Central Limit Theorems for Poisson Random Waves

Based on joint work with S. Bourguin, C. Durastanti and D. Marinucci

Anna Paola Todino

Sapienza Università di Roma

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Deterministic eigenfunctions

(M, g) closed smooth Riemannian surface
$\Delta_{(M,g)}$ Laplace-Beltrami operator on (M, g)

$$\Delta_{(M,g)} f + \lambda f = 0 \quad \text{on } M$$

Eigenvalues: $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$
Eigenfunctions: $f_0, f_1, f_2, f_3 \dots$ orthon. basis $L^2(M, dVol_g)$

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Berry's Random Wave Model

For "generic" chaotic surfaces, compare $f_j \longleftrightarrow W_j$
$W_j = (W_j(x))_{x\in\mathbb{R}^2}$ centered Gaussian field on the plane

$$Cov(W_j(x), W_j(y)) = J_0(\sqrt{\lambda_j}||x-y||)$$

$\Delta_{\mathbb{R}^2} W_j + \lambda_j W_j = 0 \quad a.s.$

Gaussianity: Random Phase Model on $\mathbb{R}^2$

$\mathbb{R}^2$: assume we observe the superposition of N waves at frequency k, that is

$$T_{k,N}(x) = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \exp(ik\langle \theta_j, x\rangle + \phi_j)$$

for $\mathbf{x} \in \mathbb{R}^2$, $\mathbf{k} \in \mathbb{R}^+$, where $\{\theta_j\}_{j=1,\dots,N}$ are random directions on the unit circle and $\{\phi_j\}_{j=1,\dots,N}$ are random phases.

- CLT $\Rightarrow$ $T_{k;N}(\mathbf{x}) \to_d \widetilde{T}_k(\cdot)$ a zero mean Gaussian field

  $$\mathbb{E}\left[\widetilde{T}_k(\mathbf{x}_1)\widetilde{T}_k(\mathbf{x}_2)\right] = J_0(k\left\|\mathbf{x}_1 - \mathbf{x}_2\right\|_2),$$

  where $J_0(\cdot)$ is the Bessel function of order 0, given by

  $$J_0(u) = \sum_{m=0}^{\infty} (-1)^m \frac{u^{2m}}{2^{2m}(m!)^2} \ .$$

GAUSSIANITY: RANDOM PHASE MODEL ON $\mathbb{R}^2$

- Double asymptotic setting:

  (1) a diverging number of random phases ensures that the behaviour of random eigenfunctions is Gaussian, due to a standard CLT;

  (2) taking Gaussianity for granted the asymptotic behaviour of random eigenfunctions is investigated, in the high-frequency/high energy sense (i.e., for diverging eigenvalues).

Q: Gaussianity has been established for a *fixed* eigenvalue k, can we justify the use of this assumption in the limit as $k \to \infty$?

Do we need some conditions that relate of divergence for the eigenvalue k to the rate of divergence of the random phases N?

The model on $\mathbb{S}^2$

Laplacian operator in $\mathbb{S}^2$:

$$\Delta_{\mathbb{S}^2} := \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \ ;$$

in the spherical case, a deterministic eigenfunction centred on $y \in \mathbb{S}^2$ can be constructed by

$$e_{\ell;y}(\cdot) : \mathbb{S}^2 \to \mathbb{R} \ , \ e_{\ell;y}(\cdot) := \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\langle \cdot, y \rangle) \ ,$$

$P_\ell(\cdot)$ Legendre polynomials

$$P_\ell(t) := \frac{1}{2^\ell \ell!}\frac{d^\ell}{dt^\ell}(t^2-1)^\ell \ , \ \ell = 0,1,2,\ldots; \ \ t \in [0,1] \ .$$

$P_\ell(1) \equiv 1$ for all ℓ and $\|e_{\ell;y}\|_{L^2(\mathbb{S}^2)} = 1$. $\{e_{\ell;y}(\cdot)\}$ satisfies the Helmholtz equation

$$\Delta_{\mathbb{S}^2} e_{\ell;y}(x) + \lambda_\ell e_{\ell;y}(x) = 0 \ , \ \ell = 0,1,2,...,$$

$\lambda_\ell = \ell(\ell+1)$ is the sequence of eigenvalues

Random Phase Model on $\mathbb{S}^2$

- Spherical Poisson Random Waves (with rate ν_t):

$$T_{\ell;t}(x) = \frac{1}{\sqrt{\nu_t}} \int_{\mathbb{S}^2} \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\langle x, \xi \rangle) dN_t(\xi)$$

where $\{N_t(\cdot)\}$ is a Poisson process on the sphere, with

$$\mathbb{E}[N_t(A)] = \nu_t \times \mu(A) \text{ for all } A \in \mathcal{B}(\mathbb{S}^2)$$

μ is the Lebesgue measure on $\mathbb{S}^2$. Our model implies that for all $A \subset \mathbb{S}^2$ and $t \geq 0$, $N_t(A)$ is a Poisson random variable with expected value equal to $\nu_t \times \mu(A)$, and for $A_1 \cap A_2 = \emptyset$ $N_t(A_1)$ and $N_t(A_2)$ are independent.

