Workshop Calculus of Variations and applications

June 19th-21st 2023

1 Practical information

All the information contained in this document are also available on the website of the workshop.

Website of the workshop: https://indico.math.cnrs.fr/event/9091/

Organizers:

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Venue:

All the talks will take place at the University Paris Cité (near Metro "Bibliotheque"), in the Amphi 4C, Halles aux Farines. It is the building **number 4 on the map** of next page.

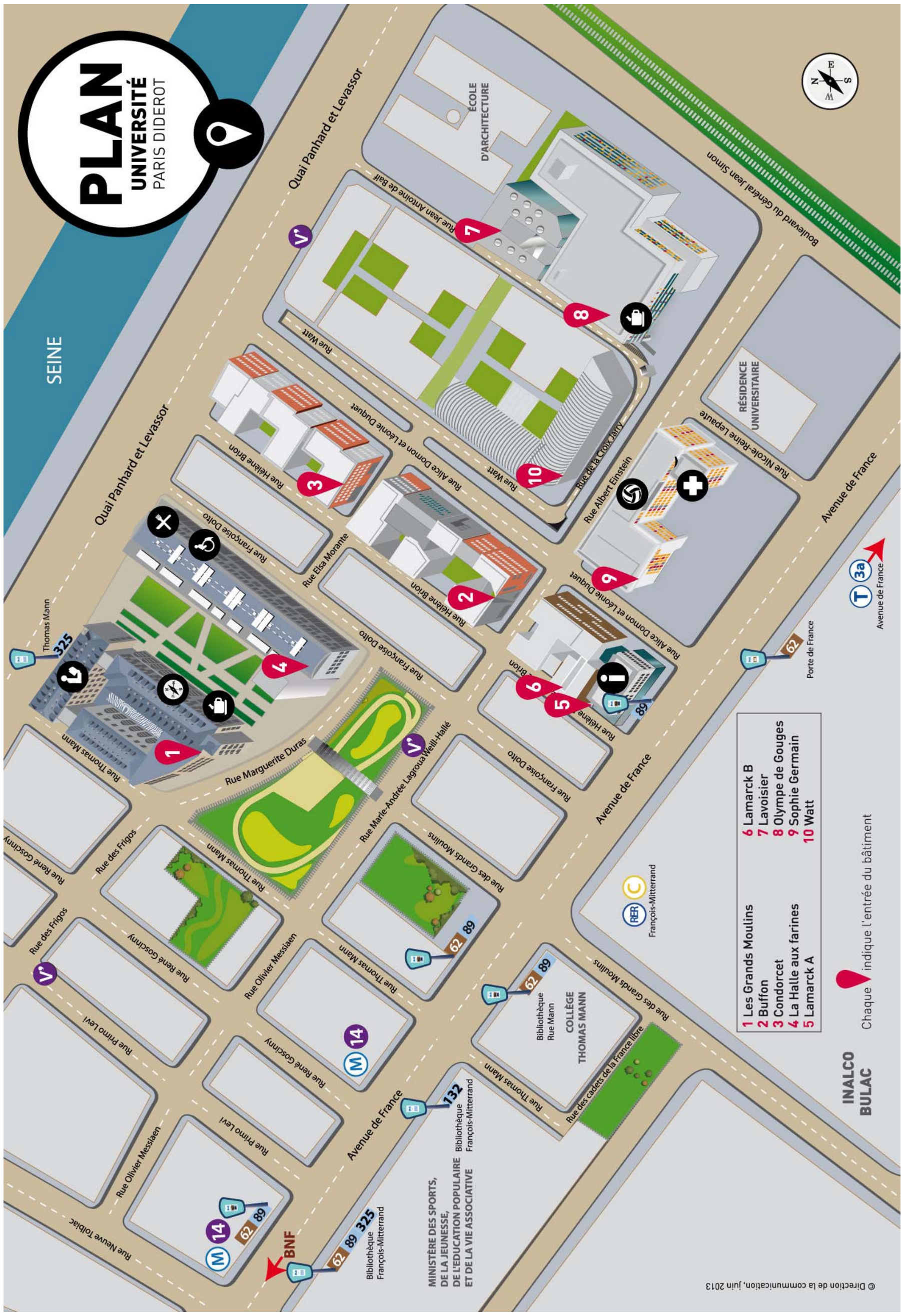
The nearest metro station is Bibliothèque François Mittérand, Line Metro 14 and Line RER C. You can also access by TRAM 3A and get out at “Avenue de France" or even by BUS.

