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# An analytic pointview on the Mock Theta functions of Ramanujan

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**Plan and key-words**

- Motivation : Dedekind eta-function, integer partitions, Ramanujan-Hardy formula, rank and co-rank, Appell-Lerch of the third order, mock theta functions of Ramanujan.
- Asymptotics found from modular formula, definition of theta-type function, Eulerian series and Ramanujan theta-functions.
- Singular $q$-difference equation, heat kernel (Gaussian) and Jacobi theta function, Stokes phenomenon, modular-like relation.
- Continued fractions, asymotototic behavior of Appell-Lerch series, some technical remarks
- Conclusion : Stokes phenomenon implies modularity.

## Number of partitions and formula of Hardy-Ramanujan

- A partition of a positive integer $n$, also called an integer partition, is a way of writing $n$ as a sum of positive integers. The number of partitions of $n$ is denoted by $p(n)$.
- Example : $4 = 4\times 1 (= 1+1+1+1) = 2\times 1 + 1\times 2 = 1\times 1 + 1\times 3 = 2\times 2 = 1\times 4$, so $p(4) = 5$.
- Since Euler, one knows that $\sum_{n\ge 0} p(n) q^n = \frac{1}{(q;q)_\infty}$, where and in the following :

$$(a;q)_\infty = \prod_{n\ge 0} (1 - aq^n)\,, \quad |q| < 1\,.$$

- An asymptotic expression for $p(n)$ is given by Hardy and Ramanujuan :

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right) \text{ as } n\to\infty\,.$$

  Rademacher has completed this asymptotic formula into an exact formula.
- A key point consists of the fact that the Dedekind $\eta$-function $\eta(\tau) := q^{1/24}(q;q)_\infty$ is modular, where $q = e^{2\pi i\tau}$, $\tau\in\mathcal{H}$.

## Rank of a partition and Appell-Lerch series of order 3

- Following Dyson, the rank of a partition is its largest part minus the number of its parts.
- Thanks to Garvan and Andrews, one knows that

$$R(a;q) := \sum_{n\geq 0} \sum_{\lambda \in \mathcal{P}_n} a^{\mathrm{rank}(\lambda)} q^n = \sum_{n\geq 0} \frac{q^{n^2}}{\prod_{m=1}^{n}(1-aq^m)(1-q^m/a)} .$$

- Given an integer $m$, knowing the number of partitions of $n$ with rank congruent to $r$ modulo $m$ for all $r \in \mathbb{Z}/m\mathbb{Z}$ and all $n$ is equivalent to knowing the specialization of $R(a;q)$ to all $m$-th roots of unity $a = e^{2\pi ik/m}$.
- Gordon and McIntosh give that

$$R(a;q) = \frac{1-a}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(3n^2+n)/2}}{1-q^n a} ,$$

where the last series is an Appell-Lerch series of order 3.

## Specialization at $a = -1$ for $R(a; q)$

- Putting $a = -1$ into the expression

$$R(a;q) = \sum_{n\geq 0} \frac{q^{n^2}}{\prod_{m=1}^{n}(1-aq^m)(1-q^m/a)}\,.$$

yields

$$R(-1;q) = 1 + \sum_{n\geq 1} \frac{q^{n^2}}{\prod_{m=1}^{n}(1+q^m)^2}\,.$$

- Ramanujan claimed that

$$\alpha_n \sim \frac{(-1)^{n+1}}{\sqrt{n}} \exp\left(\pi\sqrt{\frac{n}{6}}\right) \text{ as } n \to \infty$$

where $\alpha_n$ denotes the coefficient of $q^n$ in the power series expansion $R(-1;q) = 1 + \sum_{n\geq 1} \alpha_n q^n$.

- This asymptotic relation has first been completed into an exact formula as conjecture by Andrews-Dragonette and proved later by K. Ono and K. Bringmann : by using Cauchy + Unit circle method.

## Ramanujan mock-theta functions of order 3

- In *The final problem* (1935), G. N. Watson discussed the last letter of Ramanujuan to Hardy in which a list of $q$-series had been introduced :

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 ...(1+q^n)^2},$$

$$\phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1+q^2)(1+q^4)...(1+q^{2n})},$$

...

