The homology of groups.

Part I : basic theory

These notes review the basic facts regarding the (co)homology of groups, illustrated by a few exercises. The exercises are chosen because they prove classical results, or because they nicely illustrate some concepts/techniques, or because they are useful in the context of the stable homology of groups (sometimes for these three reasons at the same time). By "basic facts", we mean some facts which do not rely on spectral sequences – spectral sequences and their applications will be reviewed in part II.

The reader is assumed to know basic homological algebra (such as the first two chapters of Weibel's book below), and to have a basic knowledge of algebraic topology. Good introductory books on the homology of groups include:

- K. S. Brown, Cohomology of groups. Graduate Texts in Mathematics 87, Springer, 1982.
- L. Evens, The cohomology of groups. Oxford Mathematical Monographs. Oxford University Press, 1991.
- C. A. Weibel, An introduction to homological algebra. Cambridge Studies in Advanced Mathematics 38, Cambridge University Press, 1994.

In these notes, we shall often refer to some chapters, or specific statements that can be found in these books. When doing so, we just mention the author's name. For example, we write "see Brown, Chap V" for a reference to chapter V of the first book.

Contents

1 Homology

1.1 Definition

We denote by $\mathbb{Z}G$ the group ring of a group G. A representation M of G is an abelian group M on which G acts by isomorphisms of abelian groups. This is the same as a $\mathbb{Z}G$-module. The homology of G with coefficients in M is defined by

$$H_*(G,M) = \mathrm{Tor}_*^{\mathbb{Z}G}(\mathbb{Z},M)$$

where the left variable of Tor is the trivial right $\mathbb{Z}G$-module (i.e. the abelian group $\mathbb{Z}$ with the trivial right action of G). The homology of G can be concretely computed as the homology of the complex $P_* \otimes_{\mathbb{Z}G} M$ where P_* is any projective representation of $\mathbb{Z}$. The homology of degree 0 of degree zero is isomorphic to the coinvariants M_G:

$$H_0(G,M) = \mathbb{Z} \otimes_{\mathbb{Z}G} M \simeq M_G := M/\langle gm - m | g \in G, m \in M \rangle \ .$$

Higher degree homology groups are usually hard to compute, however this is easy to do for cyclic groups, because of the existence of nice small projective resolutions of the trivial module. To me more specific, if C_∞ is an infinite cyclic group with generator t, we have a resolution (where ϵ is the augmentation map, given by $\epsilon(g) = 1$ for all $g \in G$):

$$0 \to \mathbb{Z}C_\infty \xrightarrow{1-t} \mathbb{Z}C_\infty \xrightarrow{\epsilon} \mathbb{Z} \ . \tag{1}$$

If C_n is a cyclic group of order n with generator t, and if $N = 1 + t + \cdots + t^{n-1} \in \mathbb{Z}G$ is the norm element, there is a periodic resolution:

$$\ldots \xrightarrow{N} \mathbb{Z}C_n \xrightarrow{1-t} \mathbb{Z}C_n \xrightarrow{N} \mathbb{Z}C_n \xrightarrow{1-t} \mathbb{Z}C_n \xrightarrow{\epsilon} \mathbb{Z} \ . \tag{2}$$

Exercice 1. THE basic computation in group homology. Check that (**??**) and (**??**) are indeed projective resolutions of the trivial representation, and compute explicitly $H_*(C_n, M)$ and $H_*(C_\infty, M)$ for all M.

[Solution: see Weibel, Thm 6.2.2 p. 168]

Bar constructions provide nice complexes computing $H_*(G,M)$. These bar constructions are in fact defined not only for group rings, but more generally for $\Bbbk$-algebras A. If N is a right A-module and M is a left A-module, the bar complex $B(N,A,M)$ is the complex of k-modules with

$$B(N,A,M)_k = N \otimes_\Bbbk \underbrace{A \otimes_\Bbbk \cdots \otimes_\Bbbk A}_{k \text{ factors}} \otimes_\Bbbk M \ ,$$

and with differentials $d : B(N,A,M)_k \to B(N,A,M)_{k-1}$ given by

$$\begin{aligned} d(n \otimes a_1 \otimes \cdots \otimes a_k \otimes m) = \ & na_1 \otimes a_2 \otimes \cdots \otimes a_k \otimes m \\ & + \sum_{i=1}^{k-1} (-1)^i n \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_k \otimes m \\ & + (-1)^k n \otimes a_1 \otimes \cdots \otimes a_{k-1} \otimes a_k m \ . \end{aligned}$$

If A and N are projective as $\Bbbk$-modules, then it is known[1] that:

$$H_*(B(N, A, M)) \simeq \mathrm{Tor}^A_*(N, M)\ . \tag{3}$$

For $\Bbbk = \mathbb{Z}$ and $A = \mathbb{Z}G$ the complex $B(\mathbb{Z}, \mathbb{Z}G, M)$ is sometimes called the "standard complex" and its homology is $H_*(G, M)$. This standard complex is useful for many purposes, but it is usually much too big to be used in explicit computations of group homology.