Restaurant for lunch. All registered (except late registrations) participants are invited for lunch at Restaurant Buffon (17 rue Hélène Brion, Paris

13ème). Lunch time is at **12h00**.

Social-dinner. After the talks on Tuesday 20th, the participants are invited for dinner at Bouillon République, 39 Bd du Temple 75003, Paris 3ème Arrondissement. Meeting time: 18:45 at the restaurant.

Wi-Fi. In order to have access to internet during the workshop, please make sure that you have access to the **"eduroam"** network.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7 Lavoisier
8 Olympe de Gouges
9 Sophie Germain
10 Watt
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2 Schedule

Monday June 19th

- 9h–9h50: S. Serfaty :
  Vortex lines in 3D Ginzburg-Landau with magnetic field.

- 9h50–10h10: Coffee break.

- 10h10–11h: L. De Luca :
  Stability results for fractional parabolic flows.

- 11h–11h50: V. Millot :
  Torus and split solutions of the Landau-de Gennes model for nematic liquid crystals.

- 12h-14h: Lunch at Restaurant Buffon

- 14h–14h50: S. Conti :
  Laminates with variable volume fraction in shape-memory alloys.

- 14h50–15h20: L. Schmeller :
  Gel models for phase separation at finite strains.

- 15h20–15h50: Coffee break

- 15h50–16h20: A. Vikelis :
  Measure-valued solutions for non-associative finite plasticity.

- 16h20–17h10: B. Zwicknagl :
  Variational models for pattern formation: from helimagnets to shape-memory alloys.

Tuesday June 20th

- 9h–9h50: S. Di Marino :
  The shape of Kantorovich potentials: on the existence of minimizers for the Lieb-Oxford inequality

- 9h50–10h10: Coffee break.

- 10h10–11h: M. Friedrich :
  Equilibrium configurations for epitaxially strained crystalline films.

- 11h–11h50: P. Pegon :
  Asymptotics for optimal quantization in branched optimal transport.

- 12h-14h20: Lunch at Restaurant Buffon

- 14h20–14h50: Liangjun Weng :
  A constrained mean curvature type flow and isoperimetric type inequalities.

- 14h50–15h20: E. Martinet :
  Numerical maximization of Neumann eigenvalues on domains on the sphere.

- 15h20–15h50: Coffee break

- 15h50–16h20: A. Marveggio :
  Uniqueness and stability of planar multiphase mean curvature flow beyond a circular topology change.

- 16h20–17h10: T. Laux :
  The large-data limit of the MBO scheme for data clustering.

- 18h45– Social Dinner at Bouillon République

Wednesday June 21st

- 9h–9h50: M.G. Mora :
  Explicit minimizers for anisotropic Coulomb energies.

- 9h50–10h10: Coffee break.

- 10h10–10h40: J. Machado :
  1D approximation of measures in Wasserstein spaces.

- 10h40–11h10: A. Dumas :
  Existence and Lipschitz regularity of the trajectories minimizing the total variation in a congested setting.

- 11h10–12h: A. Lemenant :
  Epsilon-regularity for Griffith minimizers.

- 12h-14h: Lunch at Restaurant Buffon

3 Titles and abstract

Sergio Conti, Universität Bonn

Title. Laminates with variable volume fraction in shape-memory alloys.

