On some algorithmically undecidable problems connected with partial differential equations

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Introduction

We will consider algorithmic undecidability of the problems of testing the existence of some kinds of solutions for partial linear differential (and difference) equations with polynomial coefficients.

- Rational functions
- Formal Laurent series
- Analytic functions
- Infinitely differentiable functions
- Finitely differentiable functions

Introduction

Partial linear differential operator with polynomial coefficients:

$$L = \sum_{n \in S} p_n(x_1, \ldots, x_m) \frac{\partial^{|n|}}{\partial x_1^{n_1} \ldots \partial x_m^{n_m}} \tag{1}$$

- S — finite subset of $\mathbb{Z}^m_{\geqslant 0}$
- $x_1, \ldots, x_m$ — vector of m independent variables (further we will denote them $\mathbf{x}$)
- $p_n \in \mathbb{Z}[\mathbf{x}]$
- $|n| = n_1 + \cdots + n_m$

We will denote the ring of these operators as $\mathbb{Z}[\frac{\partial}{\partial \mathbf{x}}, \mathbf{x}]$

Introduction

Partial linear difference operator with polynomial coefficients:

$$L = \sum_{n \in S} p_n(x_1, \dots, x_m) \Delta_1^{n_1} \dots \Delta_m^{n_m} \tag{2}$$

- S — finite subset of $\mathbb{Z}_{\geqslant 0}^m$
- $p_n \in \mathbb{Z}[\mathbf{x}]$
- $\Delta_i y(x) = y(x_1, \dots, x_i + 1, \dots, x_m) - y(x_1, \dots, x_m)$

We will denote the ring of these operators as $\mathbb{Z}[\Delta, \mathbf{x}]$

Introduction

If $m = 1$:

- differential equation $Ly(\mathbf{x}) = 0 \quad (L \in \mathbb{Z}[\frac{\partial}{\partial \mathbf{x}}, \mathbf{x}])$ has form

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) = 0 \tag{3}$$

- difference equation $Ly(\mathbf{x}) = 0 \quad (L \in \mathbb{Z}[\Delta, \mathbf{x}])$ has form

$$a_n(x)y(x+n) + a_{n-1}(x)y(x+n-1) + \cdots + a_1(x)y(x+1) + a_0(x)y(x) = 0 \tag{4}$$

For such equations there are known algorithms

- for searching polynomial solutions
- for searching rational function solutions

Introduction

In case when $m \geqslant 2$ there no such universal algorithms.

Researches on algorithmic aspects of searching rational function solutions (universal denominator) for partial differential equations:

- M. Kauers, C. Schneider. *Partial Denominator Bounds for Partial Linear Difference Equations* (2010)
- M. Kauers, C. Schneider. *A Refined Denominator Bounding Algorithm for Multivariate Linear Difference Equations* (2011)

Algorithmic undecidability of testing existence of polynomial and formal power series solutions for partial differential (and difference) equations:

- J. Denef, L. Lipshitz. *Power series solutions of algebraic differential equations* (1984)
- S. Abramov, M. Petkovšek. *On polynomial solutions of linear partial differential and (q-)difference equations* (2012)

Introduction

$$\delta_i = \begin{cases} x_i \frac{\partial}{\partial x_i}, & \text{in differential case,} \\ x_i \Delta_i, & \text{in difference case,} \end{cases} \tag{5}$$

$$\delta^n = \delta_1^{n_1} \dots \delta_m^{n_m} \quad \text{when } n \in \mathbb{Z}_{\geqslant 0}^m.$$

$$\mathbf{x}^{\langle n \rangle} = \begin{cases} \mathbf{x}^n, & \text{in differential case,} \\ \mathbf{x}^{\overline{n}}, & \text{in difference case,} \end{cases} \tag{6}$$

$$n \in \mathbb{Z}^m, \qquad \mathbf{x}^n = x_1^{n_1} \dots x_m^{n_m}, \qquad \mathbf{x}^{\overline{n}} = x_1^{\overline{n_1}} \dots x_m^{\overline{n_m}}$$

$$x^{\overline{k}} = \begin{cases} x(x+1)\dots(x+k-1), & \text{if } k > 0, \\ 1, & \text{if } k = 0, \\ \frac{1}{(x-1)(x-2)\dots(x-|k|)}, & \text{if } k < 0. \end{cases} \tag{7}$$