Note that

$$T_{\ell;t}(x) = \frac{1}{\sqrt{\nu_t}} \sum_{k=1}^{N_t(\mathbb{S}^2)} \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\langle x, \xi_k \rangle)$$

Spherical Harmonics

The standard basis for the $(2\ell+1)$-dimensional space of eigenfunctions corresponding to the eigenvalue λ_ℓ; are defined as the normalized eigenfunctions $\{Y_{\ell m}\}_{m=-\ell,\ldots,\ell}$ which satisfy the further condition (in spherical coordinates)

$$Y_{\ell m} : \mathbb{S}^2 \to \mathbb{R} \ , \ \frac{\partial^2}{\partial \varphi^2} Y_{\ell m}(\theta,\varphi) = -m^2 Y_{\ell m}(\theta,\varphi) \ .$$

$$Y_{\ell m}(\theta,\varphi) = \begin{cases} \sqrt{\frac{2\ell+1}{2\pi}\frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos\theta)\cos(m\varphi) & \text{for } m \in \{1,\ldots,\ell\} \\ \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta) & \text{for } m = 0 \\ \sqrt{\frac{2\ell+1}{2\pi}\frac{(\ell+m)!}{(\ell-m)!}} P_\ell^{-m}(\cos\theta)\sin(-m\varphi) & \text{for } m \in \{-\ell,\ldots,-1\} \end{cases},$$

where

$$P_\ell^m(t) := (1-t^2)^{m/2}\frac{d^m}{dt^m}P_\ell(t) \ , \ t \in [0,1]$$

is the Legendre associated function $P_\ell^m : [-1,1] \mapsto \mathbb{R}$ of degree ℓ and order m.

- Duplication formula:

$$\int_{\mathbb{S}^2} \frac{2\ell+1}{4\pi} P_\ell(\langle x, z\rangle) \frac{2\ell+1}{4\pi} P_\ell(\langle z, y\rangle) dz = \frac{2\ell+1}{4\pi} P_\ell(\langle x, y\rangle) \ ,$$

for all $x, y \in \mathbb{S}^2$.

$$\Rightarrow \mathbb{E}[T_{\ell;t}(x) T_{\ell;t}(y)] = P_\ell(\langle x, y\rangle) \ .$$

- Addition formula:

$$\sum_{m=-\ell}^{\ell} Y_{\ell m}(x) Y_{\ell m}(y) = \frac{2\ell+1}{4\pi} P_\ell(\langle x, y\rangle) \ , \text{ for all } x, y \in \mathbb{S}^2.$$

$$\Rightarrow T_{\ell;t}(x) = \frac{1}{\sqrt{\nu_t}} \sqrt{\frac{4\pi}{2\ell+1}} \sum_{k=1}^{N_t(\mathbb{S}^2)} \sum_{m=-\ell}^{\ell} Y_{\ell m}(x) Y_{\ell m}(\xi_k) = \sum_{m=-\ell}^{\ell} \hat{a}_{\ell,m}(t) Y_{\ell m}(x),$$

Random Plane Wave is the scaling limit of random spherical harmonics. Hilb's asymptotic formula: $\forall \varepsilon > 0$, uniformly for $\theta \in [0, \pi - \varepsilon]$ as $\ell \to \infty$,

$$P_\ell(\cos\theta) \sim \sqrt{\frac{\theta}{\sin\theta}} J_0\left(\left(\ell + \frac{1}{2}\right)\theta\right)$$

where the random spherical harmonic coefficients $\{\hat{a}_{\ell,m}\}_{m=-\ell,\dots,\ell}$ are defined by

$$\hat{a}_{\ell,m}(t) := \sqrt{\frac{4\pi}{(2\ell+1)\nu_t}} \sum_{k=1}^{N_t} Y_{\ell m}(\xi_k) \ ,$$

where $\{\xi_k\}$ are the points charged by the Poisson process.

$$\mathbb{E}[\hat{a}_{\ell,m}(t)\hat{a}_{\ell',m'}(t)] = \delta_m^{m'} \delta_\ell^{\ell'} \frac{4\pi}{(2\ell+1)}$$

- Parseval's identity holds, i.e.

$$||T_{\ell;t}||^2_{L^2(\mathbb{S}^2)} = \int_{\mathbb{S}^2} T^2_{\ell;t}(x)dx = \sum_{m=-\ell}^{\ell} |\hat{a}_{\ell,m}(t)|^2.$$