Abstract. I will discuss a singularly perturbed variational model for single laminates in shape-memory alloys, with boundary conditions that induce a position-dependent volume fraction. The scaling of the minimum value of the (geometrically linear) energy with respect to the surface energy density is determined by an explicit upper bound and an ansatz-free lower bound, both for a Dirichlet and for a Neumann problem. The lower bound builds upon a rigidity estimate for functions of bounded deformation. This talk is based on joint work with R. V. Kohn and O. Misiats.

Lucia De Luca, IAC-CNR, Roma

Title. Stability results for fractional parabolic flows.

Abstract. We present an abstract method for studying the stability of parabolic flows, exploiting the Γ-convergence of the corresponding energy functionals. We apply such a result to analyse the behavior of the s-fractional heat flows, as s tends to 0 and to 1, and of the s-Riesz flows, as s tends to 0 and to d (where d is the dimension of the ambient space). Time permitting, we present also stability results for the corresponding geometric flows.

Simone Di Marino, Università di Genova

Title. The shape of Kantorovich potentials: on the existence of minimizers for the Lieb-Oxford inequality

Abstract. We explain the connection between the classical Lieb-Oxford inequality and multimarginal optimal transport with repulsive cost. We can see that the first order condition is linked with the Kantorovich potential, and we show, through a detailed analysis of the shape of the potentials, that if a minimizer exists, then it should be compactly supported, extending the case N=1 which was already settled by Lieb and Oxford in their original contribution.

This is a work in preparation with R. Lelotte (U. Paris-Dauphine)

Annette Dumas, Université Claude Bernard, Lyon

Title. Existence and Lipschitz regularity of the trajectories minimizing the total variation in a congested setting

Abstract. The problem I will present is motivated by the study of a Mean Field Game model whose theory was simultaneously introduced by Lasry and Lions ([1, 2, 3]) and Caines, Huang and Malhamé ([4]) in 2006. The model consists in studying a population in a city where each agent jumps to move from one place to another. Each inhabitant minimizes a cost composed of the number of jumps and an increasing function of the density of the population. The solution to this problem is a probability measure on the trajectories which is a Nash equilibrium.

The probability measure Q on the trajectories can be seen as a trajectory of the density of the population $\rho_t = e_t \sharp Q$ which leads us to the following variational problem:

$$\min_{\substack{\rho \in E, \rho \geq 0 \\ \forall t \in [0,T],\ \int_\Omega \rho(t,x)dx = 1}} \int_0^T \int_\Omega \left(|\partial_t \rho(t,x)| + V(t,x)\rho(t,x) + f(\rho(t,x)) \right) dxdt + \psi_0(\rho(0)) + \psi_T(\rho(T)),$$

where $E = BV([0,T], L^1(\Omega)) \cap L^2([0,T] \times \Omega)$. The number of jumps is associated with the total variation of the density and the additional cost is associated with the increasing function of the density. Density constraints are also added to the problem. We will see that the solution exists, is unique and is Lipschitz in time, despite the discontinuous trajectories taken by each agent. With additional hypothesis on the data, boundedness or continuity in space can be obtained with Dirichlet conditions in time.

The aspect of the solutions are given by the Euler-Lagrange equations which show that in space, either the solution is constant, or it follows the critical points of the cost. Numerical simulations are carried out on a simple example by using the fast dual proximal gradient method from [5] which validates the theoretical framework.

Manuel Friedrich, FAU Erlangen-Nürnberg

Title. Equilibrium configurations for epitaxially strained crystalline films

Abstract. In this talk, we revisit results obtained on the existence of minimizers and relaxation for energies related to epitaxially strained crystalline films. We first extend the analysis to the framework of three-dimensional linear elasticity. Afterwards, we discuss a rigorous relation between models in nonlinear and linearized elasticity for both continuum and atomistic energies.

Based on joint works with Vito Crismale, Leonard Kreutz, and Konstantinos Zemas.

