Computing Solutions of Linear Mahler Equations

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Joint work with Th. Dreyfus, Ph. Dumas, and M. Mezzarobba

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Mahler operator of radix b ($b \in \mathbb{N},\ b \geq 2,\ y$ a function or series):

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- enumeration of words
- partition theory
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In contrast, if $b = \operatorname{char} \mathbb{K}$ and is prime, all solutions y are algebraic: $y(x^b) = y(x)^b$. (See Dumas's talk.)

Our Goal in this Work: Solutions in Closed Form When $\operatorname{char} \mathbb{K} = 0$

Targeted classes

Given L of order r, with polynomial coefficients of degree $\leq d$, find algorithms of reasonable complexity for:

- (power/Puiseux) series solutions,
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$$q(\alpha) = 0 \text{ and } \beta^{b^r} = \alpha \implies \ell_r(\beta) = 0 \text{ or } \left(q(\alpha') = 0 \text{ for } \ \alpha' = \beta^{b^i},\ i < r \right)$$

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Geometric intuition

If α, not a root of unity, is a root of q, it generates:

- a finite sequence $\alpha_0 = \alpha, \alpha_1, \ldots, \alpha_n$ of two by two distinct roots of q,
- then, a b^rth root β_n of α_n that is a root of ℓ_r.

The Graeffe Operator: a Quasi-Inverse for the Mahler Operator

$$\begin{array}{ccc}
\gamma = \alpha^b & (Gp)(\gamma) = 0 & Gp = \mathrm{Res}_y(y^b - x, p(y)) \\
\uparrow & \uparrow & \uparrow \\
\alpha \in \mathbb{C} & p(\alpha) = 0 & p \in \mathbb{K}[x] \\
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Key lemma

Given $\ell \in \mathbb{K}[x]$:

$$\begin{cases} q \in \mathbb{K}[x] \setminus \mathbb{K} \\ x \nmid q \\ M^r q \mid \ell \bigvee_{i=0}^{r-1} M^i q \end{cases} \implies \exists u \in \mathbb{K}[x] \setminus \mathbb{K}, \quad M^r u \mid \ell \quad \text{or} \quad \begin{cases} M^{r-1} u \mid \ell \\ q \mid Gu \end{cases}$$

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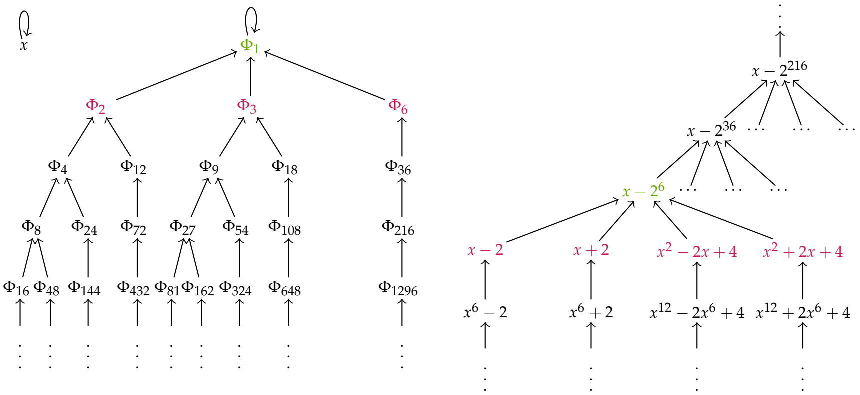
Remark: left case impossible $\implies$ q is a product of cyclotomic polynomials.

