

Computing the Lie algebra of the differential Galois group of a linear differential system (1/2)

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Functional Equations in LIMoges 2016

Motivation

◇ $k = \mathbb{C}(z)$ (for actual computations \mathbb{C} is replaced by a computable subfield of $\overline{\mathbb{Q}}$), $A \in \mathbb{M}_n(k)$, \mathbf{y} vector of unknown functions, $' = \frac{d}{dz}$

Linear differential system $[A] : \mathbf{y}' = A\mathbf{y}$

- ◇ Important object for studying $[A]$: its **differential Galois group** G
→ G measures everything that algebra can see about the solutions
- ◇ **Direct problem** in diff. Galois theory: **given $[A]$, compute G**
 - Many theoretical algorithms: *Compoint-Singer'99, Hrushovski'02 and Feng'15, van der Put-Singer'03, van der Hoeven'07*
 - None of them are either practical or implemented

Objective

Philosophy of our work (see also *Nguyen-van der Put'10*)

→ For a large class of problems, **computing the Lie algebra \mathfrak{g} of the linear algebraic group G** is enough

◇ For the computation of \mathfrak{g} , not much is known (*Aparicio's PhD thesis'10*, *Aparicio-Compoint-Weil'13*)

Goal of this talk AND the next one by T. Dreyfus

Provide a full algorithm for computing the Lie algebra \mathfrak{g} of G

- This talk (1/2): irreducible and completely reducible systems
- Next talk (2/2): reducible systems

I

Differential systems/modules/Galois group and Lie algebra

Differential modules

◇ A **differential module** \mathcal{M} over k is a finite dimensional vector space over k equipped with an additive map $\partial : \mathcal{M} \rightarrow \mathcal{M}$ s.t. $\forall f \in k, \forall m \in \mathcal{M}, \partial(f m) = f' m + f \partial(m)$

A **differential submodule** of \mathcal{M} is then a sub-vector space of \mathcal{M} which is stable under the action of ∂

◇ A differential module \mathcal{M} is

- **irreducible** if it has no non-trivial differential submodule
- **absolutely irreducible** if $\bar{k} \otimes_k \mathcal{M}$ is irreducible
- **decomposable** if $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$
- **completely reducible** if it is a direct sum of *irreducible* modules

◇ *Krull-Schmidt*: $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \cdots \oplus \mathcal{M}_r$, with \mathcal{M}_i indecomposable. It is called a **maximal decomposition** of \mathcal{M} (**Rk**: if \mathcal{M} is completely reducible, then the \mathcal{M}_i are irreducible)

Differential module \leftrightarrow Differential system

◇ Via a choice of basis, a differential module \mathcal{M} is associated with a linear differential system $[A]$ and vice versa

◇ Change of basis in $\mathcal{M} \leftrightarrow$ gauge transfo. $P \in GL_n(k)$ in $[A]$ leading to equivalent system $[P[A]]$ with $P[A] \triangleq P^{-1}(AP - P')$

$\rightarrow \mathcal{M}$ reducible: $\exists P, P[A] = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}$ block triangular

$\rightarrow \mathcal{M}$ decomposable: $\exists P, P[A] = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ block diagonal

◇ $\mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_r \leftrightarrow \exists P, P[A] = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_r \end{pmatrix}$

Maximal dec. $\leftrightarrow [A_i]$ indec. (**Rk**: irred. if \mathcal{M} is completely red.)

The differential module $\mathcal{M} \otimes_k \mathcal{M}^*$

- ◇ \mathcal{M} differential module, $[A]$ associated differential system
- ◇ Its dual $\mathcal{M}^* \triangleq \text{Hom}_k(\mathcal{M}, \mathbb{1}_k)$ is associated with $[-A^T]$
- ◇ Consider $\mathcal{M} \otimes_k \mathcal{M}^*$: elements viewed in $\mathbb{M}_n(k)$
- $\mathcal{M} \otimes_k \mathcal{M}^*$ is associated with the matrix differential system

$$F' = [A, F] \triangleq AF - FA$$

Using the classical Kronecker product of matrices:

$$F' = [A, F] \iff \text{Vect}(F)' = \left(A \otimes I_n - I_n \otimes A^T \right) \text{Vect}(F),$$

with $\text{Vect}(F) = (F_{1\bullet}^T \dots F_{n\bullet}^T)^T \in k^{n^2}$ and $F_{i\bullet}$ the i -th row of F