Tim Laux, Universität Bonn

Title. The large-data limit of the MBO scheme for data clustering

Abstract. The MBO scheme is an efficient algorithm for data clustering, the task of partitioning a given dataset into several meaningful clusters. In this talk, I will present the first rigorous analysis of this scheme in the large-data limit. The starting point for the first part of the talk is that each iteration of the MBO scheme corresponds to one step of implicit gradient descent for the thresholding energy on the similarity graph of the dataset. It is then natural to think that outcomes of the MBO scheme are (local) minimizers of this energy. We prove that the algorithm is consistent, in the sense that these (local) minimizers converge to (local) minimizers of a suitably weighted optimal partition problem. To study the dynamics of the scheme, we use the theory of viscosity solutions. The main ingredients are (i) a new abstract convergence result based on quantitative estimates for heat operators and (ii) the derivation of these estimates in the setting of random geometric graphs. To implement the scheme in practice, two important parameters are the number of eigenvalues for computing the heat operator and the step size of the scheme. Our results give a theoretical justification for the choice of these parameters in relation to sample size and interaction width. This is joint work with Jona Lelmi (U Bonn).

Antoine Lemenant, Université de Lorraine

Title. Epsilon-regularity for Griffith minimizers

Abstract. In this talk I will present some recent advances concerning the C^1 regularity of minimizers for the vectorial free-discontinuity problem of Griffith. In particular I will try to explain the strategy of proof inspired by the Reifenberg-flat theory, relying on a geometric stopping time argument on the flatness, coupled with a general extension lemma, which was employed in our latest result valid for any dimension $N > 2$. This is a recent joint work with C. Labourie, and generalizes, with a different proof, a previous 2 dimensional result obtained in collabaration with J.-F. Babadjian and F. Iurlano.

Joao Miguel Machado, Université Paris-Dauphine

Title. 1D approximation of measures in Wasserstein spaces

Abstract. Given a Borel probability measure $\rho_0 \in \mathcal{P}(\mathbb{R}^d)$, we seek to

approximate it with a measure uniformly distributed over a 1-dimensional set. We study the following variational problem

$$\inf_{\Sigma\in\mathcal{A}} W_p^p\left(\rho_0, \frac{1}{\mathcal{H}^1(\Sigma)}\mathcal{H}^1 \llcorner \Sigma\right) + \Lambda\mathcal{H}^1(\Sigma), \qquad (P_\Lambda)$$

where $\mathcal{A}$ corresponds to the class of all closed and connected subsets of $\mathbb{R}^d$. Here W_p denotes the Wasserstein distance on the space of probability measures and $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure in $\mathbb{R}^d$, which forces competitors with finite energy to be 1D.

To show existence of solution to (P_Λ) one cannot easily resort to the direct method in the calculus of variations as the set of measures of the form $\mathcal{H}^1 \llcorner \Sigma$ is not closed. Therefore, we propose a relaxed problem in $\mathcal{P}(\mathbb{R}^d)$ which always admits a solution ν with the properties:

1. If $\rho_0 \ll \mathcal{H}^1$ or has a L^∞ density w.r.t. $\mathcal{H}^1$ (*e.g.* $\rho_0 \ll \mathcal{L}^d$), then so does ν.

2. For $d = 2$, if ρ_0 does not give mass to $1D$ sets, then ν has a uniform density and therefore is a solution to the original problem (P_Λ).

3. If $\rho_0 \in L^{\frac{d}{d-1}}(\mathbb{R}^d)$, then $\Sigma := \operatorname{supp}\nu$ is Ahlfors regular, *i.e.* there is $r_0 > 0$ and $0 < C$ such that for any $x \in \Sigma$ and $r \le r_0$ it holds that

$$r \le \mathcal{H}^1(\Sigma \cap B_r(x)) \le Cr.$$

Eloi Martinet, Université Savoie Mont Blanc

Title. Numerical maximization of Neumann eigenvalues on domains on the sphere

Abstract. We consider the numerical optimization of the first three eigenvalues of the Laplace-Beltrami operator of domains on the sphere with Neumann boundary conditions. We adress two approaches : one is a shape optimization procedure via the level-set method and the other one is a relaxation of the initial problem leading to a density method. These computation gives some strong insight on the optimal shapes of those eigenvalue problems and shows a rich variety of shapes regarding the proportion of the surface area of the sphere occupied by the domain.

