# $\mathscr{C}^\infty$ convergence of densities on Wiener chaoses

Ronan Herry, Dominique Malicet & Guillaume Poly

21 mars 2023

# The Wiener space and related operators

- dimension: $n$ (none of our estimates depend on $n$)
- $n$-dimensional Wiener space : $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \gamma_n)$
- Ornstein-Uhlenbeck operator : $L(f) = \Delta f - \sum_{i=1}^n x_i \partial_i f$
- Square field operator: $\Gamma(f, g) = \nabla f \cdot \nabla g$
- $m$-th Wiener chaos:

$$\mathscr{W}_m = \mathrm{Ker}\,(L + m\mathbf{Id}) = \mathrm{Vect}\left(\prod_{i=1}^{n} H_{k_i}(x_i) \,|\, \sum_{i=1}^{n} k_i = m\right).$$

- Wiener orthogonal expansion:

$$L^2(\gamma_n) = \bigoplus_{m=0}^{\infty} \mathscr{W}_m.$$

- Integration by parts for $f \in \mathrm{Dom}(\Gamma), g \in \mathrm{Dom}(L) \subset \mathrm{Dom}(\Gamma)$:

$$\int_{\mathbb{R}^n} \Gamma(f,g) d\gamma_n := -\int_{\mathbb{R}^n} f \; L(g)\, d\gamma_n$$

- Rephrasing using expectations:

$$\mathbb{E}\left[\Gamma\left(f(\vec{X}), g(\vec{X})\right)\right] = -\mathbb{E}\left[f(\vec{X}) L\left(g(\vec{X})\right)\right],$$

where $X_i : \begin{cases} (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \gamma_n) & \to & \mathbb{R} \\ (x_1, \cdots, x_n) & \to & x_i, \end{cases}$ and $\vec{X} = (X_1, \cdots, X_n) \sim \mathcal{N}(0, \mathbf{I}_n)$.

- $\mathscr{W}_m \subset \mathbb{D}^\infty := \left(\bigcap_{i \geq 1} \mathrm{Dom}(L^i)\right) \cap \left(\bigcap_{p \geq 1} L^p(\gamma_n)\right)$

  ☞ we may skip any domain/integrability consideration on Wiener chaoses

# State of the art regarding central convergence on Wiener chaoses (univariate)

- (2005) Nualart-Peccati: $(F_n)$ sequence in $\mathscr{W}_m$

$$F_n \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0,1) \Leftrightarrow \mathbb{E}\left[F_n^2\right] \to 1 \;\&\; \mathbb{E}\left[F_n^4\right] \to 3.$$

- (2009) Nourdin-Peccati: $F \in \mathscr{W}_m$ with $\mathbb{E}\left[F^2\right] = 1$

$$\begin{aligned} d_{\mathrm{TV}}\left(F, \mathcal{N}(0,1)\right) &\leq 2\sqrt{\mathrm{Var}\left(\Gamma\left(F,F\right)\right)} \\ &\leq \frac{2}{\sqrt{3}}\sqrt{\mathbb{E}\left[F^4 - 3\right]}. \end{aligned}$$

# State of the art regarding central convergence on Wiener chaoses (multivariate)

- (2005) Peccati-Tudor: $\vec{F}_n := (F_{n,1}, \cdots, F_{n,d})$ sequence in $\mathscr{W}_{m_1} \times \cdots \times \mathscr{W}_{m_d}$:

$$\vec{F}_n \xrightarrow[n\to\infty]{\text{Law}} \mathscr{N}(0,\Sigma) \Leftrightarrow \begin{cases} \forall i & F_{n,i} \to \mathscr{N}(0,\Sigma(i,i)) \\ \forall i,j & \mathbb{E}[F_{n,i}F_{n,j}] \to \Sigma(i,j) \end{cases}$$

- (2010) Nourdin-Peccati-Reveillac:

$$d_{\text{Wass}}\left(\vec{F}_n, \mathcal{N}(0,\Sigma)\right) \le c_\Sigma \sqrt{\sum_{i,j} \mathbb{E}\left[\left(\Sigma(i,j) - \frac{1}{m_j}\Gamma(F_{n,i}, F_{n,j})\right)^2\right]}$$

# State of the art regarding central convergence on Wiener chaoses (multivariate)

- (2014) Nourdin-Peccati-Swan:
  - $\vec{F} = (F_1, \cdots, F_d) \in \mathscr{W}_{m_1} \times \cdots \times \mathscr{W}_{m_d}$ with $\mathrm{Cov}\left(\vec{F}\right) = \mathbf{I}_d$
  - $\vec{N} = (N_1, \cdots, N_d) \sim \mathscr{N}\left(0, \mathbf{I}_d\right)$

$$\begin{aligned} \mathrm{D}\left(\vec{F} \,||\, \vec{N}\right) &\leq C_{\vec{m}} \sqrt{\mathbb{E}\left[\|\vec{F}\|^4 - \|\vec{N}\|^4\right]} \\ &\times \left|\log\left(\mathbb{E}\left[\|\vec{F}\|^4 - \|\vec{N}\|^4\right]\right)\right| \end{aligned}$$