Introduction

$$\delta_i \mathbf{x}^{\langle n \rangle} = n_i \mathbf{x}^{\langle n \rangle}, \qquad n \in \mathbb{Z}^m \tag{8}$$

$$\delta^k \mathbf{x}^{\langle n \rangle} = n^k \mathbf{x}^{\langle n \rangle}, \qquad n, k \in \mathbb{Z}^m, \tag{9}$$

$$P(\delta_1, \ldots, \delta_m) \mathbf{x}^{\langle n \rangle} = P(n_1, \ldots, n_m) \mathbf{x}^{\langle n \rangle}, \qquad P \in \mathbb{Z}[x_1, \ldots, x_m]. \tag{10}$$

Equality (10) implies that differential equation

$$P(\delta_1, \ldots, \delta_m) \mathbf{x}^{\langle n \rangle} = 0, \qquad P \in \mathbb{Z}[x_1, \ldots, x_m] \tag{11}$$

has solution $n \in \mathbb{Z}^m$ if and only if n is solution of Diophantine equation

$$P(n_1, \ldots, n_m) = 0. \tag{12}$$

Theorem (Matiyasevich, 1970)

There is no such algorithm that determines for an arbitrary Diophantine equation whether it has integer solution.

Rational functions

$$o(\mathbf{x}^n) = \sum_{s \prec n,\ s \in \mathbb{Z}^m} c_s \mathbf{x}^s, \qquad n \in \mathbb{Z}^m, \qquad " \prec " \text{ is lexicographical order}$$

Lemma

If rational function $\frac{\mathbf{x}^n + o(\mathbf{x}^n)}{\mathbf{x}^d + o(\mathbf{x}^d)}$ is solution of equation

$$P(\delta_1, \ldots, \delta_m) y(\mathbf{x}) = 0, \qquad P \in \mathbb{Z}[x_1, \ldots, x_m], \tag{13}$$

then $\mathbf{x}^{\langle n-d \rangle}$ is also solution of this equation.

Theorem

There is no such algorithm that determines for an arbitrary linear differential (or difference) homogeneous equation whether it has rational function solution.

Formal Laurent Series

In case of single variable x formal Laurent series with coefficients from field $\mathbb{K}$ have the form

$$\sum_{n=z}^{\infty} a_n x^n, \quad z \in \mathbb{Z}, \quad a_n \in \mathbb{K} \tag{14}$$

Set of them is field $\mathbb{K}((x))$ that is the quotient field of the ring of formal power series $\mathbb{K}[[x]]$.

In case of several variables $x_1, ..., x_m$ there are various approaches for defining Laurent series. One of them is simple recursive:

$$\mathbb{K}((x_1))((x_2)) \ldots ((x_m)) \tag{15}$$

Formal Laurent Series

Alternative approach for defining formal Laurent series
A. Aparicio Monforte, M. Kauers. *Formal Laurent series in several variables* (2013)

C is rational line-free cone in $\mathbb{R}^m$:

$$C = \{c_1 v_1 + \cdots + c_n v_n \mid c_1, \ldots, c_n \in \mathbb{R}^m_{\geqslant 0}\}$$
$$v_1, \ldots, v_n \in \mathbb{Z}^m, \qquad \forall u \in C \setminus \{0\} : -u \notin C$$