Alice Marveggio, IST Vienna

Title. Uniqueness and stability of planar multiphase mean curvature flow beyond a circular topology change

Abstract. The evolution of a network of interfaces by mean curvature flow features the occurrence of topology changes and geometric singularities. As a consequence, classical solution concepts for mean curvature flow are in general limited to short-time existence theorems, which include singular times only for some stable shrinkers such as the circle. At the same time, the transition from strong to weak solution concepts (e.g. Brakke solutions) may lead to non-uniqueness of solutions.

Following the relative energy approach à la Fischer-Hensel-Laux-Simon and introducing a suitable notion of gradient-flow calibration for a shrinking circle, we prove a quantitative stability estimate holding up to the singular time. This implies a weak-strong uniqueness principle for weak BV solutions to planar multiphase mean curvature flow beyond circular topology changes.

Furthermore, we expect our method to have further applications to other types of shrinkers, as well as to prove quantitative convergence of diffuse-interface (Allen-Cahn) approximations for mean curvature flow.

This is work in progress with Julian Fischer, Sebastian Hensel and Maximilian Moser.

Vincent Millot, Université Paris-Est Créteil Val de Marne

Title. Torus and split solutions of the Landau-de Gennes model for nematic liquid crystals

Abstract. In this talk, I will present the Q-tensor model of Landau-de Gennes for nematic liquid crystals in the so called Lyutsyukov regime dealing with maps with values in the 4-dimensional sphere. This model describes stable configurations of a liquid crystal as minimizers of a Ginzburg-Landau type energy in which the potential well is the real projective plane, seen as a submanifold of S4. In the case where the 3D domain is the unit ball and the Dirichlet boundary data is radially symmetric (equivariantly), one may expect that a minimizer inherits such symmetry. Simulations show that this is not the case and a certain toroidal structure is expected to appear. If (equivariant) axial symmetry is imposed to reduce the complexity of the problem, another type of « singular » solutions appears, the split solutions. By means of regularity results on this model, I will discuss the existence / geometry of torus and split solutions and explain the strong dependence of the type of solutions with respect to the boundary condition and the shape of

the domain. This talk is based on recent works in collaboration with Federico Dipasquale and Adriano Pisante.

Maria Giovanna Mora, Università di Pavia

Title. Explicit minimizers for anisotropic Coulomb energies

Abstract. Nonlocal interaction energies play a pivotal role in describing the behavior of large systems of particles, in a variety of applications. Traditionally, the focus of the mathematical literature on nonlocal energies has been on radially symmetric potentials, which model interactions depending on the mutual distance between particles. The mathematical study of anisotropic potentials, despite their natural occurrence in modeling interactions where a preferred direction of interaction is present, has on the other hand been very limited until recently. In this talk we will consider a general class of anisotropic energies of Coulomb type in three dimensions and give a complete characterization of their minimizers, under the sole assumption of non-negativity for the Fourier transform of the interaction kernel.

Paul Pegon, Université Paris-Dauphine

Title. Asymptotics for optimal quantization in branched optimal transport

Abstract. The problem of optimal quantization of measures consists in finding the best approximation of a given measure by an atomic measure with a fixed number of atoms, usually expressed through Wasserstein distances. One can formulate the same problem considering instead the irrigation distances of branched optimal transport, where the transport cost behaves as a concave power of the mass and depends on all the trajectories of the particles. We study the asymptotic behaviour of optimal quantizers for absolutely continuous measures as the number of atoms grows to infinity. We compute the limit distribution of the corresponding point clouds and show in particular a branched transport version of Zador's theorem. Moreover, we establish the asymptotic quasi-uniformity of optimal quantizers in terms of separation distance and covering radius of the atoms, when the measure is uniform. This is a joint work with Mircea Petrache.