Graph of (Radical of) Graeffe Operator for $b = 6$ in $\mathbb{Q}[x]$

$$Mp = \prod_{q \text{ s.t. } \sqrt{G}(q)=p} q$$

$$M\Phi_1 = \Phi_1\Phi_2\ \Phi_3\ \Phi_6$$

$$M(x-2^6) = (x-2)(x+2)(x^2-2x+4)(x^2+2x+4)$$



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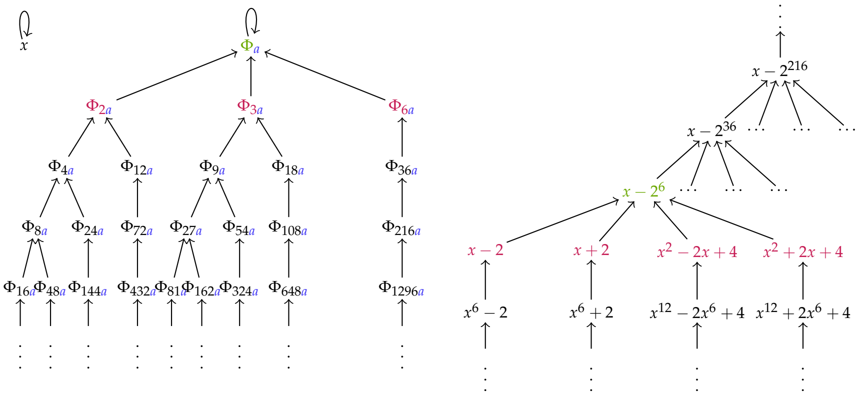

Looking for a Denominator: an Example When $b = 3$

$L = (2x^9-1)(x^{18}-x^9-1)(x^{12}+x^9+2x^6-x^3+1)(x^{18}-x^9+1)(2x^{34}-9x^{33}+10x^{32}+\cdots)(x^{54}-x^{27}-1)\,M^2+\cdots$

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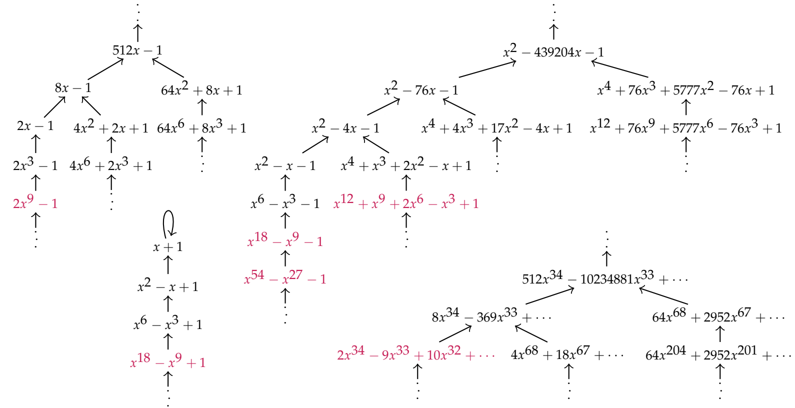

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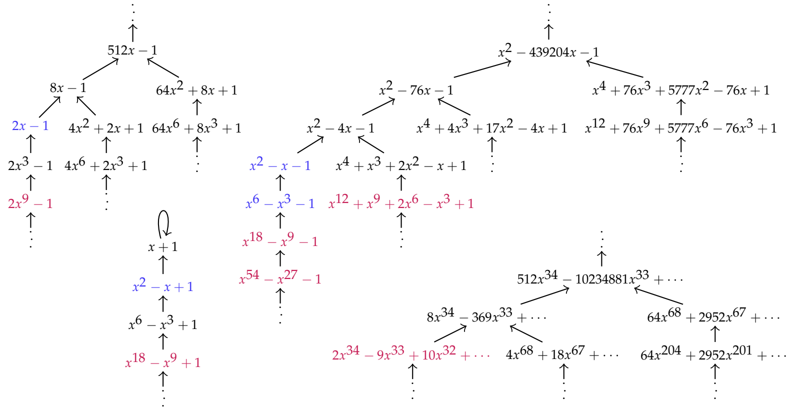

$$u_1 = (2x - 1)(x^2 - x + 1)(x^2 - x - 1)(x^6 - x^3 - 1) \implies M^2 u_1 \mid \ell_2$$