- These series are named by Ramanujuan *mock theta functions*.
- Some of these functions (NOT ALL) can be directly expressed in terms of $R(a;q)$ :

$$f(q) = R(-1;q), \quad \phi(q) = R(i;q), \quad ...$$

## What might mean *mock-theta*?

Following the lastest letter of Ramanujan to Hardy, one may guess that a mock theta function means a function $f$ of the complex variable $q$ , defined by a $q$-series of a particular type (Ramanujan calls this the Eulerian form), which converges for $|q| < 1$ and satisfies the following conditions :

1. infinitely many roots of unity are exponential singularities,
2. for every root $\zeta$ of unity, there is a theta function $T_\zeta(q)$ such that the difference $f(q) - T_\zeta(q)$ is bounded as $q \to \zeta$ radially,
3. $f$ is not the sum of two functions, one of which is a theta function and the other a function which is bounded radially toward all roots of unity.

**Question** – theta functions $=$ ?
Following Andrews and Hickerson, one can define $\theta$-products by using Jacobi's theta functions and one can guess that a theta is the sum of a finitely many theta-products.

## What might mean *theta* for Ramanuajan ?

- One finds that four functions play a particular role in his theory :

$$(q;q)_\infty, \quad (-q;q)_\infty, \quad (q;q^2)_\infty, \quad (-q;q^2)_\infty .$$

- Since $(a^2;q^2)_\infty = (a,-a;q)_\infty$ and $(a;q)_\infty = (a,aq;q^2)_\infty$, one finds that

$$(-q;q)_\infty = \frac{(q^2;q^2)_\infty}{(q;q)_\infty}, \quad (q;q^2)_\infty = \frac{(q;q)_\infty}{(q^2;q^2)_\infty}$$

and that

$$(-q;q^2)_\infty = \frac{(q^2;q^4)_\infty}{(q;q^2)_\infty} = \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty(q^4;q^4)_\infty} .$$

Thus, all the four above functions can be expressed in terms of (quotients of) $(q^k;q^k)_\infty$, where $k$ may be 1, 2 or 4.

- Remember that $q^{1/24}(q;q)_\infty$ is the Dedekind eta-modular function, so Ramanuajan's theta functions might *merely* be related to modular functions.

## A brief revisit of Jacobi's Theta functions (I)

Let $z \in \mathbb{C}^*$, $x = e(z) = e^{2\pi iz}$, $\tau \in \mathcal{H}$, $q = e(\tau) = e^{2\pi i\tau}$, and let $\theta(x; q) = \theta(z \,|\, \tau)$ with

$$\theta(z \,|\, \tau) = \sum_{n\in\mathbb{Z}} q^{n(n-1)/2} x^n = (q, -x, -\frac{q}{x}; q)_\infty.$$

The classical $\theta$-modular formula says that

$$\theta(z + \frac{1}{2} \,|\, \tau) = \frac{A}{\sqrt{m}} \sqrt{\frac{i}{\hat{\tau}}}\, e(-\frac{1}{2\hat{\tau}}(z + \frac{1}{2m} - \frac{\hat{\tau}}{2})^2)\, \theta(\frac{z}{m\hat{\tau}} + \frac{1}{2} \,|\, \tau_m)\,,$$

where $A = A(m, p, \alpha, \beta)$ denotes some unity root,

$$\begin{pmatrix} \alpha & \beta \\ -m & p \end{pmatrix} \in SL(2, \mathbb{Z}), \quad m \in \mathbb{Z}_{>0}$$

and where

$$\hat{\tau} = \tau - \frac{p}{m}, \quad \tau_m = -\frac{\alpha}{m} - \frac{1}{m^2 \hat{\tau}}\,.$$

## A brief revisit of Jacobi's Theta functions (II)

As $\tau \to \frac{p}{m}$, it follows that $\hat{\tau} \to 0$ and $\tau_m \to i\infty$ in $\mathcal{H}$. This implies that if $q_m = e(\tau_m)$, then $q_m \to 0$ exponentially.