## Can we go beyond entropy?

- (2014) Nualart-Hu: quantitative $\mathcal{C}^\infty$ -convergence of densities if $\frac{1}{\Gamma(F_n,F_n)}$ bounded in every $L^p$, $p \geq 1$
- (2016) Nourdin-Nualart: quantitative convergence in Fisher information if $\frac{1}{\Gamma(F_n,F_n)^{4+\delta}}$ integrable
- (2016) Nualart-Hu-Tyndel: $\mathcal{C}^\infty$ convergence of densities for

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} P(X_k),$$

with $P \in \mathbb{R}[X]$, $(X_k)_{k\geq 1}$ stationary Gaussian process with spectral density $f$ such that $\log(f) \in L^1$.

☞ Are the above assumptions (in red) necessary or are they 'automatic'?

# Our result

☞ The previous assumptions are unecessary. Central convergence on Wiener chaoses always implies bounded negative moments for $\Gamma(F_n, F_n)$ at any order.

<u>Assumptions:</u>

- $(F_n)$ sequence in $\mathscr{W}_m$
- $F_n \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0,1)$

<u>Conclusion:</u> $\forall p \geq 1, \limsup \mathbb{E}\left[\Gamma(F_n, F_n)^{-p}\right] < \infty$

$$\Rightarrow F_n \xrightarrow[n\to\infty]{\mathcal{C}^\infty} \mathcal{N}(0,1)$$

$\Leftrightarrow \forall q \geq 1, \forall \epsilon > 0, \exists \delta_{\epsilon,q,m} > 0$ such that $\forall F \in \mathscr{W}_m$:

$$\text{dist}(F, \mathcal{N}(0,1)) \leq \delta_{\epsilon,q,m} \Rightarrow \begin{cases} F \text{ has a } \mathscr{C}^q \text{ density } f \\ \sup_{x\in\mathbb{R}} \left| f^{(q)}(x) - \gamma^{(q)}(x) \right| \leq \epsilon \end{cases}$$

# Our result in higher dimensions

Assumptions:

- $\vec{F}_n = (F_{n,1}, F_{n,2}, \cdots, F_{n,d})$ sequence in $\mathscr{W}_{m_1} \times \cdots \times \mathscr{W}_{m_d}$
- $\vec{F}_n \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0, \mathbf{I}_d)$

Conclusion: $\forall p \geq 1, \limsup_n \mathbb{E}\left[\det\left(\Gamma\left(F_n, F_n\right)\right)^{-p}\right] < \infty$

$$\Rightarrow \vec{F}_n \xrightarrow[n\to\infty]{\mathcal{C}^\infty} \mathcal{N}(0, \mathbf{I}_d)$$

# A slightly stronger result in dimension one

Assumptions:

- $(F_n)$ sequence in $\mathscr{W}_m \bigoplus \mathscr{W}_{m-1} \bigoplus \cdots \bigoplus \mathscr{W}_0$
- $\mathcal{J}_m(F_n) \xrightarrow[n\to\infty]{\text{Law}} \mathcal{N}(0,1)$
- $\forall i \leq m-1, \mathbb{E}\left(\mathcal{J}_i(F_n)^2\right) \leq M$

Conclusion: $\forall p \geq 1, \limsup_n \mathbb{E}\left[\Gamma\left(F_n, F_n\right)^{-p}\right] < \infty$

# Exemple 1:

- $X_N = \left(\frac{X_{i,j}}{\sqrt{N}}\right)_{1\leq i,j\leq N}$ with $(X_{i,j})_{i,j\geq 1}$ are i.i.d. $\mathcal{N}(0,1)$.
- $k_1 < k_2 < \cdots < k_d$
- $D = \mathrm{Diag}\left(k_1, k_2, \cdots, k_d\right)$
- $$\left(\mathrm{Tr}\left(X_N^{k_1}\right) - \mathbb{E}\left(\mathrm{Tr}\left(X_N^{k_1}\right)\right), \cdots, \mathrm{Tr}\left(X_N^{k_d}\right) - \mathbb{E}\left(\mathrm{Tr}\left(X_N^{k_d}\right)\right)\right) \xrightarrow[n\to\infty]{\mathcal{C}^\infty} \mathcal{N}(0, D).$$

# Exemple 2

Let $A$ be a Gaussian matrix of size $N \times p$.

$$\forall q \geq 1, \exists C_{q,p} > 0, \exists \delta_{q,p} > 0,$$

$$\|{}^t A \cdot A - \mathbf{I}_p\|_2 \leq \epsilon \Rightarrow \mathbb{E}\left(\det\left({}^t A \cdot A\right)^{-q}\right) \leq C_{q,p}.$$