1.2 Homology of $\Bbbk$-linear representations

Let $\Bbbk$ be a commutative ring. The representation M is called *a $\Bbbk$-linear representation* if M is a $\Bbbk$-module and G acts by $\Bbbk$-linear isomorphisms. Equivalently, M is a module over the group algebra $\Bbbk G$.

If M is a $\Bbbk$-linear representation, then $N \otimes_{\mathbb{Z}G} M$ is a $\Bbbk$-module for all modules N: the action of $\Bbbk$ on the tensor product is given by: $\lambda \cdot (n \otimes m) = n \otimes \lambda n$. It follows that the homology groups $H_i(G, M) = H_i(P_* \otimes_{\mathbb{Z}G} M)$ are $\Bbbk$-modules.

Exercice 2. Change of ground ring for group homology. Let M be a $\Bbbk$-linear representation of G.

1. If P_* is a projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module, show that $\Bbbk \otimes_{\mathbb{Z}} P_*$ is a projective resolution of trivial $\Bbbk G$-module $\Bbbk$.

2. Show that there is an isomorphism of $\Bbbk$-modules, natural with respect to the $\mathbb{Z}G$-module N:

$$N \otimes_{\mathbb{Z}G} M \simeq (\Bbbk \otimes_{\mathbb{Z}} N) \otimes_{\Bbbk G} M\ .$$

3. Deduce that there is a $\Bbbk$-linear isomorphism:

$$H_*(G, M) \simeq \mathrm{Tor}^{\Bbbk G}_*(\Bbbk, M)\ . \tag{4}$$

The interest of the isomorphism (**??**) lies in the fact that $\Bbbk G$ may be nicer than $\mathbb{Z}G$. The next exercise gives an example of this phenomenon (note that if G is a nontrivial finite group, the first assertion is never satisfied for $\Bbbk = \mathbb{Z}$, whereas it is satisfied for many fields such as $\Bbbk = \mathbb{Q}$ or $\Bbbk = \mathbb{F}_p$ with p prime to the order of G).

Exercice 3. Semi-simplicity and homology. Let G be a finite group, and let $\Bbbk$ be a field. Show that the following assertions are equivalent.

[1] Here is a proof. First, observe that $B(N, A, A)$ is a complex of projective right A-modules, with right action of A on each $B(N, A, A)_k$ given by the formula $(n \otimes a_1 \otimes \cdots \otimes a_k \otimes a) \cdot b = n \otimes a_1 \otimes \cdots \otimes a_k \otimes ab$. Second, consider the augmented complex $B(N, A, A) \xrightarrow{\epsilon} N$ of $\Bbbk$-modules, where $\epsilon : N \otimes A \to N$ is given by $\epsilon(n \otimes a) = na$. Then this complex is acyclic: a contracting homotopy is $h(n \otimes a_1 \otimes \cdots \otimes a_k \otimes a) = (-1)^k n \otimes a_1 \otimes \cdots \otimes a_k \otimes a \otimes 1$. Thus $B(N, A, A)$ is a projective resolution of N. Finally, the isomorphism (**??**) follows from the isomorphism of complexes $B(N, A, M) \simeq B(N, A, A) \otimes_A M$.

(i) The cardinal of G is invertible in $\Bbbk$.

(ii) The augmentation map $\Bbbk G \xrightarrow{\epsilon} \Bbbk$ has a $\Bbbk G$-linear section.

(iii) The trivial $\Bbbk G$-module $\Bbbk$ is projective.

(iv) For all $\Bbbk$-linear representations M, and for all $i \geq 1$, $H_i(G, M) = 0$.

