On a CLT associated with the intrinsic volumes of a convex body

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(joint work with Valentin Garino)

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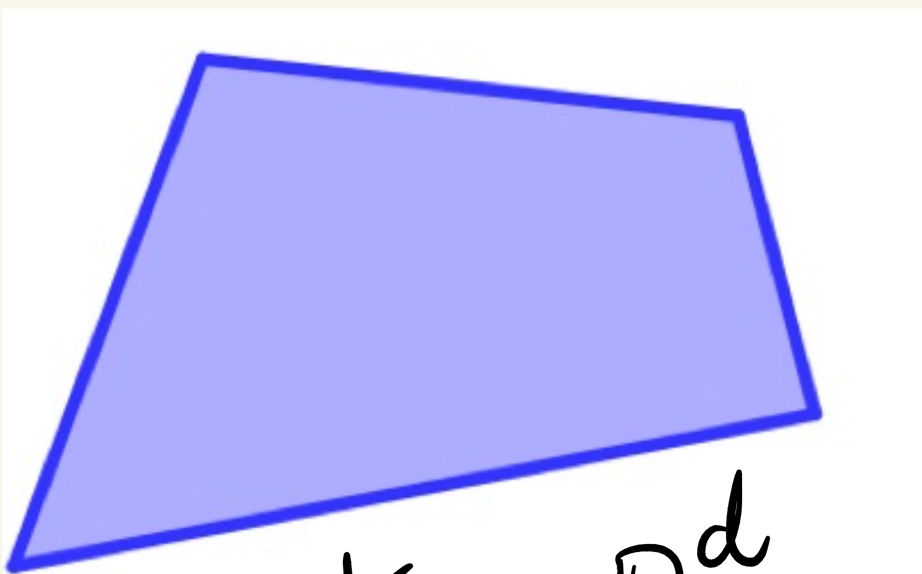
Heroes of this talk:

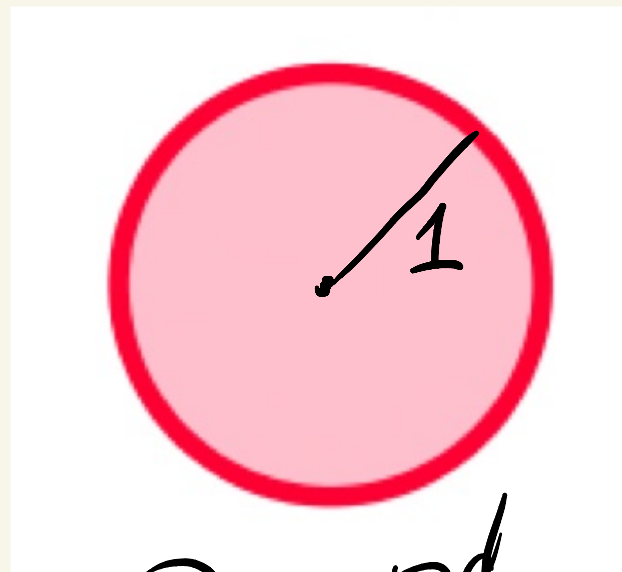
Intrinsic volumes of a **convex body***

(*compact convex set with non-empty interior)



