T \in T$, there exists $\zeta \in \mathbb{U} \cup \{0\}$ such that $f - T$ is unbounded.
A singular $q$-difference equation

From the analytic classification viewpoint of complex linear differential and $q$-difference equations, the series

$$\sum_{n \geq 0} q^{-n(n-1)/2} (-x)^n$$

is a $q$-analog of the famous Euler series $\sum_{n \geq 0} n! (-x)^n$. We are led to the $q$-difference equation

$$y(qx) + qx \, y(x) = 1,$$

that admits $x = 0$ as non-Fuchsian singular point.

As the moment problem of the sequence $(q^{-n^2/2})_n$ is undetermined, the above $q$-Euler series can be the asymptotic expansion of several solutions of this functional equation. Thus,

the $q$-analog of Borel-sum is not unique.
Instead of \( n! = \int_0^\infty e^{-t} t^{n+1} \frac{dt}{t} \)

- Let \( n \in \mathbb{Z} \) and \( a > 0 \). Since
  \[
  \int_{-\infty}^{\infty} e^{-(t+na)^2} \, dt = 2 \int_0^{\infty} e^{-t^2} \, dt = \sqrt{\pi},
  \]
  it follows that
  \[
  e^{n^2a^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2} e^{-2ant} \, dt.
  \]

Putting \( e^{a^2} = q^{-\frac{1}{2}} \) and \( \xi = e^{-2at} \) yields that
  \[
  q^{-\frac{1}{2}n^2} = \frac{1}{\sqrt{2\pi \ln 1/q}} \int_0^{\infty} e^{ \frac{1}{2\ln q} \log^2 \xi } \xi^n \frac{d\xi}{\xi}.
  \]

- Let \( \mu \not\in (-q^{\mathbb{Z}}) \), and let \( \theta(x; q) = \sum_{k \in \mathbb{Z}} q^{(k-1)/2} x^k = (q, -x, -q/x; q)_\infty \).
  Using \( \theta(q^k x; q) = q^{-k(k-1)/2} x^{-k} \theta(x; q) \) gives that
  \[
  q^{-\frac{1}{2}n(n-1)} = \sum_{\xi=\mu q^k, k \in \mathbb{Z}} \frac{\xi^n}{\theta(\xi; q)}.
  \]
**q-Borel-Laplace transforms**

- **q-Borel**:  
\[
\sum_{n \geq 0} a_n x^n \quad \Rightarrow \quad \sum_{n \geq 0} a_n q^{n(n-1)/2} \xi^n.
\]

- **q-Laplace with (Gaussian) heat kernel**  
\[
\omega(u; q) = \frac{1}{\sqrt{2\pi \ln(1/q)}} e^{\frac{\log^2(u/\sqrt{q})}{2\ln q}}.
\]

\[
\phi \quad \Rightarrow \quad \int_0^\infty \phi(\xi) \omega\left(\frac{\xi^2}{x}; q\right) \frac{d\xi}{\xi}.
\]

- **q-Laplace with Jacobi theta function**  
\[
\theta(u; q) = \sum_{k \in \mathbb{Z}} q^{k(k-1)/2} u^k.
\]

\[
\phi \quad \Rightarrow \quad \sum_{\xi \in \mu q\mathbb{Z}} \frac{\phi(\xi)}{\theta\left(\frac{\xi^2}{x}; q\right)},
\]

where \(\mu\) is a given complex number that indicates the integration path.
Appell-Lerch series and Mordell’s integral as solution of $q$-difference equation

By following the above $q$-analogs of the Borel-Laplace summation method, one can check that both expressions give solutions of $y(qx) + qx y(x) = 1$:

$$L(x, \mu; q) = \sum_{n=-\infty}^{\infty} \frac{1}{1-\mu q^n} \frac{1}{\theta(-\frac{\mu}{x} q^n; q)}$$

and

$$G(x; q) = \int_{0}^{\infty} \frac{\omega(\xi/x; q)}{1+\xi} \frac{d\xi}{\xi}.$$

**Physics** – $G(x; q)$ represents a solution for the “heat equation”

$$2q \partial_q f + x^2 \partial_x^2 f = 0, \quad f(x, q)|_{q=1} = \frac{1}{1+x},$$

which is formally satisfied by the power series $\sum_{n \geq 0} q^{-n(n-1)/2}(-x)^n$. 