[Solution: it is easy to prove that (i), (ii) and (iii) are equivalent, this relies on the fact that $\mathrm{Hom}_{\Bbbk G}(\Bbbk, \Bbbk G)$ has dimension one, with generator ϕ such that $\phi(\lambda) = \sum_{g \in G} \lambda g$. Assertion (iii) obviously implies (iv). To prove that (iv)⇒(ii), observe first that $\mathrm{Hom}_{\Bbbk G}(\Bbbk G, \Bbbk)$ has dimension one, with generator ϵ. Thus proving (ii) reduces to proving that the map $\phi^* : \mathrm{Hom}_{\Bbbk G}(\Bbbk G, \Bbbk) \to \mathrm{Hom}_{\Bbbk G}(\Bbbk, \Bbbk)$ induced by composition by ϕ is surjective. For all representations M there is a natural isomorphism $\mathrm{Hom}_{\Bbbk G}(M, \Bbbk) \simeq \mathrm{Hom}_{\Bbbk}(M_G, \Bbbk)$. Thus surjectivity of ϕ^* is equivalent to the injectivity of the map $(\Bbbk)_G \to (\Bbbk G)_G$ induced by ϕ. The latter follows from (iv) and the description of $H_0(G, -)$ as coinvariants.]

1.3 Tensor products

We keep a commutative ring $\Bbbk$. An important feature of the category of $\Bbbk G$-modules is that if M and M are two $\Bbbk$-linear representations of G, their tensor product $M \otimes_{\Bbbk} N$ is also canonically equiped with an action of G by:

$$g \cdot (m \otimes n) = (gm) \otimes (gn) \ .$$

Taking tensor products of representations is an important way to construct interesting new representations out of old ones.

Projectivity behave nicely with respect to tensor products. Namely, if M is projective as a $\Bbbk$-module and if N is projective as a $\Bbbk G$ module, then $M \otimes_{\Bbbk} N$ is projective as a $\Bbbk G$-module. (see). This important fact has many nice applications, such as the following one.

Exercice 4. Maschke's theorem.

1. Show that the trivial representation $\Bbbk$ is projective as a $\Bbbk G$-module if and only if every $\Bbbk$-linear representation is projective as a $\Bbbk G$-module.

2. Use the previous question and exercise 3 to prove Maschke's theorem: given a finite group G and a field $\Bbbk$, $\Bbbk G$ is semisimple if and only if the cardinal of G is invertible in $\Bbbk$.

1.4 Functoriality

Homology of a given group G is natural with respect to the representation M. Every morphism of $\mathbb{Z}G$-modules $f : M \to M'$ induces a graded morphism:

$$H_*(G,f) \ : \ H_*(G,M) \to H_*(G,M') \ . \tag{5}$$

To be more specific, (**??**) is induced by the chain map $P_*\otimes_{\mathbb{Z}G} f : P_*\otimes_{\mathbb{Z}G} M \to P_* \otimes_{\mathbb{Z}G} M'$. Homology is also natural with respect to G, as we shall see it now.

Functoriality with respect to G.

If $\alpha : H \to G$ is a morphism of group and if M is a representation of G, we denote by α^*M the representation of H obtained by restricting the action og G on M along α. That is, α^*M is M, acted on by H by the formula $h \cdot m := \alpha(h)m$. Then we have a graded morphism:

$$\alpha_* : \ H_*(H, \alpha^*M) \to H_*(G,M) \ . \tag{6}$$

Warning 5. When α is obvious from the context, e.g. H is a given subgroup of G and α is the inclusion, α^*M is simply denoted by M. This is a lighter notation, but it may cause confusion, e.g. if α is an automorphism of G.

To be more specific, let P_* be a projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module, and let Q_* be a projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}H$-module. The morphism (**??**) is induced by the composition of the chain maps

$$Q_* \otimes_{\mathbb{Z}H} \alpha^*M \xrightarrow{\overline{\mathrm{id}}\otimes\alpha^*M} (\alpha^*P_*) \otimes_{\mathbb{Z}H} \alpha^*M \xrightarrow{\mathrm{can}} P \otimes_{\mathbb{Z}G} M$$

where can is the canonical quotient map, which sends $p\otimes m \in \alpha^*P_i\otimes_{\mathbb{Z}H}\alpha^*M$ to $p \otimes m \in P_i \otimes_{\mathbb{Z}G} M$, and where the chain map $\overline{id} : Q_* \to \alpha^*P_*$ is a lift of the identity map $\mathrm{id} : \mathbb{Z} \to \mathbb{Z}$ (such a lift exists and is unique up to homotopy by the fundamental lemma of homological algebra, see e.g. Weibel, Thm 2.2.6 p. 35).