- ◇ \mathcal{M} completely reducible $\Rightarrow \mathcal{M} \otimes_k \mathcal{M}^*$ is completely reducible

The differential Galois group

- ◇ \mathcal{M} differential module associated with a differential system $[A]$
- ◇ K Picard-Vessiot extension for \mathcal{M} : diff. field ext. of k
→ $[A]$ admits a fundamental matrix of solutions $U \in \mathrm{GL}_n(K)$
- ◇ The differential Galois group G of \mathcal{M} is the group $\mathrm{Aut}_\partial(K/k)$ of differential k -algebra automorphisms of K :
 $\forall g \in G, \forall f \in K, \quad g(f') = g(f)'$, $f \in k \Rightarrow g(f) = f$

- ◇ G viewed as a subgroup of $\mathrm{GL}_n(\mathbb{C})$ is a linear algebraic group:

There exists a polynomial ideal $\mathcal{I} \subset \mathbb{C}[X_{1,1}, X_{1,2}, \dots, X_{n,n}, \det^{-1}]$, where \det^{-1} is the inverse of $\det((X_{i,j})_{i,j})$, such that

$$G \cong \{M = (m_{i,j})_{i,j} \in \mathrm{GL}_n(\mathbb{C}) \mid \forall P \in \mathcal{I}, P(m_{i,j}) = 0\}$$

The Lie algebra \mathfrak{g} of G

- ◇ The Lie algebra \mathfrak{g} of G is the tangent space of G at the point $\text{id} \in G$: \mathfrak{g} can be represented as a Lie sub-algebra of $\mathfrak{gl}_n(\mathbb{C})$

$$\mathfrak{g} \cong \{N \in \mathbb{M}_n(\mathbb{C}) \mid I_n + \epsilon N \in G(\mathbb{C}[\epsilon]) \text{ with } \epsilon \neq 0 \text{ and } \epsilon^2 = 0\},$$

where $G(\mathbb{C}[\epsilon])$ set of $\mathbb{C}[\epsilon]$ -points of G

- ◇ Adjoint action of G on \mathfrak{g} : $G \times \mathfrak{g} \rightarrow \mathfrak{g}$, $(g, h) \mapsto g h g^{-1}$
- ◇ $V \triangleq \mathbb{C}$ -vector space of solutions of $[A]$ in K^n , $\text{End}(V)$ endowed with a Lie algebra structure $\mathfrak{gl}(V)$ identified with $\mathfrak{gl}_n(\mathbb{C})$
 \Rightarrow We have a representation of \mathfrak{g} in $\text{End}(V)$
- ◇ Using $\text{End}(V) \cong V \otimes V^*$, \mathfrak{g} can then be viewed as a sub-vector space of $V \otimes V^*$ stable under the adjoint action of G

Tannakian correspondence and characterization of \mathfrak{g}

◇ **Tannakian correspondence**: 1-1 correspondence (compatible with all constructions of linear algebra) between sub-vector spaces of V stable under the action of G and differential submodules of \mathcal{M}

→ The representation of \mathfrak{g} in $\text{End}(V)$ corresponds to the differential submodule $\mathfrak{g}^s \triangleq (K \otimes_{\mathbb{C}} \mathfrak{g})^G$ of $\mathcal{M} \otimes_k \mathcal{M}^*$

(**Rk**: \mathfrak{g}^s is the Lie algebra considered by *Katz* in his works)

→ \mathfrak{g}^s (and thus \mathfrak{g}) can be investigated by **studying differential submodules of $\mathcal{M} \otimes_k \mathcal{M}^*$** which can all be obtained from a maximal decomposition if \mathcal{M} is completely reducible

Sketch of our algorithm

1. Compute a **maximal decomposition** of $\mathcal{M} \otimes_k \mathcal{M}^*$
(tools: *eigenring techniques & use specific structure*)
 2. Find a **candidate** for \mathfrak{g}^s
(tools: *modular approach based on Grothendieck-Katz p -curvature conjecture*)
 3. **Validation** of the candidate
(tools: *reduced form & conjugation between Lie algebras*)
- ◇ **For the ease of presentation**, in the following, we assume $\mathcal{M}/[A]$ **absolutely irreducible** (it can be checked: *Compoint-Weil'04*)
- **Completely reducible case quite similar** (only small modif.)
- Reducible case: see next talk by T. Dreyfus