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Stokes phenomenon

The functions $L(x, \mu; q)$ and $G(x; q)$ are both sum-functions of the same power series, so their difference satisfies the homogeneous $q$-difference equation $y(qx) + qxy(x) = 0$. This approach is similar to the analysis of Stokes in the theory of differential equations.

Let $L(z, w | \tau) = L(x, \mu; q)$ and $G(z | \tau) = G(x; q)$, where $x = e(z) = e^{2\pi iz}$, $\mu = e(w)$ and $q = e(\tau)$. It follows that

$$L(z, w | \tau) = G(z | \tau) + C(z | \tau) \left( L\left( \frac{z}{\tau}, \frac{w}{\tau} \bigg| -\frac{1}{\tau} \right) - 1 \right),$$

where $C(z | \tau)$ has the following alternative expressions:

$$C(z | \tau) = -i \sqrt{\frac{i}{\tau}} e\left( \frac{(z + \frac{\tau}{2} - \frac{1}{2})^2}{2\tau} \right) = \frac{\theta(-\frac{z}{\tau} + \frac{1}{2} | -\frac{1}{\tau})}{\tau \theta(-z + \frac{1}{2} | \tau)}.$$

The last expression of $C(z | \tau)$ in the above requires that $z \not\in \mathbb{Z} \oplus \tau\mathbb{Z}$. 
Key Lemma

Lemma

Let \( \phi \) to be a given germ of analytic function at \( \epsilon = 0 \in \mathbb{C} \) such that \( \phi(0) \neq 0 \), and let \( \Phi(\epsilon) = G(\phi(\epsilon); e^{-\epsilon}) \) for all enough small \( \epsilon > 0 \). Then \( \Phi \) can be continued to be an analytic function in some sector \( \Delta_R \) possessing the following properties.

1. If \( \arg(\phi(0)) \in (-\pi, \pi] \) and \( \phi(0) \neq -1 \), then \( \Phi(\epsilon) \) admits an asymptotic expansion as \( \epsilon \to 0 \) in \( \Delta_R \) and
   \[
   \Phi(\epsilon) = \frac{1}{1+\phi(0)} + O(\epsilon).
   \]

2. If \( \phi(0) = -1 = e^{i\pi} \), \( \phi(\epsilon) = -e^{\epsilon(\psi(\epsilon)+\frac{1}{2})} \) and
   \[
   \tilde{\Phi}(\epsilon) = \Phi(\epsilon) - i \sqrt{\frac{\pi}{2\epsilon}} e^{-\frac{\epsilon}{2} (\psi(\epsilon))^2},
   \]
   then \( \tilde{\Phi}(\epsilon) \) admits an asymptotic expansion as \( \epsilon \to 0 \) in \( \Delta_R \) and
   \[
   \tilde{\Phi}(\epsilon) = 1 + \phi'(0) + O(\epsilon).
   \]
Closed-forms and half-periods

**Resurgence** – The asymptotic expansions of $\Phi$ or $\tilde{\Phi}$ are generally divergent. By a Stokes’ analysis, one can find the following

**Theorem**

1. When $\phi(0) \neq -1$, the function $\Phi$ becomes analytic at $\epsilon = 0$ if, and only if, $\phi(\epsilon) = e^{(n+\frac{1}{2})\epsilon}$ for some $n \in \mathbb{Z}$.

2. The function $\tilde{\Phi}$ becomes analytic at $\epsilon = 0$ if, and only if, $\phi(\epsilon) = e^{\pi i + \frac{n}{2}\epsilon}$ for some $n \in \mathbb{Z}$.

As for the Eulerian product $(x; q)_{\infty}$, the following principle is applied:

Analytic $\iff$ Stokes factor is null $\iff$ Without finite singularities in Borel plane
Passing from a unity root to the unity

let us come back to the modular-like relation:

\[ L(z, w \mid \tau) = G(z \mid \tau) + C(z \mid \tau) \left( L\left( \frac{z}{\tau}, \frac{w}{\tau} \mid -\frac{1}{\tau} \right) - 1 \right), \]

If \( \tau \to 0 \) vertically in \( \mathcal{H} \), then \(-1/\tau \to i\infty\) and one gets easily the asymptotic behavior of \( L\left( \frac{z}{\tau}, \frac{w}{\tau} \mid -\frac{1}{\tau} \right) \) in terms of \( q_1 = e(-\frac{1}{\tau}) \) (\( \to 0 \) exponentially). So, the asymptotic behavior of \( L(z, w \mid \tau) \) is completely deduced from that of \( G(z \mid \tau) \) as stated by Key Lemma.