Exercice 6. Let M be a representation of a cyclic group C_n of order n. Let C_q be the cyclic subgroup of C_n of order q. Compute the map:

$$H_*(C_q, \iota^*M) \to H_*(C_n, M)$$

induced by the inclusion $C_q \hookrightarrow C_n$.

[Hint: Let k such that $n = kq$, let t be the generator of C_n and let $t' = t^k$ be the generator of C_q. Use the small resolutions (**??**), and the commutative diagram of $\mathbb{Z}C_q$-modules:

$$\begin{tikzcd}
\mathbb{Z} \arrow[d, "="] & \mathbb{Z}C_q \arrow[l, "\epsilon"'] \arrow[d, hook] & \mathbb{Z}C_q \arrow[l, "1-t'"'] \arrow[d, "K"] & \mathbb{Z}C_q \arrow[l, "N'"'] \arrow[d, hook] & \cdots \arrow[l, "1-t'"'] \\
\mathbb{Z} & \mathbb{Z}C_n \arrow[l, "\epsilon"'] & \mathbb{Z}C_n \arrow[l, "1-t"'] & \mathbb{Z}C_n \arrow[l, "N"'] & \cdots \arrow[l, "1-t"']
\end{tikzcd}$$

where $N = 1+t+\cdots+t^{n-1}$, $N' = 1+t'+\cdots+t'^{q-1}$ and $K = 1+t+\cdots+t^{k-1}$.]

Exercice 7. (Life is easier with bar complexes.) Show that the morphism (**??**) is induced by the chain map $B(\mathrm{id},\alpha,\mathrm{id}) : B(\mathbb{Z},\mathbb{Z}H,\alpha^*M) \to B(\mathbb{Z},\mathbb{Z}G,M)$ defined by:

$$B(\mathrm{id},\alpha,\mathrm{id})(1\otimes h_1\otimes\cdots\otimes h_k\otimes m) = 1\otimes\alpha(h_1)\otimes\cdots\otimes\alpha(h_k)\otimes m\;.$$

Small subgroups and big overgroups. Functoriality with respect to G is essential to understand $H_*(G,M)$. Indeed, the homology of G is typically to complicated to be understood directly. A classical way to obtain some information on the homology of G is then to consider small subgroups H (e.g. cyclic subgroups) which are more easily understood and to study the map $H_*(H,\iota^*M)\to H_*(G,M)$ induced by the inclusion $\iota : H\to G$.

An "opposite way" to obtain information on $H_*(G,M)$ is to compare it with the homology of bigger groups. For example, the symmetric group $\mathfrak{S}_d$ on $\{1,\dots,d\}$ identifies to the subgroup of $\mathfrak{S}_{d+1}$ stabilizing the integer $d+1$. Thus, there is a sequence of injective morphisms of symmetric groups:

$$\cdots\hookrightarrow\mathfrak{S}_d\hookrightarrow\mathfrak{S}_{d+1}\hookrightarrow\mathfrak{S}_{d+2}\hookrightarrow\cdots\cdots\hookrightarrow\mathfrak{S}_\infty=\bigcup_{d\geq 1}\mathfrak{S}_d\;.$$

Let $\Bbbk$ be a commutative ring, with trivial action of $\mathfrak{S}_d$. A theorem of Nakaoka shows that for all d the inclusion induces an injective morphism:

$$H_*(\mathfrak{S}_d,\Bbbk)\to H_*(\mathfrak{S}_{d+1},\Bbbk)$$

which is an isomorphism in degrees $\leq d/2$. By the next exercise, this implies that the map

$$H_*(\mathfrak{S}_d,\Bbbk)\to H_*(\mathfrak{S}_\infty,\Bbbk)$$

is injective and, an isomorphism in degrees $\leq d/2$. Now this may sound strange, but $H_*(\mathfrak{S}_\infty,\Bbbk)$ is easier to understand : this actually comes from the fact that $H_*(\mathfrak{S}_\infty,\Bbbk)$ has a richer structure (among other things, it is ring). One may compute the homology of $\mathfrak{S}_\infty$ and retrieve from this a complete computation of $H_*(\mathfrak{S}_d,\Bbbk)$ when $\Bbbk$ is a field. The interested reader may find details in chap V of the book "Cohomology of finite groups" by Adem and Milgram.