1. Convergence of the finite-dimensional distributions

- **Theorem 1 (Durastanti, Marinucci, T. 2022)**: For every fixed x, assuming that $\nu_t \times (\log \ell)^{-1} \to \infty$,

$$d_W(T_{\ell,t}(x), N) \leq \left(\sqrt{\frac{2}{\pi}} \frac{1}{2} + \sqrt{3} \right) \sqrt{\frac{\log \ell}{\nu_t}}.$$

$F = (T_{\ell;t}(x_1), T_{\ell;t}(x_2), \dots, T_{\ell;t}(x_d))$, $x_1, x_2, \dots, x_d$ d points on $\mathbb{S}^2$, Z a Gaussian vector.

$$d_3(F, Z) \leq Cd^2 \sqrt{\frac{\log \ell}{\nu_t}}$$

$$d_W(X, Y) := \sup_{h:||h||_{Lip} \leq 1} |\mathbb{E}h(X) - \mathbb{E}h(Y)|$$

$$d_3(A, B) := \sup_{h \in C^3} |\mathbb{E}[h(A)] - \mathbb{E}[h(B)]|$$

Comparison with Needlets Coefficients

$$\beta_j(\xi) := \sum_{\ell=2^{j-1}}^{2^{j+1}} b\left(\frac{\ell}{2^j}\right) T_{\ell;t}(\xi) = \frac{1}{\sqrt{\nu_t}} \int_{\mathbb{S}^2} \psi_j(\langle x, \xi \rangle)\, dN_t(\xi)$$

$$\psi_j(\langle x, \xi \rangle) := \sum_{\ell=2^{j-1}}^{2^{j+1}} b\left(\frac{\ell}{2^j}\right) \frac{2\ell+1}{4\pi} P_\ell(\langle x, \xi \rangle)$$

where $\{b\left(\frac{\ell}{2^j}\right)\}_{\ell=2^{j-1},\dots,2^{j+1}}$ weights normalized so that $\beta_j(\xi)$ has unit variance.

$$d_3(\beta_j(\xi), Z) = O\left(\sqrt{\frac{2^{2j}}{\nu_t}}\right)$$

(Durastanti-Marinucci-Peccati 2014)

Idea of the proof

$$T_{\ell;t}(x) = \frac{1}{\sqrt{\nu_t}} \int_{\mathbb{S}^2} \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\langle x, \xi \rangle) \, dN_t(x)$$

$\rightarrow$ FMT for integral functionals of Poisson processes!

Definition

For every deterministic function $h \in L_s^2(\rho)$ the Wiener-Ito integral of h with respect to N is given by

$$I_1(h) = \int_\Theta h(z) N(dz).$$

The Hilbert space composed of the random variables of the form $I_1(h)$ where $h \in L_s^2(\rho)$, is called the first Wiener chaos associated with the Poisson measure N.

Here $\Theta = \mathbb{R}_+ \times \mathbb{S}^2$ $\mathcal{A}$ the class of Borel subsets of Θ, labeled by $\mathcal{B}(\Theta)$.

Fourth Moment Theorem

- Theorem [Döbler-Vidotto-Zheng 2018]: For $\ell \in \mathbb{N}$, let $F \in W_1$, while $Z \sim N(0,1)$. $Var(F) = 1$ and $\mathbb{E}[F^4] < \infty$. Then it holds that

$$d_W(F,Z) \leq \left(\frac{1}{\sqrt{2\pi}} + \frac{2}{3}\right)\sqrt{\mathbb{E}[F^4] - 3}$$

- Theorem [Döbler-Vidotto-Zheng 2018]: $F = (F_1, \dots, F_d)^T$ centred random vector with covariance matrix Γ_d and s.t. $F_j \in W_1$. $Z_d \sim N(0, \Gamma_d)$. Then for every $g \in C^3(\mathbb{R}^d)$, we have that

$$|\mathbb{E}[g(F)] - \mathbb{E}[g(Z_d)]| \leq B_3(g,d) \sum_{i=1}^{d} \sqrt{\mathbb{E}[F_i^4] - 3\mathbb{E}[F_i^2]^2}$$

with

$$B_3(g,d) = \frac{\sqrt{2d}}{4} M_2(g) + \frac{2\sqrt{dTr(\Gamma_d)}}{9}$$

$$\mathbb{E}[T_{\ell;t}^4] = \sum_{m_1 m_2 m_3 m_4} \mathbb{E}[\hat{a}_{\ell,m_1}(t)\ \hat{a}_{\ell,m_2}(t)\hat{a}_{\ell,m_3}(t)\ \hat{a}_{\ell,m_4}(t)]$$
$$\times\ Y_{\ell m_1}(x) Y_{\ell m_2}(x) Y_{\ell m_3}(x) Y_{\ell m_4}(x)\, dx\ ;$$