If \( \tau \to r \in \mathbb{Q} \cap (0, 1) \), one takes the continued fractions and makes use of the fact that \( G(z \mid r) \) is analytic at \( z = r \) (see key Lemma). This implies that, finally, one will come back to the case of \( \tau \to 0 \).
In what follows, we write

$$\Omega = \left( -\frac{1}{2}, \frac{1}{2} \right] \times \left[ -\frac{1}{2}, \frac{1}{2} \right), \quad \Omega_\tau = \{ a + b\tau : (a, b) \in \Omega \}. $$

This is a fundamental domain of $\mathbb{R}^2$ for the usual action of $\mathbb{Z}^2 : (a, b) \mapsto (a + \ell, b + m)$ for all $(\ell, m) \in \mathbb{Z}^2$.

**Theorem**

Let $z, w$ to be given in $\Omega_\tau$. Assume that $w \neq 0$ and $w \neq z$, and consider

$$f(q) = L(z, w | \tau) = L(x, \mu; q).$$

Then $f \in \mathcal{M}$ except in the following cases:

1. $z \in \left\{ \frac{1}{2}, \frac{1}{2} - \frac{\tau}{2}, -\frac{\tau}{2} \right\}$ and $w \in \left\{ \frac{1}{2}, \frac{1}{2} - \frac{\tau}{2}, -\frac{\tau}{2} \right\}$, in which case $f$ is a constant function.

2. $z \in \left\{ \frac{1}{2}, \frac{1}{2} - \frac{\tau}{2}, -\frac{\tau}{2} \right\}$ and $w \notin \left\{ \frac{1}{2}, \frac{1}{2} - \frac{\tau}{2}, -\frac{\tau}{2} \right\}$, in which case $f$ is a false theta-type function.
How to find the asymptotic expansion?

Given $a \in \mathbb{U}$, consider the behavior as $q \rightarrow \zeta = e^{2\pi i \frac{p}{m}}$ of the series

$$R(a; q) := \sum_{n \geq 0} \sum_{\lambda \in \mathcal{P}_n} a^{\text{rank}(\lambda)} q^n = \sum_{n \geq 0} \frac{q^{n^2}}{\prod_{m=1}^{n}(1 - aq^m)(1 - q^m/a)}.$$

Two steps

1. Define an average sum related to the root $\zeta$.

2. Make a modular-like transform.
Average summation associated with a root $\zeta$

Let $\zeta$ be a root of unity, with

$$\zeta = e^{\frac{2}{m}\pi ip}, \quad p \in \mathbb{Z}, \quad m \in \mathbb{Z}_{\geq 1}, \quad p \wedge m = 1.$$  

Consider

$$h(x; \zeta) = \sum_{n=0}^{m-1} \zeta^{\frac{1}{2}n(n-1)} x^n, \quad H(x; \zeta) = \sum_{n \geq 0} \zeta^{\frac{1}{2}n(n-1)} x^n.$$  

Both $H(x; \zeta)$ and $h(x; \zeta)$ are related to Gauss’ sums.

For any integer $n$, we define

$$C_n(\zeta) = \frac{1}{m} h(-e^{\frac{1}{m}\pi i} \zeta^{-n}; \zeta^{-1}).$$

When $n = 0$, we shall write $C(\zeta)$ instead of $C_0(\zeta)$.  

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Compare $q$-and $|q|$-Borel-Laplace

By definition, the formal $q$-Borel transform $B_q$ is the automorphism of the $\mathbb{C}$-vector space $\mathbb{C}[[x]]$ that transforms each monomial $x^n$ into $q^{\frac{1}{2} n(n-1)} x^n$. When $q = \rho \zeta$ with $\rho = |q|$, $B_q$ is the (formal) Hadamard product of $B_\rho$ with the power series $H(x; \zeta)$.