Leonie Schmeller, WIAS Berlin

Title. Gel models for phase separation at finite strains

Abstract. Hydrogels are crosslinked polymer networks saturated in a liquid solvent and can be modeled as a two-phase system employing the phase

field approach. During swelling and squeezing, they un- dergo enormous volume changes, which requires finite strain models for realistic considerations. We analytically investigate the two-phase model for phase separation in a geometrically nonlinear elas- tic material. The coupled system of PDEs consists of a Cahn-Hilliard equation and a quasi-static mechanical force balance of the deforming gel. The phase field and the mechanics are coupled by a multiplicative decomposition of the deformation gradient, and time-dependent Dirichlet boundary conditions are imposed on the deformation field. Based on a time-incremental scheme, we derive existence theory of solutions in a suitable weak formulation. Using techniques from the calculus of variations and nonlinear PDE theory, we obtain further an existence result for the time-continuous problem under suitable assumptions.
This is a joint work with Marita Thomas within the DFG priority program SPP 2171 Dynamic wetting of flexible, adaptive and switchable substrates, project # 422786086 and within the MATH+ project AA2-9.

Sylvia Serfaty, Courant Institute, NYU

Title. Vortex lines in 3D Ginzburg-Landau with magnetic field

Abstract. In joint work with Carlos Román and Etienne Sandier, we study the onset of vortex lines in the three-dimensional Ginzburg-Landau model of superconductivity. We discuss the critical field at which the first lines appear, which is naturally connected to an "isoflux" problem. We study the optimal number of lines, their interaction, and derive a (Gamma)-limit problem for their arrangement.

Andreas Vikelis, University of Vienna

Title. Measure-valued solutions for non-associative finite plasticity

Abstract. The variational treatment of evolutionary non-associative elasto-plasticity at finite strains remains unexplored. In this direction, following the concept of energetic solutions, we present an existence result for measure-valued solutions of the quasistatic evolution problem which are stable and balance the energy. In particular, we apply a modification of the standard time-discretization scheme, considering Young measures generated by piece-wise constant interpolants of time-discrete solutions of a properly defined minimization problem. A key point in our analysis is the limit passage in the dissipation energy. The later calls for time-continuity properties of the stresses which are not expected in the quasistatic framework. To overcome this obstacle we introduce a regularization of the generalized stress in the definition of our energetic solutions. Joint work with Ulisse Stefanelli.

Liangjun Weng, University of Rome Tor Vergata

Title. A constrained mean curvature type flow and isoperimetric type inequalities

Abstract. In this talk, we will discuss the isoperimetric inequality and its high-order version – Alexandrov-Fenchel inequality, which dates back to Queen Dido in the ancient Carthage era. We introduce the quermassintegrals for compact hypersurfaces with capillary boundary from the variational viewpoint. Then by using a constrained mean curvature type flow, we can obtain some new isoperimetric type inequalities for compact hypersurfaces with capillary boundary.

Barbara Zwicknagl, HU Berlin

Title. Variational models for pattern formation: from helimagnets to shape-memory alloys

Abstract. We consider continuum variational models for pattern formation in helimagnetic compounds. The energy functional consists of a multi-well bulk energy regularized by a higher order interfacial energy, and arises from a frustrated spin model in the sense of Gamma-convergence. We derive the scaling law for the minimal energy in the case of incompatible boundary conditions. The scaling law indicates the formation of various branching-type patterns in certain parameter regimes. We in particular outline relations to well-studied variational models for martensitic microstructures. This talk is based on joint works with Janusz Ginster and Melanie Koser (both Humboldt-Universität zu Berlin).

References

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