substituting the value of $\hat{a}_{\ell,m}$ we have that

$$\mathbb{E}[T_{\ell;t}^4] = \left(\frac{4\pi}{\nu_t(2\ell+1)}\right)^2 \sum_{m_1,\ldots m_4=-\ell}^{\ell} \mathbb{E}[\sum_{k_1,\ldots,k_4=1}^{N_t}$$
$$\hat{Y}_{\ell m_1}(\xi_{k_1}) \hat{Y}_{\ell m_2}(\xi_{k_2}) \hat{Y}_{\ell m_3}(\xi_{k_3}) \hat{Y}_{\ell m_4}(\xi_{k_4})]$$
$$\times\ Y_{\ell m_1}(x) Y_{\ell m_2}(x) Y_{\ell m_3}(x) Y_{\ell m_4}(x)\ .$$

Applying the addition formula:

$$\mathbb{E}[T_{\ell;t}^4] = \left(\frac{2\ell+1}{4\pi}\right)^2 \frac{1}{\nu_t^2} \mathbb{E}\left[\sum_{k_1,\dots,k_4=1}^{N_t} P_\ell(\langle \xi_{k_1}, x\rangle) P_\ell(\langle \xi_{k_2}, x\rangle) P_\ell(\langle \xi_{k_3}, x\rangle) P_\ell(\langle \xi_{k_4}, x\rangle)\right]$$

$$= \left(\frac{2\ell+1}{4\pi}\right)^2 \frac{1}{\nu_t^2} \left(\nu_t \mathbb{E}\left[P_\ell(\langle \xi_{k_1}, x\rangle)^4\right] + 3\nu_t^2 \mathbb{E}\left[P_\ell(\langle \xi_{k_1}, x\rangle)^2\right]^2\right).$$

$$\int_0^1 P_\ell^4(t)\,dt \sim \frac{3}{2\pi^2}\frac{\log \ell}{\ell^2} \text{ , (Marinucci-Wigman (2011))}$$

$$\Rightarrow \mathbb{E}\left[P_\ell(\langle \xi_{k_1}, x\rangle)^4\right] = \int_{\mathbb{S}^2} P_\ell(\langle z, x\rangle)^4 dz \sim 4\pi\frac{3}{2\pi^2}\frac{\log \ell}{\ell^2} \text{ , as } \ell \to \infty \ .$$

Moreover, since

$$\int_0^1 P_\ell(t)^2\,dt = \frac{1}{2\ell+1},$$

$$\Rightarrow \mathbb{E}[P_\ell(\langle \xi_{k_1}, x\rangle)^2] = \int_{\mathbb{S}^2} P_\ell(\langle z, x\rangle)^2 dz = \frac{4\pi}{2\ell+1}.$$

Then

$$\mathbb{E}[T_{\ell;t}^4] = 3 + \frac{3}{2\pi^3}\frac{\log \ell}{\nu_t} + o\left(\frac{\log \ell}{\nu_t}\right) \ .$$

2. Convergence in law for the vector of spherical harmonic coefficients

Let us consider the vector

$$V_{\ell;t} := \left(\hat{a}_{\ell,-\ell}(t), \ldots, \hat{a}_{\ell,\ell}(t)\right) = \{\hat{a}_{\ell,m}(t)\}_{m=-\ell,\ldots,\ell} \ ,$$

where

$$\hat{a}_{\ell,m}(t) = \sqrt{\frac{4\pi}{(2\ell+1)}} \frac{1}{\sqrt{\nu_t}} \sum_{k=1}^{N_t} Y_{\ell,m}(\xi_k) \ .$$

$$\mathbb{E}\left[\hat{a}_{\ell,m}(t)\right] = 0 \ ;$$

$$\mathbb{E}[\hat{a}_{\ell,m}(t)\hat{a}_{\ell',m'}(t)] = \delta_m^{m'} \delta_\ell^{\ell'} \frac{4\pi}{(2\ell+1)}$$

- **Theorem 2 (Durastanti, Marinucci, T. 2022)**: Let $Z_{2\ell+1}$ a Gaussian vector of dimension $2\ell+1$ with zero mean and covariance function equal to $\frac{\delta_m^{m'} 4\pi}{2\ell+1}$. Then we have that

$$d_3(V_{\ell;t}, Z_{2\ell+1}) \leq \sup_{g \in C^3} B_3(g;\ell) \sqrt{8\sqrt{4\pi} C \frac{\log \ell}{\ell \nu_t} + O\left(\frac{1}{\ell \nu_t}\right)} ,$$

where

$$B_3(g;\ell) := \frac{\sqrt{2(2\ell+1)}}{4} M_2(g) + \frac{2}{9}\sqrt{(2\ell+1)4\pi} M_3(g).$$

- It should be noted that the resulting bound is of order $\sqrt{\frac{\log \ell}{\nu_t}}$.