**Lemma**

*Given $z_0$, $z_1 \in \mathbb{R}$, if $z = z_0 + z_1\tau$ and $\tau \to \frac{p}{m}$, then*

$$\theta(z + \frac{1}{2}\,|\,\tau) = \frac{A'}{\sqrt{m}}\sqrt{\frac{i}{\hat{\tau}}}\, e(\frac{\lambda_-}{\hat{\tau}} + \lambda_+\hat{\tau})\,\theta(\frac{z}{m\hat{\tau}} + \frac{1}{2}\,|\,\tau_m)\,,$$

*with $\hat{\tau} = \tau - \frac{p}{m} \to 0$,*

1. $|A'| = 1$, $\lambda_- = -\frac{1}{2}(z_0 + \frac{z_1 p}{m} + \frac{1}{2m})^2$, $\lambda_+ = -\frac{1}{2}(z_1 - \frac{1}{2})^2$,
2. $\theta(\frac{z}{m\hat{\tau}} + \frac{1}{2}\,|\,\tau_m) = C\, q_m^{\delta}\,(1 + O(q_m^{\kappa}))$, *where $\delta \in \mathbb{R}$ and $\kappa > 0$.*

This revisit about Jacobi's theta functions shows that :

- All modular function $f(\tau)$ has a similar asymptotic behavior at every root of unity $e(p/m) = e^{2\pi ip/m}$ :

$$f(\tau) = \frac{A'}{\sqrt{m}} \sqrt{\frac{i}{\hat{\tau}}}\, e(\frac{\lambda_-}{\hat{\tau}} + \lambda_+ \hat{\tau})\, (1 + O(e^{-\kappa/|\hat{\tau}|}))\,,$$

  with $\hat{\tau} = \tau - \frac{p}{m} \to 0$.

- Moreover, the factor $1 + O(e^{-\kappa/|\hat{\tau}|})$ can be expressed as a germ of analytic function of the NEW (modular) variable $e(-1/\hat{\tau})$ at $0 \in \mathbb{C}$.
- Modularity $\Longrightarrow$ Exponential scale for the asymptotics : **transseries !**.

## Radial limit behavior at each root of unity

For simplify, a function $f$ will be called **theta-type** at a given root $\zeta = e^{2\pi i \frac{p}{m}}$ of unity, and **one writes** $f \in \mathfrak{T}_\zeta$**,** if there exists a quadruplet $(\upsilon, \lambda, I, \gamma)$, composed of a couple $(\upsilon, \lambda) \in \mathbb{Q} \times \mathbb{R}$, a strictly increasing and unbounded sequence $I$, and a $\mathbb{C}^*$-valued map $\gamma$ defined on $I$, such that the following relation holds for any $N \in \mathbb{Z}_{\geq 0}$ as $\tau \xrightarrow{a.v} r$ :

$$f(q) = \left(\frac{i}{\hat{\tau}}\right)^{\upsilon} q^{\lambda} \Big( \sum_{k \in I \cap (-\infty, N]} \gamma(k)\, q_m^k + o(q_m^N) \Big) .$$

In the above, $\hat{\tau} = \tau - \frac{p}{m}$ and $q_m$ denotes the modular variable related to $\zeta$ :

$$q_m = e^{-2\pi i \frac{1}{m^2 \hat{\tau}}}\ e^{2\pi i \frac{p'}{m}}$$

with $pp' = -1 \pmod{m}$.

## When Euler products is theta-type ?

The infinite product $(x;q)_\infty = (1-x)(1-xq)(1-xq^2)\ldots$ is called Euler product.

### Theorem

*The following conditions are equivalent for any Eulerian product $(x;q)_\infty$ while $x = e^{2\pi i a} q^b$ with $a$, $b \in \mathbb{R}$, $q = e^{2\pi i \tau}$, $\tau \in \mathcal{H}$.*

1. $(x;q)_\infty \in \mathfrak{T}_1$,
2. $(x;q)_\infty \in \mathfrak{T}_\zeta$ *for some root of unity,*
3. $(x;q)_\infty \in \mathfrak{T}_\zeta$ *for every root of unity,*
4. $x \in \{q, -q, \sqrt{q}, -\sqrt{q}\}$.