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Maximal decomposition of $\mathcal{M} \otimes_k \mathcal{M}^*$

Maximal decomposition: general method

Problem: given $[A]$, find $P \in GL_n(k)$ s.t. $P[A]$ block diagonal

- ◇ Already studied in computer algebra: *Singer'96, Barkatou'07*
→ Compute the **eigenring** (rational solutions - *Barkatou'99*)

$$\mathcal{E}(A) \triangleq \{F \in \mathbb{M}_n(k) \mid F' = [A, F] = AF - FA\}$$

- ◇ If $F \in \mathcal{E}(A)$, $P^{-1}FP = \text{diag}(F_1, \dots, F_r)$ (F_i constant matrices with distinct eigenvalues), then $P[A] = \text{diag}(A_1, \dots, A_r)$
- ◇ This corresponds to $\mathcal{M} = \mathcal{M}_1 \oplus \dots \oplus \mathcal{M}_r$, where the **bases of the submodules \mathcal{M}_i** are given by the columns of P
- ◇ Maximal dec. given by a random element of $\mathcal{E}(A)$ (*Barkatou'07*)

Maximal decomposition of $\mathcal{M} \otimes \mathcal{M}^*$: specific methods (1)

- ◇ We can apply the **previous method** to $\mathcal{A} \triangleq A \otimes I_n - I_n \otimes A^T$
 - Computing $\mathcal{E}(\mathcal{A})$: rational solutions of $\mathcal{A} \otimes I_{n^2} - I_{n^2} \otimes \mathcal{A}^T$ of size n^4 ! *BarkatouCluzeauElBachaWeil'12* → $O(n^{20})$ arithm. op.
 - We should take into account the **specific form of \mathcal{A}**
- ◇ **Problem**: compute rational solutions of $\overline{\mathcal{A}} \triangleq \mathcal{A} \otimes I_{n^2} - I_{n^2} \otimes \mathcal{A}^T$
- ◇ **First approach**: algorithm in *Barkatou'99* proceeds in two steps:
 1. Local data at each singularity → universal denominator
 2. Polynomial solutions of an auxiliary system

Adapt ideas of *BarkatouPfluegel'98* ⇒ **local datas needed for rational solutions of $\overline{\mathcal{A}}$ (size n^4) can be computed from $[A]$ (size n)**

(Rk: the second step can also be accelerated)

Maximal decomposition of $\mathcal{M} \otimes \mathcal{M}^*$: specific methods (2)

◇ **Second approach**: use structural decompositions

$[\bar{\mathcal{A}}]$ associated with $\text{End}(\text{End}(\mathcal{M})) \triangleq (\mathcal{M} \otimes \mathcal{M}^*) \otimes (\mathcal{M} \otimes \mathcal{M}^*)^*$

→ Decompose $\text{End}(\text{End}(\mathcal{M}))$ and rational sol. of smaller systems

Theorem 1 (BCdVW'16): we have $\mathcal{M} \otimes \mathcal{M}^* = \mathbb{1}_k \oplus \mathcal{W}$ and the explicit iso.

$$\text{End}(\text{End}(\mathcal{M})) \cong \underbrace{\mathbb{1}_k \oplus \mathcal{W} \oplus \text{Sym}^2(\mathcal{W})}_{\text{Sym}^2(\mathbb{1}_k \oplus \mathcal{W})} \oplus \underbrace{\mathcal{W} \oplus \Lambda^2(\mathcal{W})}_{\Lambda^2(\mathbb{1}_k \oplus \mathcal{W})}$$

Theorem 2 (BCdVW'16): we have the explicit iso.

$$\text{End}(\text{End}(\mathcal{M})) \cong \underbrace{\mathbb{1}_k \oplus \mathcal{N}_{S^2}}_{\text{End}(S^2)} \oplus \underbrace{\mathbb{1}_k \oplus \mathcal{N}_{\Lambda^2}}_{\text{End}(\Lambda^2)} \oplus \text{Hom}(S^2, \Lambda^2) \oplus \text{Hom}(\Lambda^2, S^2)$$

→ Rational solutions of systems of smaller size **still having specific structures** (Sym^2 , Λ^2 , Hom) that can be used for rational solutions (*AparicioBarkatouSimonWeil'11, BarkatouPfluegel'98*)