Exercice 8. Homology and colimits. Let G_∞ be a group, and consider an increasing exhaustive chain of subgroups G_n:

$$G_1\subset G_2\subset\cdots\subset G_n\subset\cdots\bigcup_{n\geq 1}G_n = G_\infty\;.$$

Assume that we are given for each n a representation M_n of G_n, together with a G_n-equivariant map $f_n : M_n\to M_{n+1}$.

1. Show that $M_\infty := \mathrm{colim}_{n\geq 1} M_n$ is endowed with an action of G_∞.

2. Show that $H_*(G_\infty, M_\infty)$ is the colimit of the diagram (whose morphisms are induced by the inclusions $G_n \hookrightarrow G_{n+1}$ and the maps f_n):

$$H_*(G_1, M_1) \to H_*(G_2, M_2) \to \cdots \to H_*(G_n, M_n) \to \cdots .$$

   [Hint: use the bar complex, and use that homology of complexes commutes with filtered colimits (cf. Weibel, Thm 2.6.15 p. 57).]

1.5 Schapiro's lemma

Let H be a subgroup of G. From a representation M of H, one defines the induced representation $\mathrm{ind}_H^G M$ of G by:

$$\mathrm{ind}_H^G M = \mathbb{Z}G \otimes_{\mathbb{Z}H} M$$

with action $g \cdot (g' \otimes m) := (gg') \otimes m$. Induction is left adjoint to restriction along the inclusion $\iota : H \hookrightarrow G$. The unit of adjunction is the H-equivariant map:

$$\begin{array}{rccc} \eta_M : & M & \to & \mathbb{Z}G \otimes_{\mathbb{Z}H} M \\ & m & \mapsto & 1 \otimes m \end{array} .$$

Shapiro's lemma asserts that the following composition is an isomorphism:

$$H_*(H, M) \xrightarrow{H_*(H,\eta_M)} H_*(H, \mathrm{ind}_H^G M) \xrightarrow{\iota_*} H_*(G, \mathrm{ind}_H^G M) \tag{7}$$

Schapiro's isomorphism (**??**) provides useful information on the homology of G from the homology of its subgroups.

Exercice 9. Groups with finite homological dimension[2]. Let G be a group such that the trivial representation $\mathbb{Z}$ admits a finite projective resolution. Show that G has no element of finite order.

1.6 Action on the homology of normal subgroups.

Let M be a representation of a group G and let H be a normal subgroup of G. Then $H_*(H, M)$ is endowed with an action of G, which coincides in degree 0 with the canonical action of G on M_H. This action plays a crucial role for computations with the Lyndon-Hochschild-Serre spectral sequence (see part II), so we recall the details here.

[2] Using the fact that tensoring any representation with a projective representation is projective, one sees that the trivial representation $\mathbb{Z}$ has a projective resolution of length n if and only if every representation has a projective representation of length n. The latter property says exactly that G has homological dimension n (see e.g. Weibel, chap 4).

For all $g \in G$ we consider the conjugation map: $c_g : H \to H$ such that $c_g(h) = ghg^{-1}$. The action of G on $H^*(H, M)$ is defined by letting $g \in G$ act as the composition:

$$H_*(H, M) \xrightarrow{\simeq} H_*(H, c_g^* M) \xrightarrow{(c_g)_*} H_*(H, M) \tag{8}$$

where the isomorphism on the left hand-side is induced by the morphism of representations $M \to c_g^* M$, $m \mapsto gm$.

Exercice 10. The action of G on $H_*(G, M)$ is trivial. Let P_* be a projective resolution of $\mathbb{Z}$ as a $\Bbbk G$-module.

1. Show that the restriction of P_* to H yields a projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}H$-module, and that the map (**??**) is induced by the chain map $P_* \otimes_{\mathbb{Z}H} M \to P_* \otimes_{\mathbb{Z}H} M$ which sends $p \otimes m \in P_k \otimes_{\mathbb{Z}H} M$ to $pg^{-1} \otimes gm \in P_k \otimes_{\mathbb{Z}H} M$.

2. In the case $H = G$, deduce that the action of H on $H_*(H, M)$ is trivial. In particular, the action of G on $H_*(H, M)$ factors through an action of G/H.

Exercice 11. Let $\alpha_1, \alpha_2 : H \to G$ be two conjugate morphisms of groups (i.e. there are $g \in G$ and $h \in H$ such that $\alpha_1 = c_g \circ \alpha_2 \circ c_h$). Show that α_1 and α_2 induce the *same* map $H_*(H, \Bbbk) \to H_*(G, \Bbbk)$ in homology with trivial coefficients.