---

for fixed $k \geq 1$ and $g \in C^{k-1}(\mathbb{R}^d)$

$$M_k(g) := \sup_{x \neq y} \frac{||D^{k-1}g(x) - D^{k-1}g(y)||_{op}}{||x-y||_{L^2(\mathbb{R}^d)}}$$

Idea of the proof

- Step 1:

$$\mathbb{E}[a_{\ell,m}(t)^4] = \left(\frac{4\pi}{(2\ell+1)}\right)^2 \frac{1}{\nu_t}\mathbb{E}[Y_{\ell m}(\xi_1)^4] + 3\left(\frac{4\pi}{(2\ell+1)}\right)^2$$

- Step 2:

$$\mathbb{E}[Y_{\ell m}(\xi_1)^4] = \frac{(2\ell+1)}{\sqrt{4\pi}}\left[\frac{(C^{L0}_{\ell 0;\ell 0})^2 (C^{L0}_{\ell - m \ell m})^2}{2L+1}\right]$$

- Step 3:

$$\sum_{m=-\ell}^{\ell} cum_4(\hat{a}_{\ell m}(t)) \leq 8\sqrt{4\pi}C\frac{\log \ell}{\ell(2\ell+1)\nu_t} + O\left(\frac{1}{\ell^2 \nu_t}\right)$$

Clebsch-Gordan Coefficients

They are involved in the evaluation of multiple integrals of spherical harmonics, the so-called *Gaunt integrals*, given by

$$\int_{\mathbb{S}^2} Y_{\ell_1 m_1}(x) \dots\dots Y_{\ell_n m_n}(x)\, dx$$

$$= \sqrt{\frac{4\pi}{2\ell_n+1}} \sum_{L_1 \dots L_{n-3}} \sum_{M_1 \dots M_{n-3}} \Big[C^{L_1,M_1}_{\ell_1,m_1;\ell_2,m_2} C^{L_2,M_2}_{L_1,M_1;\ell_3,m_3} \dots C^{\ell_n,-m_n}_{L_{n-3},M_{n-3};\ell_{n-1},m_{n-1}}$$

$$\times \sqrt{\frac{\prod_{i=1}^{n-1}(2\ell_i+1)}{(4\pi)^{n-1}}} \{ C^{L_1,0}_{\ell_1,0;\ell_2,0} C^{L_2,0}_{L_1,0;\ell_3,0} \dots C^{\ell_n,0}_{L_{n-3},0;\ell_{n-1},0} \} \Big] .$$

3. QCLT in $L^2(\mathbb{S}^2)$

$\{T_{\ell;t}\}$ as random elements $T_{\ell;t} : \Omega \to L^2(\mathbb{S}^2)$, i.e. as measurable applications with the topology induced on $L^2(\mathbb{S}^2)$ by the standard metric

$$d^2(f,g) := ||f-g||_{L^2(\mathbb{S}^2)} = \int_{\mathbb{S}^2} |f(x)-g(x)|^2 \, dx$$

- **Theorem 3 (Durastanti, Marinucci, T. 2022)**: Let Z be a centred Gaussian process with the same covariance operator as $T_{\ell;t}$. We have that

$$d_3(T_{\ell;t}, Z) \leq \left(\frac{1}{4} + 4\sqrt{\pi}\right)\sqrt{\frac{4\pi}{\nu_t}}$$

- Asymptotic gaussianity holds under the simple condition that $\nu_t \to \infty$ no matter how fast the sequence of eigenvalues diverge to infinity.

QCLT in $L^2(\mathbb{S}^2)$: proof

- Theorem [Bourguin-Campese-Dang 2021]: X is a $K-$valued random variable who belongs to the first Wiener chaos with finite fourth moment, i.e. $\mathbb{E}[||X||_K^4] < \infty$, and with covariance operator S. We denote by Z a Gaussian process taking values in the same separable Hilbert space of X and having the same covariance operator S. Then

$$d_3(X,Z) \leq \left(\frac{1}{4} + \sqrt{4\mathbb{E}[||X||_K^2]}\right)\sqrt{\mathbb{E}[||X||_K^4] - \mathbb{E}[||X||_K^2]^2 - 2||S||_{HS(K)}^2}.$$

$|| \cdot ||_{HS}$ denotes the Hilbert-Schmidt norm

$\Rightarrow$ We need to compute the quantity

$$\mathbb{E}[\| T_{\ell;t}\|_{L^2(\mathbb{S}^2)}^4] - (\mathbb{E}[\| T_{\ell;t}\|_{L^2(\mathbb{S}^2)}^2])^2 - 2\,\|S_{\ell;t}\|_{HS}^2\,,$$

where $S_{\ell;t}$ is the covariance operator of $T_{\ell;t}$.