# Ideas of proof: (1) a decoupling procedure

$$F = \phi(X_1, \cdots, X_n)$$

Sharp operator:

$$\sharp F = \sum_{k=1}^{n} \frac{\partial F}{\partial X_k} \hat{X}_k, \ \left(X_k, \hat{X}_k\right)_{k \geq 1} \text{ i.i.d. } \sim \ \mathcal{N}(0,1).$$

$$\mathbb{E}\left(e^{it\sharp F}\right) = \mathbb{E}\left(\exp\left(-\frac{t^2}{2}\Gamma\left(F,F\right)\right)\right).$$

$\Gamma(F,F)$ has negative moments

$\Leftrightarrow \sharp F$ has a smooth law

# Ideas of proof: (2) decoupling twice

$$\begin{aligned} \sharp F \text{has a smooth law} \quad &\Leftrightarrow \quad \Gamma[\sharp F, \sharp F] \text{ has negative moments} \\ &\Leftrightarrow \quad \sharp[\sharp F] \text{ has a smooth law} \end{aligned}$$

☞ One is left to study the smoothness of the law of

$$\sum_{i,j \leq n} \frac{\partial^2 F}{\partial X_i \partial X_j} \hat{X}_i \hat{X}_j.$$

# A random quadratic form

- Set $M = \left(\frac{\partial^2 F}{\partial X_i \partial X_j}\right)_{1 \leq i,j \leq n}$
- if $F \in \mathscr{W}_m$ then $M \in \mathcal{M}_n(\mathscr{W}_{m-2})$.

$$\left|\mathbb{E}\left(e^{it\sum_{i,j\leq n}\frac{\partial^2 F}{\partial X_i \partial X_j}\hat{X}_i\hat{X}_j}\right)\right| = \left|\mathbb{E}\left(e^{it\sum_{k=1}^{n}\lambda_k \hat{X}_k^2}\right)\right|$$

$$= \left|\mathbb{E}\left(\prod_{k=1}^{n}\frac{1}{\sqrt{1+2it\lambda_k}}\right)\right| \leq \mathbb{E}\left(\prod_{k=1}^{n}\frac{1}{(1+4t^2\lambda_k^2)^{\frac{1}{4}}}\right)$$

☞ One is left to prove negative moments for

$$\mathcal{R}_q(M) = \sum_{k_1 \neq k_2 \cdots \neq k_q} \lambda_{k_1}^2 \cdots \lambda_{k_q}^2$$

☞ It is equivalent to get negative moments for

$$\mathcal{D}_q(M) = \inf_{\mathrm{rank}(B) \leq q-1} \|M - B\|_2$$

# A kind of compress sensing argument

For any matrix $A$ we have $\|A\|_2^2 = q\,\mathbb{E}\left(\|AX\|^2\right)$ for $X$ random matrix with standard Gaussian independent entries of dimensions $n \times q$

☞ For a typical realization of $X$ one should have

$$\begin{aligned} \mathbb{P}\left(D_q(M) \leq \epsilon\right) &\approx_q \mathbb{P}\left(D_q(MX) \leq \epsilon\right) \\ &\approx_q \mathbb{P}\left(\inf_{a \in \mathcal{S}^{q-1}} \|MX^t a\| \leq \epsilon\right) \end{aligned}$$

Dealing with a fixed $a \in \mathcal{S}^{q-1}$:

We consider $Y \sim \mathcal{N}(0, \mathbf{I}_n)$ then

$$F \approx \mathcal{N}(0,1) \;\&\; F \in \mathscr{W}_m \Rightarrow \frac{1}{\sqrt{m(m-1)}} {}^tYMX \approx \mathcal{N}(0, \mathbf{I}_q)$$

$$\underbrace{\Rightarrow}_{\text{conditionning}} \gamma_{n,q}\left(\chi \in \mathcal{M}_{n,q}(\mathbb{R}) \mid \frac{1}{\sqrt{m(m-1)}} {}^tYM\chi \approx \mathcal{N}(0, \mathbf{I}_q)\right) \approx 1$$

Then for $\chi$ typical and $a \in \mathcal{S}^{q-1}$ one should have

$$\frac{1}{\sqrt{m(m-1)}} {}^tYM\chi {}^ta \approx \mathcal{N}(0,1).$$

# Reasoning by induction on $m$

Set $M\chi a = (z_1, \cdots, z_n)$,

$$\mathbb{E}\left(e^{it\frac{1}{\sqrt{m(m-1)}}{}^tYM\chi^t a}\right) \approx \mathbb{E}\left(\exp\left(-\frac{t^2}{2}\sum_i z_i^2\right)\right)$$
$$= \mathrm{O}\left(\frac{1}{t^p}\right),\ p >> 1.$$

Where we use that ${}^tYM\chi^t a$ may be seen as an element of $\mathscr{W}_{m-1}$ so we can proceed by induction.

☞ $\mathbb{P}\left(\|M\chi^t a\| \leq \epsilon\right) \leq \epsilon^p$ for large $p$.

☞ We manage the inversion of inf and $\mathbb{P}$ using a discretization of the sphere, the union bound and optimizing the constants.