III

Candidate for g^s

Candidate for \mathfrak{g}^S : reduction modulo p

Problem: In the max. dec. $\mathcal{M} \otimes_k \mathcal{M}^* = \bigoplus_{i=1}^r \mathcal{W}_i$, find \mathfrak{g}^S

→ **Idea:** use a **modular approach** to find a candidate for \mathfrak{g}^S

◇ Crucial object for studying diff. systems $[A_p]$ / modules $(\mathcal{M}_p, \partial)$ in characteristic $p > 0$: **the p -curvature $\chi_p \triangleq \partial^p$** acting on \mathcal{M}_p

◇ In terms of matrices: χ_p is given by the p th iterate of the sequence $\chi_1 = A_p$ and, for $i > 1$, $\chi_{i+1} = \chi_i' - A_p \chi_i$

→ **Algorithms:** *Katz'82, van der Put'95-96, Cluzeau'03* and recently *BostanCarusoSchost'15* for a fast algorithm

Grothendieck-Katz p -curvature conjecture: The Lie algebra \mathfrak{g}^S is the smallest (algebraic) Lie sub-algebra of $\mathfrak{gl}_n(k)$ whose reduction modulo p contains the p -curvature χ_p for almost all p .

Candidate for \mathfrak{g}^S : algorithm MODULARSELECTION

◇ $\mathcal{M} \otimes_k \mathcal{M}^* = \bigoplus_{i=1}^r \mathcal{W}_i$ given by gauge transfo. $T \in \mathrm{GL}_{n^2}(k)$
(the columns $T_{\bullet j}$ of T provide bases of the submodules \mathcal{W}_i)

1. Choose a prime $p \rightarrow \bigoplus_{i=1}^r \mathcal{W}_{i,p}$ given by $T_p = T \pmod p$;
2. Compute the p -curvature χ_p of $[A_p]$;
3. Compute $V = T_p^{-1} \mathrm{Vect}(\chi_p)$;
4. From the non-zero entries of V , deduce a basis of the submodule of $\bigoplus_{i=1}^r \mathcal{W}_i$ whose reduction mod p contains χ_p .

◇ From G-K conjecture, the submodule found can be used as a reasonable guess for \mathfrak{g}^S

◇ **Remark:** this may select a bigger or smaller submodule

→ we need to check whether our guess is correct or not

IV

Validation of the candidate

Reduced form of a linear differential system

◇ **Definition:** $[A]$ in **reduced form** if $A \in \bar{k} \otimes \mathfrak{g}$.

→ \mathfrak{g} viewed as a \mathbb{C} -vector space generated by $N_1, \dots, N_d \in \mathbb{M}_n(\mathbb{C})$

→ reduced form iff $\exists f_1, \dots, f_d \in \bar{k}$ s.t. $A = f_1 N_1 + \dots + f_d N_d$

Theorem (Kolchin-Kovacic): There exists a **reduction matrix** $P \in GL_n(\bar{k})$ such that $[P[A]]$ is in reduced form.

◇ Reduced forms \Rightarrow **invariants** (\approx rational sol. of “constructions”) have **constant** coefficients in \mathbb{C} : *Aparicio-Compoin-Weil'13*

Theorem (Aparicio-Compoin-Weil'13): For all ordinary point $z_0 \in \mathbb{C}$ of $[A]$, there exists a reduction matrix $P \in GL_n(\bar{k})$ for $[A]$ that sends every invariant \mathbf{f} of $[A]$ to its evaluation at z_0 .

A Lie algebra conjugation problem

◇ **Definition:** Two Lie sub-algebras $\mathfrak{g}_1, \mathfrak{g}_2 \subset \mathfrak{gl}_n(k)$ are **conjugated** if \exists a **conjugation matrix** $P \in \text{GL}_n(\bar{k})$ s.t. $\mathfrak{g}_2 = P^{-1} \mathfrak{g}_1 P$.

Theorem (BCdVW'16):

- M_i ($i = 1, \dots, d$) basis of **candidate Lie algebra** \mathfrak{g}^s ,
- z_0 ordinary point of $[A]$,
- \mathfrak{g}^t Lie sub-algebra of $\mathfrak{gl}_n(\mathbb{C})$ with basis $M_i(z_0)$ ($i = 1, \dots, d$).

Then, there exists a reduction matrix $P \in \text{GL}_n(\bar{k})$ for $[A]$ that is a conjugation matrix between the Lie algebra \mathfrak{g}^s and \mathfrak{g}^t .