$$
\begin{aligned}
\mathbb{E}[||T_{\ell;t}||^2] &= \mathbb{E}\left[\int_{\mathbb{S}^2} |T_{\ell;t}(x)|^2\, dx\right] \\
&= \int_{\mathbb{S}^2} \sum_{m_1=-\ell}^{\ell} \sum_{m_2=-\ell}^{\ell} \mathbb{E}[\hat{a}_{\ell,m_1}(t)\ \hat{a}_{\ell,m_2}(t)] Y_{\ell m_1}(x) Y_{\ell m_2}(x)\, dx \\
&= \int_{\mathbb{S}^2} \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell m}(x) Y_{\ell m}(x)\, dx \\
&= \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} \int_{\mathbb{S}^2} Y_{\ell m}(x) Y_{\ell m}(x) dx = 4\pi\ .
\end{aligned}
$$

It follows that $(\mathbb{E}[||T_{\ell;t}||^2])^2 = (4\pi)^2$.

Now we compute $\mathbb{E}[||T_{\ell;t}||^4]$, which gives

$$\begin{aligned}\mathbb{E}[||T_{\ell;t}||^4] &= \mathbb{E}\left[||T_{\ell;t}||^2||T_{\ell;t}||^2\right] \\ &= \mathbb{E}\left[\sum_{m_1=-\ell}^{\ell}|\hat{a}_{\ell,m_1}(t)|^2\sum_{m_2=-\ell}^{\ell}|\hat{a}_{\ell,m_2}(t)|^2\right] \\ &= \left(\frac{4\pi}{(2\ell+1)\nu_t}\right)^2\mathbb{E}\Bigg[\sum_{m_1=-\ell}^{\ell}\sum_{k_1k_2}Y_{\ell m_1}(\xi_{k_1})Y_{\ell m_1}(\xi_{k_2}) \\ &\times\sum_{m_2=-\ell}^{\ell}\sum_{k_3k_4}Y_{\ell m_2}(\xi_{k_3})Y_{\ell m_2}(\xi_{k_4})\Bigg].\end{aligned}$$

$$||T_{\ell;t}||^2_{L^2(\mathbb{S}^2)} = \int_{\mathbb{S}^2}T^2_{\ell;t}(x)dx = \sum_{m=-\ell}^{\ell}|\hat{a}_{\ell,m}(t)|^2$$

Applying the addition formula we get

$$\mathbb{E}[||T_{\ell;t}||^4] = \left(\frac{1}{\nu_t}\right)^2 \mathbb{E}\left[\sum_{k_1=1}^{N_t}\sum_{k_2=1}^{N_t} P_\ell(\langle \xi_{k_1}, \xi_{k_2}\rangle) \sum_{k_3=1}^{N_t}\sum_{k_4=1}^{N_t} P_\ell(\langle \xi_{k_3}, \xi_{k_4}\rangle)\right]$$

$$= \left(\frac{1}{\nu_t}\right)^2 \mathbb{E}\left[\sum_{k_1=1}^{N_t} P_\ell(\langle \xi_{k_1}, \xi_{k_1}\rangle)^2\right]$$

$$+ \left(\frac{1}{\nu_t}\right)^2 \mathbb{E}\left[\sum_{k_1=k_2\neq k_3=k_4} P_\ell(\langle \xi_{k_1}, \xi_{k_2}\rangle) P_\ell(\langle \xi_{k_3}, \xi_{k_4}\rangle)\right]$$

$$+ \left(\frac{1}{\nu_t}\right)^2 \mathbb{E}\left[\sum_{k_1=k_3\neq k_2=k_4} P_\ell(\langle \xi_{k_1}, \xi_{k_2}\rangle) P_\ell(\langle \xi_{k_3}, \xi_{k_4}\rangle)\right]$$

$$+ \left(\frac{1}{\nu_t}\right)^2 \mathbb{E}\left[\sum_{k_1=k_4\neq k_3=k_2} P_\ell(\langle \xi_{k_1}, \xi_{k_2}\rangle) P_\ell(\langle \xi_{k_3}, \xi_{k_4}\rangle)\right]$$

and since $P_\ell(0) = 1$ for all ℓ we obtain

$$\begin{aligned}
\mathbb{E}[||T_{\ell;t}||^4] &= \left(\frac{1}{\nu_t}\right)^2 \mathbb{E}\left[\sum_{k_1=1}^{N_t} 1\right] + \left(\frac{1}{\nu_t}\right)^2 \mathbb{E}\left[\sum_{k_1=k_2\neq k_3=k_4} 1\right] \\
&\quad + 2\left(\frac{1}{\nu_t}\right)^2 \mathbb{E}\left[\sum_{k_1=k_3\neq k_2=k_4} P_\ell(\langle \xi_{k_1}, \xi_{k_2}\rangle)^2\right] \\
&= \frac{4\pi}{\nu_t} + (4\pi)^2 \left(\frac{1}{\nu_t}\right)^2 \nu_t^2 \\
&\quad + \left(\frac{1}{\nu_t}\right)^2 2\nu_t^2 \int_{(\mathbb{S}^2)^2} P_\ell(\langle \xi_{k_1}, \xi_{k_2}\rangle)^2 \, d\xi_{k_1}\, d\xi_{k_2} \\
&= \frac{4\pi}{\nu_t} + (4\pi)^2 + 2(4\pi)\frac{4\pi}{2\ell+1}.
\end{aligned}$$