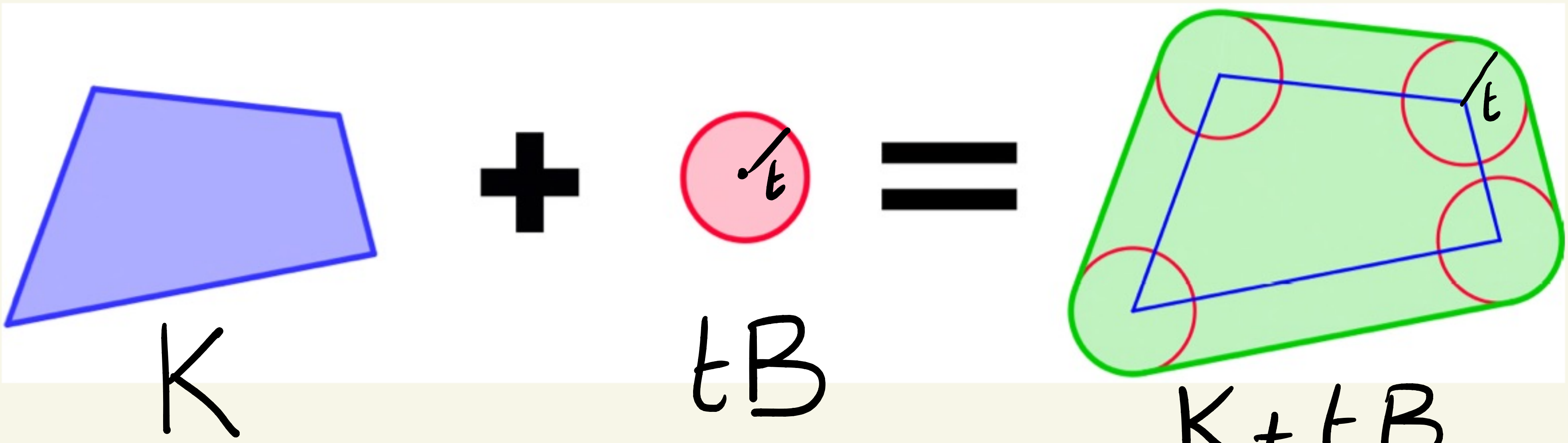
$K \subset \mathbb{R}^d$

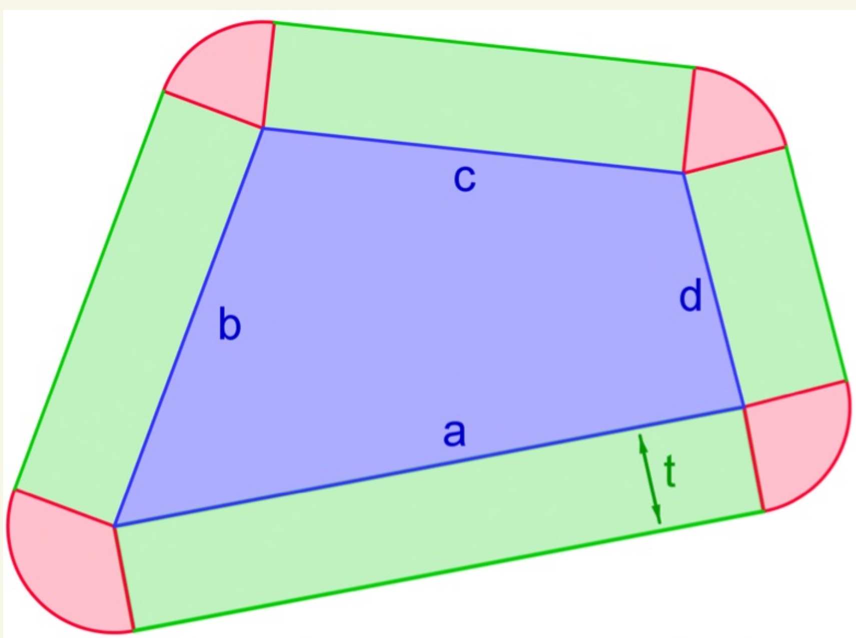
convex body



$B \subset \mathbb{R}^d$

unit ball





$$\mathrm{Vol}(K + tB)$$
$$= \mathrm{Vol}(K)$$
$$+ t\,\mathrm{Per}(K)$$
$$+ \pi t^2$$

Steiner formula in $\mathbb{R}^d$

$\mathrm{Vol}(K + tB)$ is a polynomial in t of degree d

$$\mathrm{Vol}(K+tB) = \sum_{k=0}^{d} \boxed{a_k(K)}\, t^k$$

$$a_k(K) = \boxed{\kappa_k \times v_{d-k}(K)}$$

$(v_k(K))_{0\le k\le d}$ intrinsic volumes of K

$\kappa_k = \dfrac{\pi^{k/2}}{\Gamma(1+k/2)}$ = volume of unit ball $\subset \mathbb{R}^k$

Intrinsic volumes measure how "big" is K.