## Strategy for the proof

- Let $x_1 = e^{2\pi i \frac{a+b\tau}{\tau}}$ and $q_1 = e^{-2\pi i \frac{1}{\tau}}$. Comparing $(x;q)_\infty$ with $(x_1;q_1)_\infty$ yields that
$$(x;q)_\infty = e^{T(x;q)}\,(x_1;q_1)_\infty\,,$$
where $T$ can be expressed with dilogarithm plus a Laplace integral.
- $(x;q)_\infty \in \mathfrak{T}_1$ means that the above Laplace integral defines a polynomial so we have NO divergence as $q \to 1$.
- Continued fractions $\Longrightarrow$ other roots of unity from 1.

**Counterexample** – Knowing $(q;q)_\infty \in \mathfrak{T}_1$, one can see that $(q^2;q)_\infty$ does not belong to $\mathfrak{T}_1$.
Indeed, $(q^2;q)_\infty = \frac{(q;q)_\infty}{1-q}$ and $\frac{1}{1-q} \notin \mathfrak{T}_1$ ! For

$$\frac{1}{1-e^{2\pi i\tau}} \neq C\left(\frac{i}{\tau}\right)^\nu e(\frac{\lambda_-}{\tau} + \lambda_+\tau)\left(1 + O(e^{-\kappa/|\tau|})\right).$$

The above-introduced **theta-type** property can be extended as follows $(0 = e(i\infty))$ :

## Definition

Let $q = e(\tau)$, $\tau \in \mathcal{H}$, $\zeta \in \mathbb{U} \cup \{0\}$, where $\mathbb{U}$ is the set of the roots of unity.

1. One says that $f$ is of **almost theta-type as** $q \to \zeta$ and one writes $f \in \tilde{\mathfrak{T}}_\zeta$, if there exists $f_\zeta \in \mathfrak{T}_\zeta$ such that $f - f_\zeta = O(1)$ as $q \to \zeta$.
2. One says that $f$ is a **theta-type function** and one writes $f \in \mathfrak{T}$, if $f \in \mathfrak{T}_\zeta$ for all $\zeta \in \mathbb{U} \cup \{0\}$.
3. One says that $f$ is a **false theta-type function** and one writes $f \in \mathfrak{F}$, if $f \notin \mathfrak{T}$ and there exists $T \in \mathfrak{T}$ such that $f(q) - T(q) = O(1)$ for all $\zeta \in \mathbb{U} \cup \{0\}$.
4. One says that $f$ is a **mock theta-type function** and one writes $f \in \mathfrak{M}$, if $f \in \tilde{\mathfrak{T}}_\zeta$ for all $\zeta = \mathbb{U} \cup \{0\}$ and, moreover, for each given $T \in \mathfrak{T}$, there exists $\zeta \in \mathbb{U} \cup \{0\}$ such that $f - T$ is unbounded.

## A singular $q$-difference equation

From the analytic classification viewpoint of complex linear differential and $q$-difference equations, the series

$$\sum_{n\geq 0} q^{-n(n-1)/2}(-x)^n$$

is a $q$-analog of the famous Euler series $\sum_{n\geq 0} n!\,(-x)^n$. We are led to the $q$-difference equation

$$y(qx) + qx\,y(x) = 1\,,$$

that admits $x = 0$ as non-Fuchsian singular point.

As the moment problem of the sequence $(q^{-n^2/2})_n$ is undetermined, the above $q$-Euler series can be the asymptotic expansion of several solutions of this functional equation. Thus,

**the $q$-analog of Borel-sum is not unique.**

**Instead of $n! = \int_0^\infty e^{-t}\, t^{n+1}\, \frac{dt}{t}$**

- Let $n \in \mathbb{Z}$ and $a > 0$. Since

$$\int_{-\infty}^{\infty} e^{-(t+na)^2}\, dt = 2\int_0^\infty e^{-t^2} dt = \sqrt{\pi}\,,$$

it follows that

$$e^{n^2a^2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}\, e^{-2ant}\, dt\,.$$