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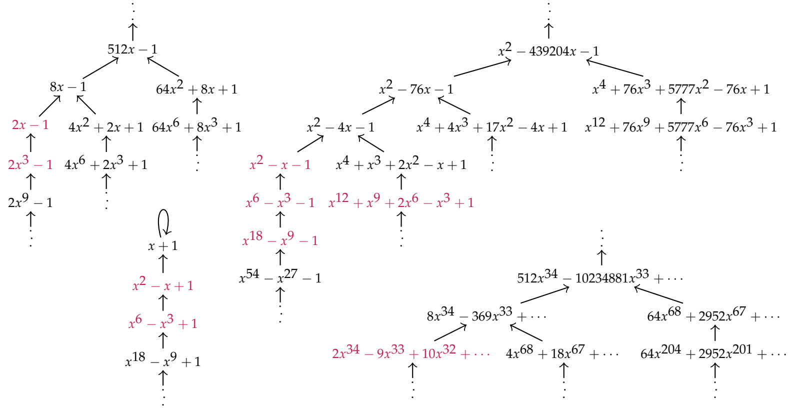

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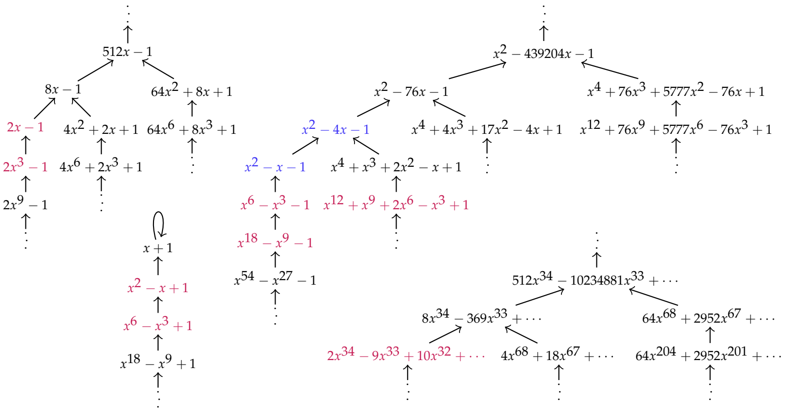

$$u_2 = (x^2 - x - 1)(x^2 - 4x - 1) \implies M^2 u_2 \mid \ell_2$$

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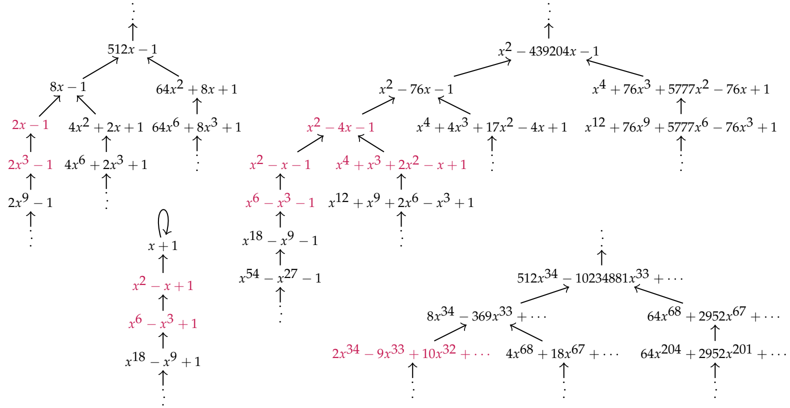

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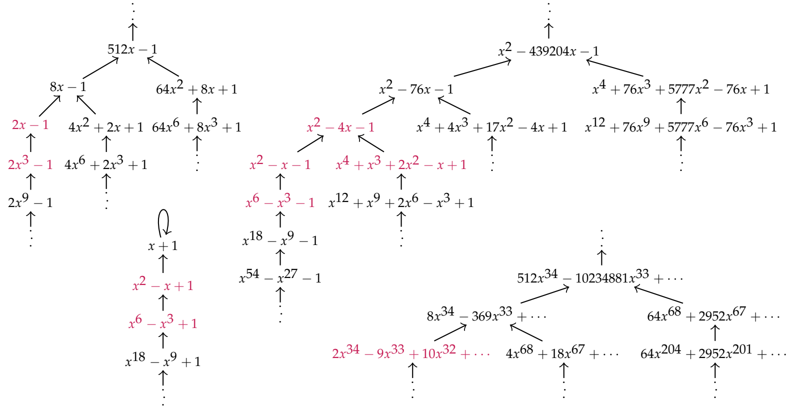