Commonly encountered intrinsic volumes

$v_0(K) = 1$

$v_1(K) =$ intrinsic width (ex: $\underbrace{v_1(B) \sim \sqrt{2\pi d}}_{\text{will be used later on!}}$)

$\vdots$

$v_{d-1}(K) = \frac{1}{2} \times$ surface area of K

$v_d(K) =$ d-dim. volume of K

Some properties satisfied by intrinsic volumes:

- $v_k(K) \geq 0$ [positivity]
- $v_k(tK) = t^k v_k(K)$ [homog.]
- $v_k(K \cup L) + v_k(K \cap L) = v_k(K) + v_k(L)$ [valuation]
- $K \subset L \Rightarrow v_k(K) \leq v_k(L)$ [$\nearrow$]
- $v_k(\underset{\mathbb{R}^d}{\underset{\cap}{K}} \times \underset{\mathbb{R}^m}{\underset{\cap}{\{0\}}}) = v_k(\underset{\mathbb{R}^d}{\underset{\cap}{K}})$ [intrinsic]

raison d'être of κ_k!

Many nice (and deep) inequalities are available for intrinsic volumes

Ex: From the isoperimetric inequality stating that «among all sets $A \subset \mathbb{R}^d$ with a given volume, the ball has the minimal surface area» Otherwise stated:

if $r \geq 0$ is s.t. $V_d(K) = V_d(rB)$ then $V_{d-1}(K) \geq V_{d-1}(rB)$

$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

$$r = \left(\frac{V_d(K)}{V_d(B)}\right)^{\frac{1}{d}} \qquad\qquad r \leq \left(\frac{V_{d-1}(K)}{V_{d-1}(B)}\right)^{\frac{1}{d-1}}$$

More generally [Alexandrov]

$$\left(\frac{V_j(K)}{V_j(B)}\right)^{\frac{1}{j}} \leq \left(\frac{V_i(K)}{V_i(B)}\right)^{\frac{1}{i}}$$

$$\forall j \geq i$$

Chevet, Mc Mullen (independently):

