IHES Summer school: Arbeitsgemeinschaft on syntomic cohomology

Some references:

- [BMS2] Bhatt–Morrow–Scholze *THH and integral p-adic Hodge theory.*
- [BS] Bhatt–Scholze *Prisms and prismatic cohomology.*
- [AMMN] Antieau–Mathew–Morrow–Nikolaus *On the Beilinson fiber square.*
- [BM] Bhatt–Mathew *Syntomic complexes and p-adic étale Tate twists.*
- [APC] Bhatt–Lurie *Absolute prismatic cohomology.*

The goal of this Arbeitsgemeinschaft is to explore the *syntomic cohomology* theory introduced in [BMS2], studied in [BS], [AMMN], [BM], etc and re-developed and extended beyond p-complete rings in [APC]. This cohomology theory associates to any qcqs scheme X a multiplicative family of p-complete cochain complexes:

$$\mathbb{Z}_p(j)^{\mathrm{syn}}(X) = R\Gamma_{\mathrm{syn}}(X, \mathbb{Z}_p(j)) \in D(\mathbb{Z}_p) \qquad \text{for } j \in \mathbb{Z}.$$

(Both notations are found in the literature.)

When p is invertible on X, syntomic cohomology identifies with the p-adic étale cohomology $R\Gamma_{\text{ét}}(X, \mathbb{Z}_p(j))$. The simplest point of view on syntomic cohomology is therefore that it provides a "good" extension of p-adic étale cohomology to schemes on which p is no longer necessarily invertible.

From a motivic perspective, $\mathbb{Z}_p(j)^{\mathrm{syn}}(X)$ should be seen as the p-adic completion of the weight-j étale motivic cohomology of X. If X is smooth over a field or over a mixed characteristic Dedekind domain, so that its motivic cohomology already makes sense, then this point of view is justified by precise comparison isomorphisms.

The Arbeitsgemeinschaft will essentially follow sections 7 and 8 of [APC]. In addition, foundations for the AG will be provided by a mini-course by Johannes Anschütz; he will explain the basics of prismatic cohomology via the stacky approach, including (transversal) prisms, quasiregular semiperfect rings, Nygaard filtration, Frobenius, Breuil–Kisin twists, etc.

Arbeitsgemeinschaft talks

The following are rough plans for the talks. Precise content is at the discretion of the speaker.

(1) Overview talk about syntomic cohomology in characteristic p, defined in terms of the étale cohomology of the sheaf $W_r\Omega^j_{X,\log}$. Relation to K-theory (Geisser–Hesselholt) and topological cyclic homology (BMS2 filtration). Chern classes. The overall goal of the AG is to extend this theory from characteristic p to mixed characteristic.

(2) Follow [APC, §7.4]. Definition of syntomic cohomology of $\operatorname{Spf} R$, where R is any commutative ring (even animated), as

$$R\Gamma_{\text{ét}}(\operatorname{Spf} R, \mathbb{Z}_p(j)) = \operatorname{hofib}\big(\operatorname{Fil}^j_N \Delta_R \xrightarrow{\varphi\{j\}-\iota} \Delta_R\big).$$

Present some of the main results of [APC, §7.4], especially the facts the syntomic cohomology satisfies p-complete fpqc descent and is p-completely left Kan extended. Mention that the former property means that syntomic cohomology globalises to p-adic formal schemes. Example: weight 0 following §8.1; also vanishing in negative weights.

(3) For any commutative ring R, use the prismatic logarithm to define the syntomic first Chern class [APC, 7.5.2/3]

$$c_1^{\mathrm{syn}} : R\Gamma_{\text{ét}}(R, \mathbb{G}_m)[-1] \longrightarrow R\Gamma_{\mathrm{syn}}(\mathrm{Spf}(R), \mathbb{Z}_p(1))$$

The **first main topic**, which will take three talks, is to sketch a proof of the theorem [APC, Thm. 7.5.6]: namely that c_1^{syn} identifies the target with the p-completion of the source.

Sketch Thm. 7.1.1, namely the logarithm sequence for quasiregular semiperfect $\mathbb{F}_p$-algebras; as we will later see, this essentially proves Thm. 7.5.6 for such rings.

(4) The next step towards Thm. 7.5.6 is an analysis of the functor $R\Gamma_{\text{ét}}(R, \mathbb{G}_m)$. Present [APC, §7.2], especially Prop. 7.2.15, which reduces the study of $R\Gamma_{\text{ét}}(-, \mathbb{G}_m)^{\wedge}_p$ to characteristic p.

(5) Complete the proof of Thm. 7.5.6 as follows: first explain §7.3, which culminates in Thm. 7.3.5 (the crystalline analogue of Thm. 7.5.6), and then cover Prop. 7.5.5 to the end of §7.5 (namely the reduction of Thm. 7.5.6 to the crystalline analogue).

Note: at some point in this reduction we need Prop. 5.5.24, which is a trick to reduce prismatic cohomology to characteristic p. It uses diffracted Hodge cohomology and the cotangent complex.

(6) The **second main topic** is the construction of the syntomic-to-étale comparison maps

$$\gamma_{\mathrm{syn}}^{\text{ét}}\{j\} : R\Gamma_{\mathrm{syn}}(\mathrm{Spf}(R), \mathbb{Z}_p(j)) \to R\Gamma_{\text{ét}}(R[\tfrac{1}{p}], \mathbb{Z}_p(j))$$

(which relies on the previous main theorem that c_1^{syn} is an isomorphism after p-completion) and their use in defining syntomic cohomology of arbitrary schemes.

Prove Theorem 8.3.1 (and whatever you need from §8.2 and §8.3): namely the existence of the syntomic-to-étale comparison maps.

(7) Cover §8.4: namely the definition of syntomic cohomology of arbitrary schemes, descent for fpqc topology, that it is left Kan extended from smooth algebras, and the examples of low weights.

(8) The motivic filtration on Selmer K-theory.