$\preceq$ is total order on $\mathbb{Z}^m$ compatible with C:

$$\forall i, j, k \in \mathbb{Z}^m : i \preceq j \Rightarrow i + k \preceq j + k, \qquad \forall u \in C \cap \mathbb{Z}^m : 0 \preceq u$$

$$\mathbb{K}_C[[x]] = \{\sum_{n \in C \cap \mathbb{Z}^m} a_n x^n \mid a_n \in \mathbb{K}\},$$

$$\mathbb{K}_\preceq[[x]] = \bigcup_{C \in \Upsilon} \mathbb{K}_C[[x]], \qquad \mathbb{K}_\preceq((x)) = \bigcup_{\gamma \in \mathbb{Z}^m} x^\gamma \mathbb{K}_\preceq[[x]]$$

Formal Laurent Series

Second approach of Laurent series definition is more flexible in some sense: if $m > 1$ there are infinite number of possible orders $\preceq$ and only $m!$ possible permutations of variables.

Example

- Let $f(x,y) = \sum_{n=0}^{\infty} (x^{2n}y^{-n} + x^{-n}y^{2n})$, then $f(x,y) \in \mathbb{K}_{\preceq}((x,y))$, but $f(x,y) \notin \mathbb{K}((x))((y))$
- Let $f(x,y) = \sum_{n=0}^{\infty} (x^{n^2}y^{n} + x^{-n^2}y^{n})$, then $f(x,y) \in \mathbb{K}((x))((y))$, but $f(x,y) \notin \mathbb{K}_{\preceq}((x,y))$

Formal Laurent Series

To generalize two considered approaches, we define the field of formal Laurent series for variables $x_1, x_2, \ldots, x_m$ over the field $\mathbb{K}$ as such field Λ that

$$\mathbb{K}[[\mathbf{x}^{\pm 1}]] \subset \Lambda \subset \mathbb{K}[[\mathbf{x}, \mathbf{x}^{-1}]], \tag{16}$$

$\mathbb{K}[[\mathbf{x}^{\pm 1}]] = \mathbb{K}[[x_1^{d_1}, \ldots, x_m^{d_m}]]$ where $d_i = \pm 1$,
$\mathbb{K}[[\mathbf{x}, \mathbf{x}^{-1}]] = \mathbb{K}[[x_1, ..., x_m, x_1^{-1}, ..., x_m^{-1}]]$ is the ring of all formal sums $\sum\limits_{n \in \mathbb{Z}^m} a_n \mathbf{x}^n$.

Proposal

Field $\Lambda = \mathbb{K}_{\preceq}((\mathbf{x}))$ satisfies condition (16) for some $d_i = \pm 1,\ i = 1, 2, \ldots, m$ for any additive order $\preceq$ in $\mathbb{Z}^m$.

Theorem

If Λ is some field of formal Laurent series for m variables $(m \geqslant 11)$ then the problem of testing the existence of solution in Λ for an arbitrary differential equation $Ly(\mathbf{x}) = 0,\ L \in \mathbb{Z}[\frac{\partial}{\partial \mathbf{x}}, \mathbf{x}]$ is algorithmically undecidable.

Equations with boundary conditions

Let $\mathbb{K}$ be the field of complex numbers $\mathbb{C}$ or field of real numbers $\mathbb{R}$.

Let $f(\mathbf{x})$ be some function defined in U.
$\overline{f}(\mathbf{x})$ is $f(\mathbf{x})$ extended with zero to boundary of U:

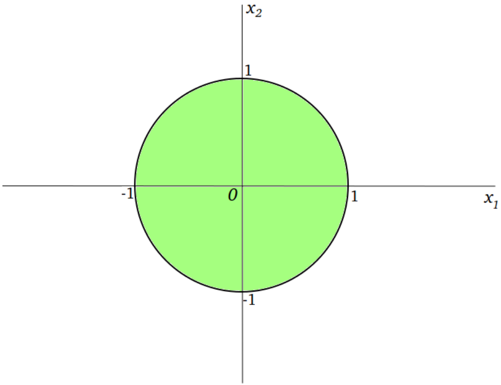
$$\overline{f}(\mathbf{x}) = \begin{cases} f(\mathbf{x}), & \text{if } (\mathbf{x}) \in U \\ 0, & \text{if } \mathbf{x} \in \overline{U} \backslash U \end{cases} \tag{17}$$