$\{ j! \, V_j(K) \}_{0 \leq j \leq d}$ is log-concave

$\downarrow$

$[(x_j)$ LC if $x_j^2 \geq x_{j+1} x_{j-1} \ \forall j]$

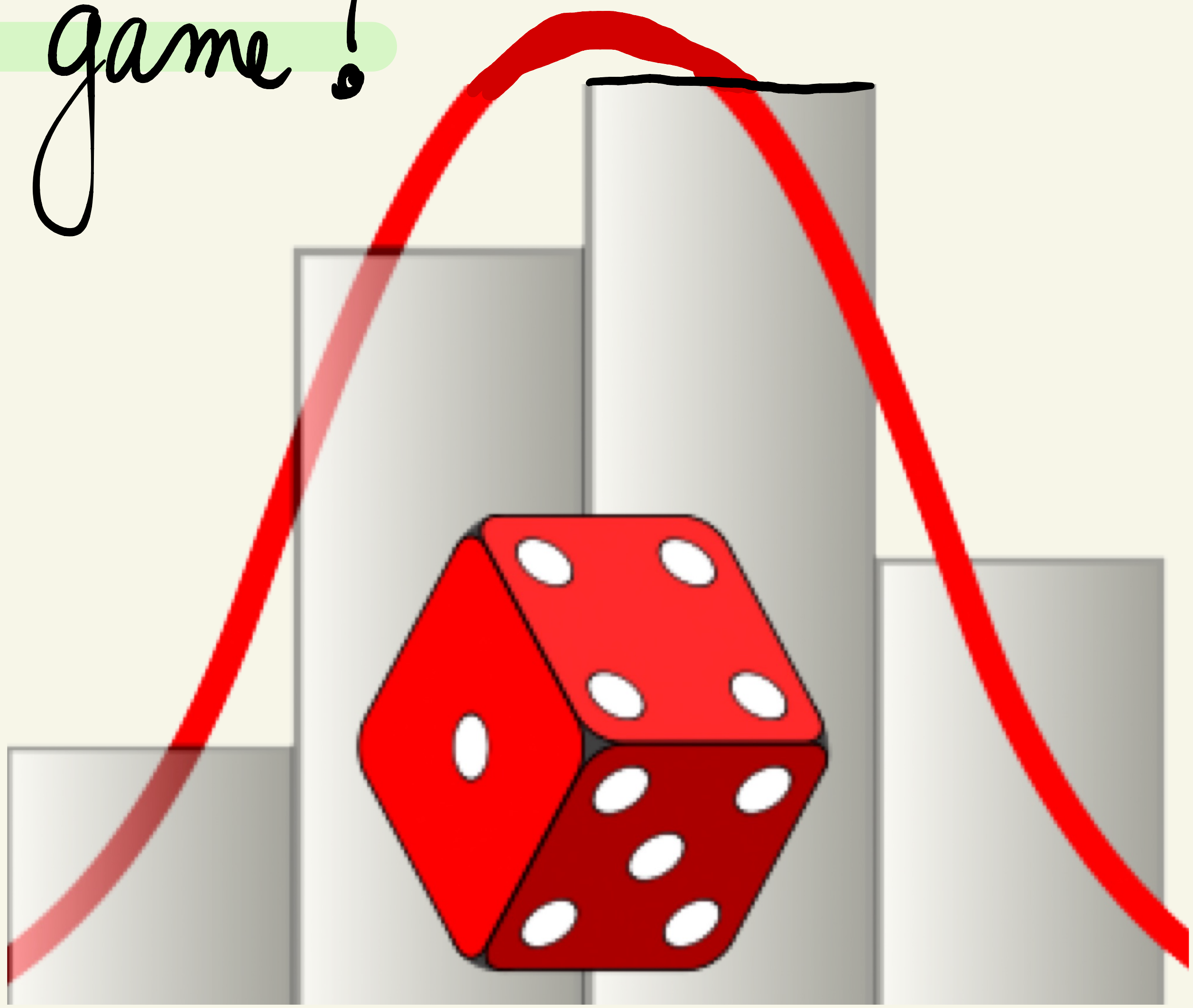
$\Longleftrightarrow \{ V_j(K) \}_{0 \leq j \leq d}$ is ultra log-concave

$\downarrow$

$[(x_j)$ ULC if $j x_j^2 \geq (j+1) x_{j+1} x_{j-1} \ \forall j]$

END of the geometric part of my talk! //

Now, let's probability enter the game!



Set $W(K) = \sum_{k=0}^{d} v_k(K)$ $\geqslant 0$

for Wills $\left(= \int_{\mathbb{R}^d} e^{-\pi \, \mathrm{dist}^2(x,K)} \, dx \right)$

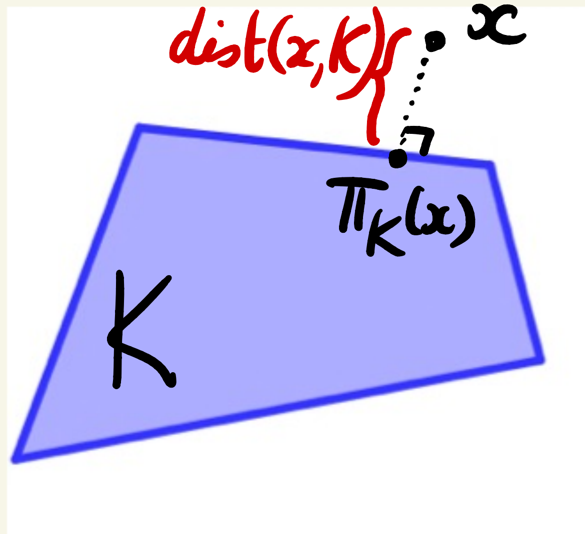
Consider $Z_K : \Omega \to \mathbb{R}_+$ distributed as

$$\mathbb{P}(Z_K = k) = \frac{v_k(K)}{W(K)}, \quad 0 \leq k \leq d$$

Qu: Asymptotic behavior of Z_K as $d \to \infty$?