The covariance operator $S_{\ell;t}$ is such that

$$||S_{\ell;t}||_{HS}^2 = \sum_{m=-\ell}^{\ell} \sum_{m'=-\ell}^{\ell} \mathbb{E}[a_{\ell,m}(t) a_{\ell,m'}(t)]^2$$

$$= \sum_{m=-\ell}^{\ell} \sum_{m'=-\ell}^{\ell} \left(\delta_m^{m'} \frac{4\pi}{2\ell+1} \right)^2 = \frac{(4\pi)^2}{2\ell+1} \ ,$$

and then we finally obtain

$$\mathbb{E}[\| T_{\ell;t}\|^4] - (\mathbb{E}[\| T_{\ell;t}\|^2])^2 - 2\, \|S_{\ell;t}\|_{HS}^2$$

$$= \quad \frac{4\pi}{\nu_t} + (4\pi)^2 + 2\frac{(4\pi)^2}{2\ell+1} - (4\pi)^2 - 2\frac{(4\pi)^2}{2\ell+1} = \frac{4\pi}{\nu_t} \ .$$

Comments:

- It may come at first sight as a suprise that the rate of convergence in this functional setting (i.e., $1/\sqrt{\nu_t}$) does not depend on the index ℓ and it is indeed faster than in the finite-dimensional case. The apparent paradox is solved noting that the topology that we consider here is too coarse to imply convergence of the finite-dimensional distributions.

4. QCLT IN $W_{\alpha,2}(\mathbb{S}^2)$

Now we consider the random eigenfunctions belonging to the Sobolev space on the sphere, i.e., the space of functions $f \in L^2(\mathbb{S}^2)$, $f = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell,m} Y_{\ell,m}$, with finite norm

$$\|f\|^2_{W_{\alpha,2}(\mathbb{S}^2)} = \sum_{\ell \geq 0} \sum_{m=-\ell}^{\ell} \left(1 + \sqrt{\ell(\ell+1)}\right)^{2\alpha} |a_{\ell,m}|^2 .$$

Theorem 4 (Bourguin, Durastanti, Marinucci, T. 2023): Let Z be a centred Gaussian process with the same covariance operator as $T_{\ell;t}$. We have that

$$d_{3,W_{\alpha,2}}(T_{\ell;t}, Z) \leq \left(\frac{1}{4} + 4\sqrt{\pi}\right) \sqrt{\frac{4\pi(1 + \sqrt{\ell(\ell+1)})^{4\alpha}}{\nu_t}}$$

QCLT IN $W_{\alpha,2}(\mathbb{S}^2)$: PROOF

First note that

$$\mathbb{E}\left[\| T_{\ell;t}\|^4_{W_{\alpha,2}}\right] = \left(1+\sqrt{\ell(\ell+1)}\right)^{4\alpha} \mathbb{E}\left[|| T_{\ell;t}||^4_{L^2(\mathbb{S}^2)}\right]$$

and

$$\mathbb{E}\left[\| T_{\ell;t}\|^2_{W_{\alpha,2}}\right] = \left(1+\sqrt{\ell(\ell+1)}\right)^{2\alpha} \mathbb{E}\left[|| T_{\ell;t}||^2_{L^2(\mathbb{S}^2)}\right]. \qquad (1)$$

Indeed, we have that

$$\begin{aligned}
\mathbb{E}\left[\| T_{\ell;t}\|^4_{W_{\alpha,2}}\right] &= \mathbb{E}\left[\| T_{\ell;t}\|^2_{W_{\alpha,2}} \| T_{\ell;t}\|^2_{W_{\alpha,2}}\right] \\
&= \mathbb{E}\left[\sum_{m=-\ell}^{\ell}\sum_{m'=-\ell}^{\ell}\left(1+\sqrt{\ell(\ell+1)}\right)^{4\alpha} |\widehat{a}_{\ell,m}|^2 |\widehat{a}_{\ell,m'}|^2\right] \\
&= \left(1+\sqrt{\ell(\ell+1)}\right)^{4\alpha} \mathbb{E}\left[\sum_{m=-\ell}^{\ell}\sum_{m'=-\ell}^{\ell} |\widehat{a}_{\ell,m}|^2 |\widehat{a}_{\ell,m'}|^2\right] \\
&= \left(1+\sqrt{\ell(\ell+1)}\right)^{4\alpha} \mathbb{E}\left[|| T_{\ell;t}||^4_{L^2(\mathbb{S}^2)}\right]
\end{aligned}$$