Lemma

One has

$$B_q \hat{f}(\xi) = \sum_{n=0}^{m-1} C_n(\zeta^{-1}) B_\rho \hat{f}(-e^{-\frac{1}{m} \pi i} \zeta^{-n} \xi)$$

and

$$B_\rho \hat{f}(\xi) = \sum_{n=0}^{m-1} C_n(\zeta) B_q \hat{f}(-e^{-\frac{1}{m} \pi i} \zeta^n \xi).$$
Definition of $Gq$-summation associated with a root

**Definition**

Given a power series $\hat{f} \in \mathbb{C}[[x]]$ and a real $d \in \mathbb{R}$, one says that $\hat{f}$ is $Gq$-summable in the direction $[0, \infty e^{id})$ if its $q$-Borel transform $B_q(\hat{f})$ represents a germ of analytic function $\phi$ at $0 \in \mathbb{C}$ that can be continued in a sector $|\arg \xi - d| < \delta$ with $\phi(\xi) = O(\theta(A|\xi|; \rho)$ for some suitable $\delta > 0$ and $A > 0$. In this case, the $Gq$-sum $S^d_q(\hat{f})$ of $\hat{f}$ is defined by the following expression:

$$S^d_q(\hat{f}) = \sum_{n=0}^{m-1} C_n(\zeta) \int_0^{\infty e^{id}} \phi(\xi) \omega\left(\frac{\xi}{e^{(1-\frac{1}{m})\pi i}}  \zeta^n x; \rho \right) \frac{d\xi}{\xi}.$$ 

**Important remark** – If $\hat{f}$ is a convergent power series, then $S^d_q(\hat{f})$ equals to the usual sum of $\hat{f}$.
Generalized Appell-Lerch series and Mordell integrals

The power series $\sum_{n \geq 0} q^{-\frac{1}{2} n(n-1)} (-x)^n$ is transformed into $\frac{1}{1+\xi}$ by $q$-Borel transform. So, its $Gq$-sum is

$$\sum_{n=0}^{m-1} C_n(\zeta) \int_0^{\infty} e^{id} \frac{1}{1 + \xi} \omega \left( \frac{\xi}{e^{(1-\frac{1}{m})\pi i} \zeta^n x}; \rho \right) \frac{d\xi}{\xi}.$$

This is to say:

$$\sum_{n=0}^{m-1} C_n(\zeta) G(x\zeta_n; \rho).$$

In the above,

$$\zeta_n = e^{(1-\frac{1}{m})\pi i} \zeta^n, \quad \arg(\zeta_n) = 2\pi \alpha_n, \quad \alpha_n = \frac{1}{2} - \frac{1}{2m} + n \frac{p}{m}. \quad (1)$$
For all \( x \in \mathbb{C}^* \), we define:

\[
G(x, \alpha; q) = \int_0^\infty \frac{(-\alpha \xi; q)_\infty}{1 + \xi} \omega\left(\frac{\xi}{x}; \rho\right) \frac{d\xi}{\xi}
\]

provided that the integral in the right-hand side is convergent, where \( q = \rho \zeta \), \( \rho \in (0, 1) \), and where the integration-path is any half straight-line \((0, e^{i \theta} \infty)\) with \( |\theta| < \pi \).

**Important fact** – The \( Gq \)-sum of \( {}_2\phi_0(q, \alpha; \mu, q, x) \) equals to

\[
(\alpha qx; q)_\infty G(x, \mu; q)
\]
Theorem

1. If \( q \in (0, 1) \), then:

\[
R(a; q) = -a(a, \frac{1}{a}; q) \sum_{n=0}^{\infty} \left( L\left( \frac{a^2}{q}, a; q \right) - G\left( \frac{a^2}{q}; q \right) \right) \\
+ (1 - a)(a; q) \sum_{n=0}^{\infty} G\left( \frac{a^2}{q}, \frac{q}{a}; q \right).
\]

2. If \( q = \rho \zeta \) with \( \rho \in (0, 1) \) and \( \zeta = e^{2\pi i \frac{p}{m}} \), then:

\[
R(a; q) = -a(a, \frac{1}{a}; q) \sum_{n=0}^{m-1} C_n(\zeta) \sum_{n=0}^{m-1} \left( L\left( \frac{a^2}{q}, a; q \right) - \sum_{n=0}^{m-1} C_n(\zeta) \left( \frac{a^2}{q} \zeta_n; \rho \right) \right) \\
+ (1 - a)(a; q) \sum_{n=0}^{m-1} C_n(\zeta) \sum_{n=0}^{m-1} \left( \frac{a^2}{q} \zeta_n, \frac{q}{a}; q \right) .
\]
Conclusion

1. Ramanujan’s third order Mock-theta functions can be expressed by means of Mordell integrals and Appell-Lerch series.
2. This comes from an analysis of Stokes phenomenon for a second order confluent basic hypergeometric equation.
3. The average sum associated to a root yields an Gevrey asymptotic expansion when $q$ tends to this root: real parameter asymptotics.
4. Modular like transform gives exponential smallness or bigness: theta part!
Thank you for your attention!