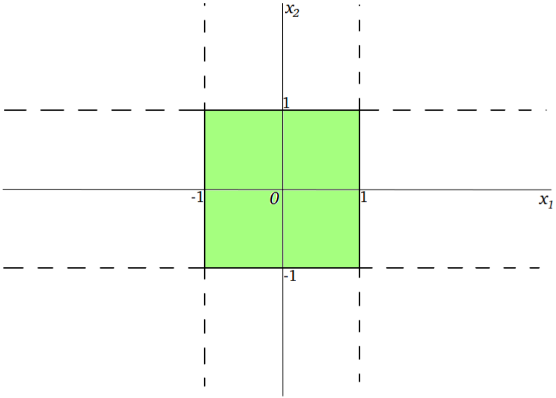
$f_\alpha(\mathbf{x})$ is partial derivative of $f(\mathbf{x})$

$$f_\alpha(\mathbf{x}) = \frac{\partial^{|\alpha|} f(\mathbf{x})}{\partial x_1^{\alpha_1} \ldots \partial x_m^{\alpha_m}}, \qquad \alpha \in \mathbb{Z}^m_{\geqslant 0}, \qquad |\alpha| = \alpha_1 + \ldots + \alpha_m \tag{18}$$

Further we will consider polynomials $q(\mathbf{x}) \in \mathbb{Z}[x_1, \ldots, x_m]$ *compatible with set* U that will mean that $q(\mathbf{x})$ has zero values on the boundary of U.

Examples





$q(x_1, x_2) = x_1^2 + x_2^2 - 1$

$q(x_1, x_2) = (x_1 + 1)(x_2 + 1)(x_1 - 1)(x_2 - 1)$

Equations with boundary conditions

Problem ZC (zero condition)

For the following input:

- non-empty finite set $A \subset \mathbb{Z}^m_{\geqslant 0}$,
- open set $U \in \mathbb{K}^m$ and polynomial $q(\mathbf{x}) \in \mathbb{Z}[x_1, ..., x_m]$ compatible with this set,
- differential operator $L \in \mathbb{Z}[\frac{\partial}{\partial \mathbf{x}}, x]$,

to determine whether differential equation $L(f) = 0$ has such non-zero solution $f(\mathbf{x})$ that
(a) function $f(\mathbf{x})$ is analytic in U,
(b) for any $\alpha \in A$ function $\overline{f_\alpha}(\mathbf{x})$ is continuous in $\overline{U}$.

Example

- $A = \{(0, \dots, 0)\}$
- U is ball in $\mathbb{R}^m$ with radius 1 and center in the origin
- $q(\mathbf{x}) = x_1^2 + \dots + x_m^2 - 1$
- $L = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_m^2}$

We get the following problem

$$\frac{\partial^2 y(\mathbf{x})}{\partial x_1^2} + \dots + \frac{\partial^2 y(\mathbf{x})}{\partial x_m^2} = 0, \tag{19}$$

$$y(\mathbf{x})|_{x_1^2 + \dots + x_m^2 = 1} = 0,$$

which has no non-zero solution.

Example

- $A = \{(0, \ldots, 0)\}$
- U is ball in $\mathbb{R}^m$ with radius 1 and center in the origin
- $q(\mathbf{x}) = x_1^2 + \cdots + x_m^2 - 1$
- $L = \frac{\partial^3}{\partial x_1^3} + \cdots + \frac{\partial^3}{\partial x_m^3}$

We get the following problem

$$\frac{\partial^3 y(\mathbf{x})}{\partial x_1^3} + \cdots + \frac{\partial^3 y(\mathbf{x})}{\partial x_m^3} = 0, \qquad (20)$$

$$y(\mathbf{x})|_{x_1^2 + \cdots + x_m^2 = 1} = 0,$$

which has solution $y(\mathbf{x}) = x_1^2 + \cdots + x_m^2 - 1$.