Putting $e^{a^2} = q^{-\frac{1}{2}}$ and $\xi = e^{-2at}$ yields that

$$q^{-\frac{1}{2}n^2} = \frac{1}{\sqrt{2\pi \ln 1/q}} \int_0^\infty e^{\frac{1}{2\ln q}\log^2\xi}\, \xi^n \frac{d\xi}{\xi}\,.$$

- Let $\mu \notin (-q^{\mathbb{Z}})$, and let $\theta(x;q) = \sum_{k\in\mathbb{Z}} q^{k(k-1)/2}x^k = (q, -x, -q/x; q)_\infty$. Using $\theta(q^kx;q) = q^{-k(k-1)/2}x^{-k}\theta(x;q)$ gives that

$$q^{-\frac{1}{2}n(n-1)} = \sum_{\xi=\mu q^k, k\in\mathbb{Z}} \frac{\xi^n}{\theta(\xi;q)}\,.$$

## $q$-Borel-Laplace transforms

- $q$-Borel :

$$\sum_{n\geq 0} a_n x^n \quad \Longrightarrow \quad \sum_{n\geq 0} a_n q^{n(n-1)/2} \xi^n .$$

- $q$-Laplace with (Gaussian) heat kernel $\omega(u;q) = \frac{1}{\sqrt{2\pi \ln(1/q)}} e^{\frac{\log^2(u/\sqrt{q})}{2\ln q}}$ :

$$\phi \quad \Longrightarrow \quad \int_0^\infty \phi(\xi)\omega(\frac{\xi}{x};q)\frac{d\xi}{\xi} .$$

- $q$-Laplace with Jacobi theta function $\theta(u;q) = \sum_{k\in\mathbb{Z}} q^{k(k-1)/2} u^k$ :

$$\phi \quad \Longrightarrow \quad \sum_{\xi\in\mu q^{\mathbb{Z}}} \frac{\phi(\xi)}{\theta(\frac{\xi}{x};q)} ,$$

where $\mu$ is a given complex number that indicates the integration path.

## Appell-Lerch series and Mordell's integral as solution of $q$-difference equation

By following the above $q$-analogs of the Borel-Laplace summation method, one can check that both expressions give solutions of $y(qx) + qx\, y(x) = 1$ :

$$L(x, \mu; q) = \sum_{n=-\infty}^{\infty} \frac{1}{1 - \mu q^n} \frac{1}{\theta(-\frac{\mu}{x} q^n; q)}$$

and

$$G(x; q) = \int_0^{\infty} \frac{\omega(\xi/x; q)}{1 + \xi} \frac{d\xi}{\xi} \, .$$

**Physics** – $G(x; q)$ represents a solution for the "heat equation"

$$2q\partial_q f + x^2 \partial_x^2 f = 0, \quad f(x, q)|_{q=1} = \frac{1}{1 + x} \, ,$$

which is formally satisfied by the power series $\sum_{n\geq 0} q^{-n(n-1)/2}(-x)^n$ .

## Stokes phenomenon

The functions $L(x, \mu; q)$ and $G(x; q)$ are both sum-functions of the same power series, so their difference satisfies the homogeneous $q$-difference equation $y(qx) + qxy(x) = 0$. This approach is similar to the analysis of Stokes in the theory of differential equations.

Let $L(z, w \,|\, \tau) = L(x, \mu; q)$ and $G(z \,|\, \tau) = G(x; q)$, where $x = e(z) = e^{2\pi iz}$, $\mu = e(w)$ and $q = e(\tau)$. It follows that

$$L(z, w \,|\, \tau) = G(z \,|\, \tau) + C(z \,|\, \tau) \left( L(\frac{z}{\tau}, \frac{w}{\tau} \,|\, -\frac{1}{\tau}) - 1 \right),$$

where $C(z \,|\, \tau)$ has the following alternative expressions :

$$C(z \,|\, \tau) = -i \sqrt{\frac{i}{\tau}}\, e(\frac{(z + \frac{\tau}{2} - \frac{1}{2})^2}{2\tau}) = \frac{\theta(-\frac{z}{\tau} + \frac{1}{2} \,|\, -\frac{1}{\tau})}{\tau\, \theta(-z + \frac{1}{2} \,|\, \tau)}.$$

The last expression of $C(z \,|\, \tau)$ in the above requires that $z \notin \mathbb{Z} \oplus \tau\mathbb{Z}$.