$$
\begin{aligned}
\|S_{\ell;t}\|_{HS(W_{\alpha,2})} &= \|\mathbb{E}\,[T_{\ell;t}\otimes T_{\ell;t}]\|^2_{W_{\alpha,2}} \\
&= \left\| \frac{4\pi}{2\ell+1}\frac{1}{\nu_t}\sum_{m=-\ell}^{\ell}\sum_{m'=-\ell}^{\ell}\mathbb{E}\,[Y_{\ell,m}(\xi_{k_1})Y_{\ell,m'}(\xi_{k_2})]\,Y_{\ell,m}\otimes Y_{\ell,m'}\right\|^2_{W_{\alpha,2}(\mathbb{S}^2)} \\
&= \frac{(4\pi)^2}{(2\ell+1)^2}\left\|\sum_{m=-\ell}^{\ell}Y_{\ell,m}\otimes Y_{\ell,m}\right\|^2_{W_{\alpha,2(\mathbb{S}^2)}} \\
&= \frac{(4\pi)^2}{2\ell+1}\left(1+\sqrt{\ell(\ell+1)}\right)^{4\alpha}
\end{aligned}
$$

Then, it follows that

$$
\mathbb{E}\left[\|T_{\ell;t}\|^4_{W_{\alpha,2}}\right]-\mathbb{E}^2\left[\|T_{\ell;t}\|^2_{W_{\alpha,2}}\right]-2\,\|S_{\ell;t}\|^2_{HS(W_{\alpha,2})}=4\pi\frac{\left(1+\sqrt{\ell(\ell+1)}\right)^{4\alpha}}{\nu_t},
$$

Comments

For $\alpha > \frac{3}{2}$, a quantitative Central Limit Theorem in Sobolev space does imply the quantitative Central Limit Theorem for the marginal distribution at every given location on the sphere.

Note first that

$$\begin{aligned} \|f\|_{L^\infty(S^2)} \quad : & = \sup_x \Big| \sum_\ell \sum_m a_{\ell m}(f) Y_{\ell m}(x) \Big| \\ & \leq \sum_\ell \sum_m |a_{\ell m}(f)| \sup_x |Y_{\ell m}(x)| \\ & \leq \sum_\ell \sum_m |a_{\ell m}(f)| \sqrt{\frac{2\ell+1}{2\pi}} \ , \end{aligned}$$

whence

$$\|f\|^2_{L^\infty(S^2)} \leq \frac{1}{2\pi} \left\{ \sum_\ell \sum_m |a_{\ell m}(f)| \sqrt{2\ell+1} \right\}^2$$

Multiplying and dividing by $(1+\sqrt{\ell(\ell+1)})^{\alpha}\sqrt{2\ell+1}$ and then applying twice Cauchy-Schwarz inequality we get

$$\begin{aligned}\|f\|^2_{L^\infty(S^2)} &\le \frac{1}{2\pi}||f||^2_{W_{\alpha,2}}\sum_{\ell}\frac{(2\ell+1)^2}{(1+\sqrt{\ell(\ell+1)})^{2\alpha}} \\ &\le \frac{2}{\pi}||f||^2_{W_{\alpha,2}}\zeta(2\alpha-2),\end{aligned}$$

where as usual

$$\zeta(2\alpha-2)=\sum_{\ell=1}^{\infty}\frac{1}{\ell^{2\alpha-2}}<\infty\ ,$$

because $\alpha>\frac{3}{2}$.

$$\Rightarrow \|f\|^2_{L^\infty(S^2)}<\frac{2}{\pi}\zeta(2\alpha-2)\times\|f\|^2_{W_{\alpha,2}}\ .$$

$\Rightarrow$ the topology generated by the norm $\|.\|_{W_{\alpha,2}}$ is finer than the topology generated by $\|.\|_{L^\infty(S^2)}$

$$\Rightarrow \sup_{h \text{ continuous w.r.t.} \|.\|_{L^\infty(S^2)}} |\mathbb{E}h(X) - \mathbb{E}h(Y)| \leq \sup_{h \text{ continuous w.r.t. } \|\cdot\|_{W_{\alpha,2}}} |\mathbb{E}h(X) - \mathbb{E}h(Y)| \; .$$

Corollary

For $\alpha > \frac{3}{2}$, we have that

$$d_3(X_\ell(x), Z_\ell(x)) = \sup_{g \in C_b^3(\mathbb{R})} |\mathbb{E}g(X_\ell(x)) - \mathbb{E}g(Z_\ell(x))| \leq C(\alpha) d_{3,W_{\alpha,2}}(X_\ell, Z_\ell) \; ,$$

where the term $C(\alpha)$ does not depend on ℓ.

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