Problem ZCD (Zero Condition for infinitely Differentiable solutions)

For the following input:

- non-empty finite set $A \subset \mathbb{Z}^m_{\geqslant 0}$,
- open bounded set $U \in \mathbb{R}^m$ and polynomial $q(\mathbf{x}) \in \mathbb{Z}[x_1, ..., x_m]$ compatible with this set,
- differential equation

$$Ly = b(\mathbf{x}), \quad L \in \mathbb{Z}[\frac{\partial}{\partial x}, x], \quad b(\mathbf{x}) \in \mathbb{Z}[\mathbf{x}], \tag{21}$$

to determine whether differential equation $L(f) = 0$ has such solution $f(\mathbf{x})$ that
(a) function $f(\mathbf{x})$ is infinitely differentiable in U,
(b) for any $\alpha \in A$ function $\overline{f_\alpha}(\mathbf{x})$ is continuous in $\overline{U}$.

Equations with boundary conditions

Theorem

Problems ZC and ZCD are algorithmically undecidable.

Remark. Proof of undecidability of problems ZC and ZCD is based on the bijection between all Diophantine equations and some special subset of differential equations: Diophantine equation has integer solution if and only if corresponding differential equation has solution. It implies that problems ZC and ZCD are undecidable even for fixed sets U and A, i.e. if the input of an algorithm is differential equation only (but it is supposed than m is big enough in this case).

Finitely differentiable

Proof of undecidability of problem ZCD use the fact that Diophantine equation $C(n_1, \dots, n_m) = 0$ has integr non-negative solution if and only if differential equation

$$C\left(x_1 \frac{\partial}{\partial x_1}, \dots, x_m \frac{\partial}{\partial x_m}\right) y(\mathbf{x}) = b(x_1, \dots x_m), \tag{22}$$

$$b(x_1, \dots x_m) = \sum_{n \in \mathbb{Z}^m_{\geqslant 0}} a_n x_1^{n_1} \dots x_m^{n_m}, \qquad a_n \neq 0,$$

doesn't have infinitely differentiable solution in the domain containing 0.

This is based on the fact that equation

$$P\left(x_1 \frac{\partial}{\partial x_1}, \dots, x_m \frac{\partial}{\partial x_m}\right) y(\mathbf{x}) = x^n, \qquad P \in \mathbb{Z}[x_1, \dots, x_m], \qquad n \in \mathbb{Z}^m \tag{23}$$

has infinitely differentiable solution if and only if $P(n_1, \dots, n_m) \neq 0$.

Finitely differentiable

But if we consider finitely differentiable solutions then similar statements appears to be false even in the univariate case.

Example

Let $P(n) = n - k$ where $k \geqslant 2$.
Corresponding differential equation $xy'(x) - ky(x) = x^k$ has $k-1$ times differentiable solution

$$y(x) = \begin{cases} x^k \ln |x|, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \tag{24}$$

Finitely differentiable

Example

Let

- $k \geqslant 2$ be some integer number,
- $b(x) = \sum\limits_{n \geqslant 0} a_n x^n$ be power series that is convergent in $(-t, t)$.

Diophantine equation $u - k = 0$ has solution $u = k$.
Corresponding differential equation $xy'(x) - ky(x) = b(x)$ also has solution

$$y(x) = \begin{cases} a_k x^k \ln|x| + \sum\limits_{n \geqslant 0,\ n \neq k} \frac{a_n}{n-k} x^n, & \text{if } x \neq 0, \\ -\frac{a_0}{k}, & \text{if } x = 0, \end{cases} \tag{25}$$

that is differentiable $k-1$ times in $(-t, t)$.

Thank you for your attention!