→ A reduction matrix can be found among the conjugation matrices between \mathfrak{g}^t and \mathfrak{g}^s

Semi-simple Lie algebras

◇ \mathcal{M} absolutely irreducible $\Rightarrow \mathfrak{g}^t$ and \mathfrak{g}^s semi-simple Lie algebras

◇ Central objects in the study of a semi-simple Lie algebra \mathfrak{g} : set of canonical generators (and Chevalley bases)

→ Matrices $H_1, \dots, H_r, X_1, \dots, X_r, Y_1, \dots, Y_r$ which satisfies:

$$[H_i, H_j] = 0, [X_i, Y_j] = \delta_{i,j} H_i, [H_i, X_j] = c_{j,i} X_j, [H_i, Y_j] = -c_{j,i} Y_j$$

- This is associated with a root space decomposition of \mathfrak{g}
- H_1, \dots, H_r are generators of a Cartan sub-algebra of \mathfrak{g}
- $C = (c_{i,j})_{1 \leq i,j \leq r}$ is a Cartan matrix of \mathfrak{g} ($c_{i,i} = 2$)

→ Algorithms for computing set of canonical generators and Chevalley bases: *deGraaf'00*

Algorithm CONJUGATIONMATRICES

Input: $\{M_i\}_i$ basis of \mathfrak{g}^s , $\{M_i(z_0)\}$ basis of \mathfrak{g}^t

Output: Conjugation matrices P between \mathfrak{g}^t and \mathfrak{g}^s

1. Compute a set of canonical generators $\{H_i^t, X_i^t, Y_i^t\}$ of \mathfrak{g}^t ;
2. Compute generators \tilde{H}_i^s of a “split” Cartan sub-algebra \mathfrak{h}^s of \mathfrak{g}^s s.t. $\chi(\tilde{H}_i^s) = \chi(H_i^t)$ (ansatz \rightarrow solving algebraic equations)
3. Compute a set of canonical generators $\{H_i^s, X_i^s, Y_i^s\}$ of \mathfrak{g}^s having the same Cartan matrix as $\{H_i^t, X_i^t, Y_i^t\}$;
4. Compute the matrices $P \in \mathrm{GL}_n(\bar{k})$ such that $\forall i, P X_i^t = X_i^s P$ and $P Y_i^t = Y_i^s P$ (overdetermined linear system).

Theorem (BCdVW'16): If our choice for \mathfrak{g}^s is correct, then output of the form $P = c \tilde{P}$, with $\tilde{P} \in \mathrm{GL}_n(\bar{k})$, c arbitrary element of \bar{k}

Algorithm REDUCTIONMATRIX

- Let $P = c \tilde{P}$ and $(N_i^t)_{i=1, \dots, d}$ be a Chevalley basis of \mathfrak{g}^t
- $\exists f_i \in \bar{k}$ such that $P[A] = \sum_{i=1}^d f_i N_i^t$ implies:

$$\tilde{P}^{-1} A \tilde{P} - \frac{c'}{c} I_n - \tilde{P}^{-1} \tilde{P}' = \sum_{i=1}^d f_i N_i^t \quad (1)$$

$$\frac{c'}{c} = \frac{1}{n} \left(\text{Tr}(A) - \frac{\det(\tilde{P}')}{\det(\tilde{P})} - \sum_{i=1}^d f_i \text{Tr}(N_i^t) \right) \quad (2)$$

- Plug (2) into (1) and solve the **linear system for the $f_i \in \bar{k}$** .
If the system has no solution, then Return "Fail".
- Plug the solution found into (2) and solve the **scalar order one linear differential equation for c** .
If algebraic solution, then Return $P = c \tilde{P}$, Else Return "Fail".

IV

Full algorithm and example

Full algorithm and remarks

1. Compute a **maximal decomposition of $\mathcal{M} \otimes \mathcal{M}^*$** ;
2. Apply **MODULARSELECTION** to get a candidate for \mathfrak{g}^S ;
3. Apply **CONJUGATIONMATRICES**;
If it fails, go back to Step 2 and choose another prime p
4. Compute a **Chevalley basis $(N_i^t)_i$ of \mathfrak{g}^t** ;
5. Apply **REDUCTIONMATRIX**.
If it fails, go back to Step 2 and choose another prime p ,
Else Return $(N_i^t)_i$.