Ex: $K = [0,1]^d \rightsquigarrow v_k(K) = \binom{d}{k} \rightsquigarrow Z_K$ satisfies a CLT.

FACT:
$$\mathrm{dist}^2(x,K) = \|x - \pi_K(x)\|^2$$



In Lotz, McCoy, Nourdin, Peccati, Tropp (2019): Steiner $\Longleftrightarrow$

$$\boxed{\begin{aligned}&\int_{\mathbb{R}^d} g(\pi\,\mathrm{dist}^2(x,K))\, e^{-\pi\,\mathrm{dist}^2(x,K)}\,dx\\ &= g(0)\,v_d(K) + \sum_{j=0}^{d-1} \frac{v_j(K)}{\Gamma\left(\frac{d-j}{2}\right)} \int_0^\infty g(r)\, r^{-1+\frac{d-j}{2}} e^{-r}\,dr\end{aligned}} \quad \text{(LM+)}$$

For $g \equiv 1$, we recover Wills' result: $\displaystyle\int_{\mathbb{R}^d} e^{-\pi\,\mathrm{dist}^2(x,K)}\,dx = W(K)$

Let $X_K : \Omega \to \mathbb{R}^d$ have the (log-concave) density

$$\frac{1}{W(K)} e^{-\pi \operatorname{dist}^2(x,K)}, \quad x \in \mathbb{R}^d.$$

Define $H_K : \Omega \to \mathbb{R}_+$ as $\boxed{H_K := \pi \operatorname{dist}^2(X_K, K)}$

Hadwiger–Wills information content of X_K

Law of H_K (continuous log-concave) is equivalent to Law of Z_K (discrete)

<u>Q4</u>: Asymptotic behavior of H_K?

Lemma (probalistic version of LM+)

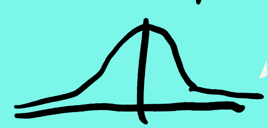
$$H_K \overset{(\mathcal{L})}{=} \sum_{j=1}^{d-Z_K} \gamma_j$$

where • $\gamma_j \sim \Gamma\left(\frac{1}{2}, 1\right)$ are $\perp\!\!\!\perp$ (Rk. $\sum_{j=1}^{k} \gamma_j \sim \Gamma\left(\frac{k}{2}, 1\right)$)

$\perp\!\!\!\perp$ •• $P(Z_K = j) = \dfrac{v_j(K)}{W(K)}$, $0 \leq j \leq d$

Prop

If $\dfrac{\mathrm{Var}(Z_K)}{d - E(Z_K)} \simeq 0$ then $\dfrac{H_K - E H_K}{\sqrt{\mathrm{Var}\, H_K}} \simeq N(0,1)$



LM (Garino, N.) If $K \subset d^{\alpha} B$ with $0 < \alpha < \frac{1}{2}$ then $\frac{\mathrm{Var}(Z_K)}{d - \mathbb{E}(Z_K)} \simeq 0$

(as $d \to \infty$)

Proof: • Set $x_j = \frac{v_j(K)}{W(K)}$. (x_j) is ULC, i.e. $j x_j^2 \geqslant (j+1) x_{j-1} x_{j+1}$

$$\Leftrightarrow \frac{1}{j+1} \frac{x_j}{x_{j+1}} \geqslant \frac{1}{j} \frac{x_{j-1}}{x_j}$$

We deduce

$$\frac{x_j}{x_{j+1}} - \frac{x_{j-1}}{x_j} \geqslant \left(\frac{j+1}{j} - 1\right) \frac{x_{j-1}}{x_j} = \frac{1}{j} \frac{x_{j-1}}{x_j} \geqslant \frac{x_0}{x_1}$$

Summing (dominos ☺): $\frac{x_j}{x_{j+1}} - \frac{x_0}{x_1} \geqslant j \frac{x_0}{x_1} \Rightarrow \frac{x_j}{x_{j+1}} \geqslant (j+1) \frac{x_0}{x_1} = \frac{j+1}{v_1(K)}$

$$\Rightarrow \mathbb{E}(Z_K) = \sum_{j=0}^{d-1} (j+1) x_{j+1} \leq v_1(K) \sum_{j=0}^{d-1} x_j \leq v_1(K)$$

• For the variance (Johnson) let $(x_j) \subset \mathbb{R}_+$ be such that $\sum x_j = 1$.