## Key Lemma

**Lemma**

*Let $\phi$ to be a given germ of analytic function at $\epsilon = 0 \in \mathbb{C}$ such that $\phi(0) \neq 0$, and let $\Phi(\epsilon) = G(\phi(\epsilon); e^{-\epsilon})$ for all enough small $\epsilon > 0$. Then $\Phi$ can be continued to be an analytic function in some sector $\Delta_R$ possessing the following properties.*

1. *If $\arg(\phi(0)) \in (-\pi, \pi]$ and $\phi(0) \neq -1$, then $\Phi(\epsilon)$ admits an asymptotic expansion as $\epsilon \to 0$ in $\Delta_R$ and $\Phi(\epsilon) = \frac{1}{1+\phi(0)} + O(\epsilon)$.*
2. *If $\phi(0) = -1 = e^{i\pi}$, $\phi(\epsilon) = -e^{\epsilon(\psi(\epsilon)+\frac{1}{2})}$ and*

$$\tilde{\Phi}(\epsilon) = \Phi(\epsilon) - i\sqrt{\frac{\pi}{2\epsilon}}\, e^{-\frac{\epsilon}{2}\,(\psi(\epsilon))^2}\,,$$

*then $\tilde{\Phi}(\epsilon)$ admits an asymptotic expansion as $\epsilon \to 0$ in $\Delta_R$ and*

$$\tilde{\Phi}(\epsilon) = 1 + \phi'(0) + O(\epsilon)\,.$$

## Closed-forms and half-periods

**Resurgence** – The asymptotic expansions of $\Phi$ or $\tilde{\Phi}$ are generally divergent. By a Stokes' analysis, one can find the following

**Theorem**

1. *When* $\phi(0) \neq -1$, *the function* $\Phi$ *becomes analytic at* $\epsilon = 0$ *if, and only if,* $\phi(\epsilon) = e^{(n+\frac{1}{2})\epsilon}$ *for some* $n \in \mathbb{Z}$.
2. *The function* $\tilde{\Phi}$ *becomes analytic at* $\epsilon = 0$ *if, and only if,* $\phi(\epsilon) = e^{\pi i + \frac{n}{2}\epsilon}$ *for some* $n \in \mathbb{Z}$.

As for the Eulerian product $(x;q)_\infty$, the following principle is applied :

Analytic $\Leftrightarrow$ Stokes factor is null
$\Leftrightarrow$ Without finite singularities in Borel plane

## Passing from a unity root to the unity

let us come back to the modular-like relation :

$$L(z,w\,|\,\tau)=G(z\,|\,\tau)+C(z\,|\,\tau)\left(L(\frac{z}{\tau},\frac{w}{\tau}\,|\,-\frac{1}{\tau})-1\right),$$

If $\tau\to 0$ vertically in $\mathcal{H}$, then $-1/\tau\to i\infty$ and one gets easily the asymptotic behavior of $L(\frac{z}{\tau},\frac{w}{\tau}\,|\,-\frac{1}{\tau})$ in terms of $q_1=e(-\frac{1}{\tau})$ ($\to 0$ exponentially). So, the asymptotic behavior of $L(z,w\,|\,\tau)$ is completely deduced from that of $G(z\,|\,\tau)$ as stated by Key Lemma.

If $\tau\to r\in\mathbb{Q}\cap(0,1)$, one takes the continued fractions and makes use of the fact that $G(z\,|\,r)$ is analytic at $z=r$ (see key Lemma). This implies that, finally, one will come back to the case of $\tau\to 0$.

In what follows, we write

$$\Omega = (-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}), \quad \Omega_\tau = \{a + b\tau : (a, b) \in \Omega\}.$$

This is a fundamental domain of $\mathbb{R}^2$ for the usual action of $\mathbb{Z}^2$ : $(a, b) \mapsto (a + \ell, b + m)$ for all $(\ell, m) \in \mathbb{Z}^2$.