◇ **Remarks on the successive choices of p :**

- There may exist an infinite number of “bad” primes: good strategy for choosing prime numbers \rightsquigarrow deterministic algo.
- if $\mathcal{W}_1, \mathcal{W}_2$ are proved not correct: try $\mathcal{W}_1 + \mathcal{W}_2$ before new p
- If candidate decomposable: check each submodule

Remark on complexity / Implementation

- ◇ Arithmetic complexity **polynomial in n** except algebraic systems solved in CONJUGATIONMATRICES
→ Significant diff. compared to the exponential (several levels) complexity obtained in *Feng'15* for computing the Galois group
- ◇ We have a **prototype Maple implementation**
- ◇ We manage to apply it to many examples **up to order $n = 7$**
- ◇ **In practice, the most costly step is the dec. of $\mathcal{M} \otimes \mathcal{M}^*$**

Example (1)

- ◇ Consider the system given by

$$A := \begin{bmatrix} \frac{x-1}{x} & x & -1 \\ -x^3 + 1 & 0 & -1 \\ \frac{x-1}{x} + x^2 & x + 1 & -1 \end{bmatrix}$$

- ◇ $\mathcal{M} \otimes \mathcal{M}^* = \mathbb{1}_k \oplus \mathcal{W}_1 \oplus \mathcal{W}_2$, with $\mathcal{W}_1, \mathcal{W}_2$ of resp. dim. 3 and 5
- ◇ p -curvature \rightarrow candidate for \mathfrak{g}^s is \mathcal{W}_1 irred with basis

$$M_1 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -x^2 - 1 & 0 & 1 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 1 & 0 \\ -x^2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, M_3 = \begin{bmatrix} 0 & 1 & 0 \\ -x^2 - 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Example (2)

◇ $x_0 = 1$ ordinary point for $[A]$: set of canonical gen. of g^t :

$$H^t = \begin{bmatrix} 2i & 0 & -2i \\ 0 & 0 & 0 \\ 4i & 0 & -2i \end{bmatrix}, X^t = \begin{bmatrix} 0 & -i & 0 \\ 1+i & 0 & -1 \\ 0 & 1-i & 0 \end{bmatrix}, Y^t = \begin{bmatrix} 0 & -i & 0 \\ -1+i & 0 & 1 \\ 0 & -1-i & 0 \end{bmatrix}.$$

◇ Computing an “aligned” set of canonical gen. of g^s , we get:

$$H^s = \begin{bmatrix} \frac{-2i}{x} & 0 & \frac{2i}{x} \\ 0 & 0 & 0 \\ \frac{-2i(x^2+1)}{x} & 0 & \frac{2i}{x} \end{bmatrix}, X^s = \begin{bmatrix} 0 & \frac{i}{x} & 0 \\ -ix+1 & 0 & -1 \\ 0 & \frac{i+x}{x} & 0 \end{bmatrix}, Y^s = \begin{bmatrix} 0 & \frac{i}{x} & 0 \\ -ix-1 & 0 & 1 \\ 0 & \frac{i-x}{x} & 0 \end{bmatrix}.$$

◇ Conjugation matrices P s.t. $X^t P = P X^s$ and $Y^t P = P Y^s$:

$$P = c \tilde{P}, \quad c \in \bar{k}, \quad \tilde{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -x & 0 \\ x+1 & 0 & -x \end{bmatrix}.$$

Example (3)

◇ Solving **linear system** obtained from $P[A] = f_1 H^t + f_2 X^t + f_3 Y^t$

$$\left\{ f_1 = \frac{i}{2x}, f_2 = -\frac{i}{2}(x^2 + i), f_3 = \frac{i}{2}(-x^2 + i) \right\}$$

◇ Solving the **scalar diff. equation for c** , we get $c = a/x$, $a \in \mathbb{C}^*$

→ **Reduction matrix P** and **reduced form R** given by:

$$P = \begin{bmatrix} \frac{a}{x} & 0 & 0 \\ 0 & -a & 0 \\ \frac{(x+1)a}{x} & 0 & -a \end{bmatrix}, \quad R = \begin{bmatrix} -x & -x^2 & x \\ x^2 + 1 & 0 & -1 \\ -2x & -x^2 + 1 & x \end{bmatrix}.$$

→ The Lie algebra \mathfrak{g} viewed as a Lie sub-algebra of $\mathfrak{gl}_3(\mathbb{C})$ admits the basis H^t, X^t, Y^t