Let $Z : \Omega \to \mathbb{R}_+$ be distributed according to $P(Z=j) = x_j$.

If (x_j) is c-logconcave (that is, $\frac{x_j}{x_{j+1}} - \frac{x_{j-1}}{x_j} \geq c$)

[discrete Bakry–Emery]

then $\mathrm{Var}(f(Z)) \leq \frac{1}{c} \sum_{j=0}^{\infty} x_j \left(f(j+1) - f(j) \right)^2$.

In particular: $\mathrm{Var}(Z) \leq \frac{1}{c}$

COR $\mathrm{Var}(Z_K) \leq v_1(K)$

Proof: $\left(\frac{v_j(K)}{W(K)}\right)$ is $\frac{1}{v_1(K)}$-logconcave & Johnson $\Rightarrow$

CQFD.

Summary:

We assume $K \subset d^{\alpha} B$ with $\alpha < \frac{1}{2}$

We need to prove $\frac{\mathrm{Var}(Z_K)}{d - \mathbb{E} Z_K} \simeq 0$

We have shown that $\begin{cases} \mathbb{E} Z_K \leq v_1(K) \\ \mathrm{Var}\, Z_K \leq v_1(K) \end{cases}$

$\bullet\bullet\bullet$ $K \subset d^\alpha B$ with $\alpha < \frac{1}{2}$

$$\Rightarrow v_1(K) \le v_1(d^\alpha B) \underset{\text{homogeneity}}{=} d^\alpha v_1(B) \underset{d\to\infty}{\sim} \sqrt{2\pi}\, d^{\overbrace{\alpha + \frac{1}{2}}^{<1}}$$

$$\Rightarrow \mathbb{E}(Z_K) = o(d) \Rightarrow d - \mathbb{E}(Z_K) \sim d$$

$$\Rightarrow \frac{\mathrm{Var}(Z_K)}{d - \mathbb{E}(Z_K)} = O\left(d^{\alpha + \frac{1}{2} - 1}\right) = O\left(d^{\alpha - \frac{1}{2}}\right) \longrightarrow 0$$

$\blacksquare$

COR. If $K \subset d^\alpha B$ with $\alpha < \frac{1}{2}$ then $\dfrac{H_K - \mathbb{E}H_K}{\sqrt{\mathrm{Var}\, H_K}} \xrightarrow{\mathcal{L}} N(0,1)$

Qu: Can we prove convergence in TV?

$\rightsquigarrow$ Stein's method!

Notation: $X \sim e^{-\phi(x)}dx$ with $\phi : \mathbb{R}^d \to \mathbb{R}$

$Y = \nabla\phi(X)$

($\hat{Y}$ ind. copy / $Y_t = e^{-t}Y + \sqrt{1-e^{-2t}}\,\hat{Y}$)

$H : \mathbb{R}^d \to \mathbb{R} \rightsquigarrow F = \dfrac{H(Y) - \mathbb{E}H(Y)}{\sqrt{\underbrace{\mathrm{Var}\,H(Y)}_{=\sigma^2}}}$

Prop: $\boxed{d_{TV}(F, N(0,1)) \leq A + B}$ where

$$A = \frac{CST}{\sigma^2}\sqrt{\mathrm{Var}\left(\int_0^\infty e^{-t}\langle (\mathrm{Hess}\,\phi)(X)\,\nabla H(Y), E_\infty(\nabla H(Y_t))\rangle dt\right)}$$

$$B = \frac{CST}{\sigma}\int_0^\infty e^{-2t}\sqrt{\mathbb{E}\left[\mathrm{Tr}\left(\left((\mathrm{Hess}\,\phi)(X) - (\mathrm{Hess}\,\phi)(X_\infty)\right)(\mathrm{Hess}\,H)(Y_t)\right)^2\right]}\,dt$$