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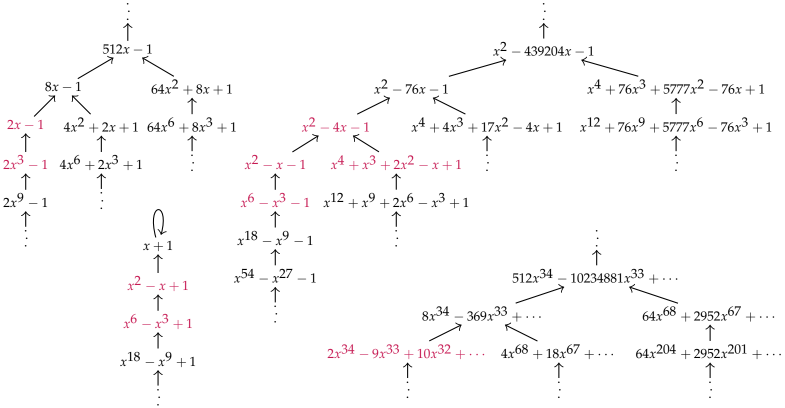

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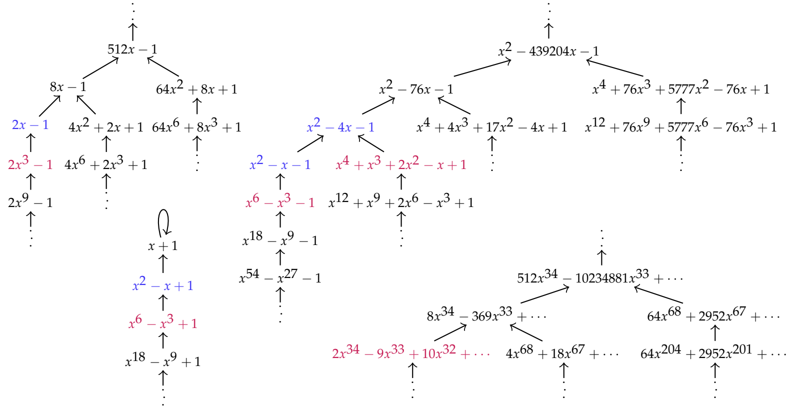

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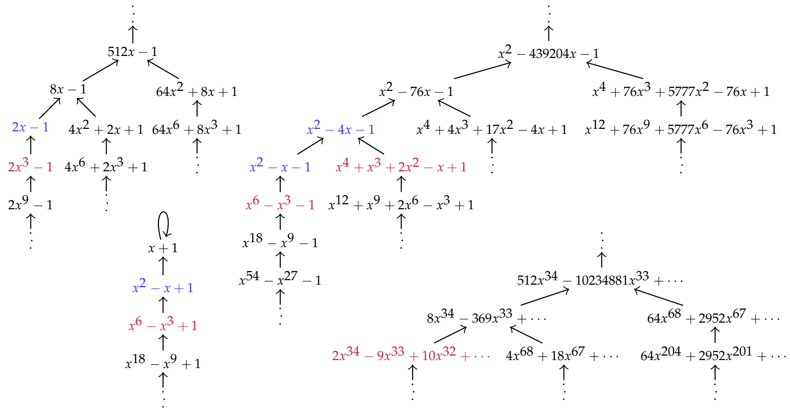

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Operator to solve:

$$L = (2x^9-1)(x^{18}-x^9-1)(x^{12}+x^9+2x^6-x^3+1)(x^{18}-x^9+1)(2x^{34}-9x^{33}+10x^{32}+\cdots)(x^{54}-x^{27}-1)\,M^2$$
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$$+x^2(2x-1)(x^2-x+1)^2(x^2-x-1)(x^2+x+1)(x^2-4x-1)(2x^{102}-9x^{99}+10x^{96}+\cdots)$$

Denominator bound found:

$$q^\star = u_1u_2G\tilde{u} = (2x-1)(x^2-x+1)^2(x^2-x-1)^2(x^2-4x-1)(x^6-x^3-1)$$