**Theorem**

*Let $z$, $w$ to be given in $\Omega_\tau$. Assume that $w \neq 0$ and $w \neq z$, and consider*

$$f(q) = L(z, w \,|\, \tau) = L(x, \mu; q).$$

*Then $f \in \mathfrak{M}$ except in the following cases :*

1. *$z \in \{\frac{1}{2}, \frac{1}{2} - \frac{\tau}{2}, -\frac{\tau}{2}\}$ and $w \in \{\frac{1}{2}, \frac{1}{2} - \frac{\tau}{2}, -\frac{\tau}{2}\}$, in which case $f$ is a constant function.*
2. *$z \in \{\frac{1}{2}, \frac{1}{2} - \frac{\tau}{2}, -\frac{\tau}{2}\}$ and $w \notin \{\frac{1}{2}, \frac{1}{2} - \frac{\tau}{2}, -\frac{\tau}{2}\}$, in which case $f$ is a false theta-type function.*

### How to find the asymptotic expansion ?

Given $a \in \mathbb{U}$, consider the behavior as $q \to \zeta = e^{2\pi i \frac{p}{m}}$ of the series

$$R(a;q) := \sum_{n\geq 0} \sum_{\lambda\in\mathcal{P}_n} a^{\mathrm{rank}(\lambda)} q^n = \sum_{n\geq 0} \frac{q^{n^2}}{\prod_{m=1}^{n}(1-aq^m)(1-q^m/a)} .$$

**Two steps**

1. Define an average sum related to the root $\zeta$.

2. Make a modular-like transform.

## Average summation associated with a root $\zeta$

Let $\zeta$ be a root of unity, with

$$\zeta = e^{\frac{2}{m}\pi i p}, \quad p \in \mathbb{Z}, \quad m \in \mathbb{Z}_{\geq 1}, \quad p \wedge m = 1 .$$

Consider

$$h(x;\zeta) = \sum_{n=0}^{m-1} \zeta^{\frac{1}{2}n(n-1)} x^n , \quad H(x;\zeta) = \sum_{n\geq 0} \zeta^{\frac{1}{2}n(n-1)} x^n .$$

Both $H(x;\zeta)$ and $h(x;\zeta)$ are related to Gauss' sums.
For any integer $n$, we define

$$C_n(\zeta) = \frac{1}{m}\, h(-e^{\frac{1}{m}\pi i}\zeta^{-n};\zeta^{-1}) .$$

When $n = 0$, we shall write $C(\zeta)$ instead of $C_0(\zeta)$.

## Compare $q$-and $|q|$-Borel-Laplace

By definition, the formal $q$-Borel transform $\mathcal{B}_q$ is the automorphism of the $\mathbb{C}$-vector space $\mathbb{C}[[x]]$ that transforms each monomial $x^n$ into $q^{\frac{1}{2}n(n-1)}x^n$. When $q = \rho\zeta$ with $\rho = |q|$, $\mathcal{B}_q$ is the (formal) Hadamard product of $\mathcal{B}_\rho$ with the power series $H(x;\zeta)$

**Lemma**

*One has*

$$\mathcal{B}_q\hat{f}(\xi) = \sum_{n=0}^{m-1} C_n(\zeta^{-1})\,\mathcal{B}_\rho\hat{f}(-e^{-\frac{1}{m}\pi i}\zeta^{-n}\xi)$$

*and*

$$\mathcal{B}_\rho\hat{f}(\xi) = \sum_{n=0}^{m-1} C_n(\zeta)\,\mathcal{B}_q\hat{f}(-e^{-\frac{1}{m}\pi i}\zeta^{n}\xi)\,.$$