Prop: $\boxed{d_{TV}(F, N(0,1)) \le A + B}$ where

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1er cas : $\phi(x) = \frac{1}{2}\|x\|^2 + \frac{d}{2}\ln(2\pi)$, that is $X \sim N(0, I_d)$

$Y = X$ and $F = \frac{H(X) - \mathbb{E}H(X)}{\sqrt{\mathrm{Var}\,H(X)}}$. In this case :

$$A = \frac{CST}{\sigma^2}\sqrt{\mathrm{Var}\left(\int_0^\infty e^{-t}\langle \nabla H(Y), \mathbb{E}_\infty(\nabla H(Y_t))\rangle dt\right)}$$

$$B = 0$$

Malliavin - Stein

$\mathrm{Var}(\langle DF, -DL^{-1}F\rangle)$

Prop: $\boxed{d_{TV}(F, N(0,1)) \le A + B}$ where

$$A = \frac{CST}{\sigma^2}\sqrt{\mathrm{Var}\left(\int_0^\infty e^{-t}\langle (\mathrm{Hess}\,\phi)(X)\,\nabla H(Y), E_\infty(\nabla H(Y_t))\rangle dt\right)}$$

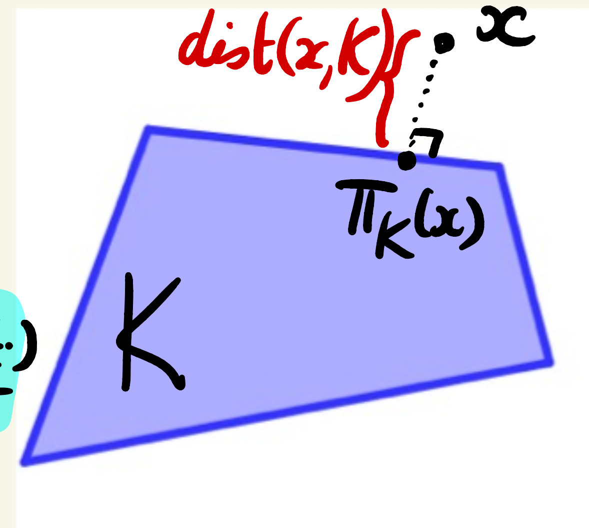
$$B = \frac{CST}{\sigma}\int_0^\infty e^{-2t}\sqrt{E\left[\mathrm{Tr}\left(\left((\mathrm{Hess}\,\phi)(X) - (\mathrm{Hess}\,\phi)(X_\infty)\right)(\mathrm{Hess}\,H)(Y_t)\right)^2\right]}dt$$

2^e cas [Our case!] $H(x) = \|x\|^2$

$\hookrightarrow \phi(x) = \pi\, \mathrm{dist}^2(x, K) + \ln(W(K))$

$y = \nabla\phi(x) = \pi(x - \pi_K(x))$

$H(\nabla\phi(x)) = \pi^2\, \mathrm{dist}^2(x, K) \rightsquigarrow F = \dfrac{\mathrm{dist}^2(X_K, K) - E(\cdot\cdot)}{\sigma(\cdot\cdot)}$



$$A = \frac{CST}{\sigma^2}\sqrt{\mathrm{Var}\left(\int_0^\infty e^{-t}\langle Y, E_\infty(Y_t)\rangle dt\right)}$$

$$B \le \frac{CST}{\sigma}\sqrt{\mathrm{Var}(\mathrm{Tr}(\mathrm{Hess}\,\phi(x)))}$$

THM. (Garino, Nourdin)

If $K \subset d^{\alpha} B$ with $\alpha < \frac{1}{2}$

then $d_{TV}\left(\frac{H_K - \mathbb{E} H_K}{\sqrt{\mathrm{Var}\, H_K}}, N(0,1) \right) = O(d^{\alpha - \frac{1}{2}})$

Proof: Second order Brascamp-Lieb inequality & (many) other things!

Open question: do we have a CLT for Z_K?

(In LM+, we studied concentration inequalities)

END!