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Explicit rational solutions:

$$\frac{2x}{(2x-1)(x^2-x-1)} \quad \text{and} \quad \frac{x-3}{(x^2-x+1)(x^2-4x-1)(x^6-x^3-1)}$$

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$$L = (2x^9-1)(x^{18}-x^9-1)(x^{12}+x^9+2x^6-x^3+1)(x^{18}-x^9+1)(2x^{34}-9x^{33}+10x^{32}+\cdots)(x^{54}-x^{27}-1)\,M^2$$
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$$+x^2(2x-1)(x^2-x+1)^2(x^2-x-1)(x^2+x+1)(x^2-4x-1)(2x^{102}-9x^{99}+10x^{96}+\cdots)$$

Denominator bound found:

$$q^\star = u_1u_2G\tilde{u} = (2x-1)(x^2-x+1)^2(x^2-x-1)^2(x^2-4x-1)(x^6-x^3-1)$$

Explicit rational solutions:

$$\frac{2x}{(2x-1)(x^2-x-1)} \quad \text{and} \quad \frac{x-3}{(x^2-x+1)(x^2-4x-1)(x^6-x^3-1)}$$

Remark: $\deg \ell_2 = 145 > \deg \ell_2[u_1] = 84 > \deg \ell_2[u_1u_2] = 62$.

Computing Denominator Bounds

$$\ell = \sum_{n\geq 0} c_n x^n \xrightarrow{i\text{th section in radix } b} \sum_{n\geq 0} c_{b^r n+i} x^n =: f_i$$

Algorithm

1. Set $\ell = \ell_r$, then repeat for $k = 1, 2, \dots$:
   1. set $u_k = \bigwedge_{i=0}^{b^r-1} f_i$ where $\ell = \sum_{i=0}^{b^r-1} x^i M^r f_i$ with $f_i \in \mathbb{K}[x]$;
   2. set $\ell = (\ell / M^r u_k) \bigvee_{i=0}^{r-1} M^i u_k$

   until $\deg u_k = 0$.
2. Set $\tilde{u} = \bigwedge_{i=0}^{b^{r-1}-1} f_i$ where $\ell = \sum_{i=0}^{b^{r-1}-1} x^i M^{r-1} f_i$ with $f_i \in \mathbb{K}[x]$.
3. Return $q^\star = u_1 \cdots u_{k-1} G \tilde{u}$.

Theorem (correctness)

Assume $Ly = 0$ with $y = \frac{p}{x^{\bar{v}} q}$ and $q(0) \neq 0$. Then, $q \mid q^\star$.

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Theorem (complexity and degree bounds)

- algorithm runs in $\mathrm{O}(d\ \mathrm{M}(d) \log d)$ ops
- $\deg q^\star \leq d$ if $b = 2$, $\quad \deg q^\star \leq d / b^{r-1}$ if $b \geq 3$

Series and Polynomial Solutions

L of order r, with polynomial coefficients ℓ_k of valuation v_k and degree $\leq d$.

Valuation and degree bounds

Introducing suitable Newton polygons:

- valuation v of series solutions: $-\dfrac{v_r}{b^{r-1}(b-1)} \leq v \leq \dfrac{v_0}{b-1}$,
- degree δ of polynomial solutions: $\delta \leq \dfrac{d}{b^{r-1}(b-1)}$.

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Complexity

- basis of approximate series solutions in $\mathrm{O}(rv_0^2 + r^2\,\mathrm{M}(v_0))$ ops,
- basis of polynomials solutions in $\tilde{\mathrm{O}}(d^2/b^r + \mathrm{M}(d))$ ops.

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In particular: reasonable complexity for numerators.

Conclusions

Summary

- theory of Newton polygons
- algorithms for denominator bounds:
  - good: complexity is polynomial in r and d,
  - bad: quadratic in output size.
- other algorithms for series solutions, polynomial solutions, and rational-function solutions.

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- theory of Newton polygons
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Future work

- the case $\ell_0 = 0$,
- infinite-product solutions,
- rational solutions of Riccati-type equation.