## Definition of $Gq$-summation associated with a root

### Definition

Given a power series $\hat{f} \in \mathbb{C}[[x]]$ and a real $d \in \mathbb{R}$, one says that $\hat{f}$ is $Gq$-summable in the direction $[0, \infty e^{id})$ if its $q$-Borel transform $\mathcal{B}_q(\hat{f})$ represents a germ of analytic function $\phi$ at $0 \in \mathbb{C}$ that can be continued in a sector $|\arg \xi - d| < \delta$ with $\phi(\xi) = O(\theta(A|\xi|;\rho)$ for some suitable $\delta > 0$ and $A > 0$. In this case, the $Gq$-sum $\mathcal{S}_q^d(\hat{f})$ of $\hat{f}$ is defined by the following expression :

$$\mathcal{S}_q^d(\hat{f}) = \sum_{n=0}^{m-1} C_n(\zeta) \int_0^{\infty e^{id}} \phi(\xi)\, \omega(\frac{\xi}{e^{(1-\frac{1}{m})\pi i}\zeta^n x};\rho)\, \frac{d\xi}{\xi}\,.$$

**Important remark** – *If $\hat{f}$ is a convergent power series, then $\mathcal{S}_q^d(\hat{f})$ equals to the usual sum of $\hat{f}$.*

## Generalized Appell-Lerch series and Mordell integrals

The power series $\sum_{n\geq 0} q^{-\frac{1}{2}n(n-1)}(-x)^n$ is transformed into $\frac{1}{1+\xi}$ by $q$-Borel transform. So, its $Gq$-sum is

$$\sum_{n=0}^{m-1} C_n(\zeta) \int_0^{\infty e^{id}} \frac{1}{1+\xi}\, \omega\Big(\frac{\xi}{e^{(1-\frac{1}{m})\pi i}\zeta^n x}; \rho\Big) \frac{d\xi}{\xi}\,.$$

This is to say :

$$\sum_{n=0}^{m-1} C_n(\zeta)\, G(x\zeta_n; \rho)\,.$$

In the above,

$$\zeta_n = e^{(1-\frac{1}{m})\pi i}\,\zeta^n\,, \quad \arg(\zeta_n) = 2\pi\,\alpha_n\,, \quad \alpha_n = \frac{1}{2} - \frac{1}{2m} + n\,\frac{p}{m}\,. \qquad (1)$$

For all $x \in \tilde{\mathbb{C}}^*$, we define :

$$G(x, \alpha; q) = \int_0^\infty \frac{(-\alpha\xi; q)_\infty}{1+\xi}\, \omega(\frac{\xi}{x}; \rho)\, \frac{d\xi}{\xi}$$

provided that the integral in the right-hand side is convergent, where $q = \rho\zeta$, $\rho \in (0, 1)$, and where the integration-path is any half straight-line $(0, e^{id}\infty)$ with $|d| < \pi$.

**Important fact** – The $Gq$-sum of ${}_2\phi_0(q, \alpha; \mu, q, x)$ equals to

$$(\alpha q x; q)_\infty\, G(x, \mu; q)$$

## Theorem

1. *If $q \in (0,1)$, then :*

$$R(a;q) = -a(a,\frac{1}{a};q)_\infty \left(L(\frac{a^2}{q},a;q) - G(\frac{a^2}{q};q)\right) + (1-a)\,(a;q)_\infty\, G(\frac{a^2}{q},\frac{q}{a};q)\,.$$

2. *If $q = \rho\zeta$ with $\rho \in (0,1)$ and $\zeta = e^{2\pi i \frac{p}{m}}$, then :*

$$R(a;q) = -a(a,\frac{1}{a};q)_\infty \left(L(\frac{a^2}{q},a;q) - \sum_{n=0}^{m-1} C_n(\zeta)\, G(\frac{a^2}{q}\zeta_n;\rho)\right) + (1-a)\,(a;q)_\infty \sum_{n=0}^{m-1} C_n(\zeta)\, G(\frac{a^2}{q}\zeta_n,\frac{q}{a};q)\,.$$

## Conclusion

1. Ramanujan's third order Mock-theta functions can be expressed by means of Mordell integrals and Appell-Lerch series.
2. This comes from an analysis of Stokes phenomenon for a second order confluent basic hypergeometric equation.
3. The average sum associated to a root yields an Gevrey asymptotic expansion when $q$ tends to this root : real parameter asymptotics.
4. Modular like transform gives exponential smallness or bigness : theta part !

Thank you for your attention !