Exercice 12. Dihedral groups. The dihedral group D_{2n} is the semidirect product of cyclic groups $C_n \rtimes C_2$, where the generator of C_2 acts nontrivially on C_n. (If $n \geq 3$, this is the group of isometries of the regular n-gon.) Let $g \in D_{2n} \setminus C_n$. Show that g acts as multiplication by $(-1)^i$ on $H_{2i-1}(C_n, \Bbbk)$ and $H_{2i}(C_n, \Bbbk)$.

[Hint: see Weibel, ex. 6.7.10 page 191.]

2 Cohomology

2.1 Basic properties

Let M be a representation of G. The cohomology of G with coefficients in M is defined by

$$H^*(G,M) = \mathrm{Ext}^*_{\mathbb{Z}G}(\mathbb{Z},M)$$

where $\mathbb{Z}$ is a trivial representation. Thus, if P_* is a projective resolution of $\mathbb{Z}$ in the category of $\mathbb{Z}G$-modules, $H^*(G,M)$ is the homology of the cochain complex $\mathrm{Hom}_{\mathbb{Z}G}(P_*,M)$. Degree zero cohomology is isomorphic to the invariants M^G

$$H^0(G,M) = \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z},M) \simeq M^G := \{m\in M \,|gm = m \;\forall g\in G\}\;.$$

Much of what has been said for the homology of groups has an analogue for cohomology.

The standard complex $C^*(G,M)$. It is the cochain complex with

$$C^n(G,M) = \mathrm{Map}(G^{\times n},M),$$

where $\mathrm{Map}(X,Y)$ stands for the set of set-theoretic functions $f:X\to Y$, and whose differential $d:C^n(G,M)\to C^{n+1}(G,M)$ sends f to the function df such that

$$\begin{aligned} df(g_1,\dots,g_{n+1}) = {} & g_1 f(g_2,\dots,g_{n+1}) \\ & + \sum_{i=1}^n (-1)^i f(g_1,\dots,g_ig_{i+1},\dots,g_{n+1}) \\ & + (-1)^{n+1} f(g_1,\dots,g_n)\;. \end{aligned}$$

The standard complex is isomorphic to the complex $\mathrm{Hom}_{\mathbb{Z}G}(P_*,M)$ with $P_* = B(\mathbb{Z}G,\mathbb{Z}G,\mathbb{Z})$, hence it computes $H^*(G,M)$.

The cohomology of cyclic groups. Using the nice small projective resolutions (**??**) and (**??**), one can compute the cohomology of cyclic groups for all representations M. For the finite cyclic groups C_n, one finds that

$$H^i(C_n,M) = \begin{cases} M^G & \text{for } i=0,\\ \mathrm{Ker}\,\overline{N} & \text{for } i \text{ odd},\\ \mathrm{Coker}\,\overline{N} & \text{for } i \text{ even and positive,}\end{cases} \tag{9}$$

where $\overline{N}: M_G\to M^G$ is the norm map, which sends the class $[m]$ in M_G of an element $m\in M$ to $Nm = \sum_{g\in G} gm \in M^G$.

Exercice 13. Cohomology of infinite cyclic groups. Compute the cohomology of the infinite cyclic group C_∞ for all representations M.

Exercice 14. Another application of the norm map[3]. Let G be a finite group of order n. We define $\rho_M : M \to M^G$ by $\rho_M(m) = \overline{N}([m])$.

1. Let M' be a subrepresentation of M and let $q : M \twoheadrightarrow M/M'$ denote the quotient morphism. Show that for all $x \in M/M'^G$, nx is in the image of $q : M^G \to (M/M')^G$.

   [Hint: use the diagram:
$$\begin{tikzcd} & M \arrow[r, "\rho_M"] \arrow[d, two heads, "q"] & M^G \arrow[d, "q"] \\ (M/M')^G \arrow[r, hook] \arrow[rr, bend right, "n\mathrm{id}"'] & M/M' \arrow[r, "\rho_M"] & (M/M')^G \end{tikzcd}$$
   ]

2. By taking M injective, show that multiplication by n kills the abelian group $H^1(G, M')$ for all M'. Deduce that multiplication by n kills $H^i(G, M')$ for all M' and all $i > 0$.

   [Hint: use dimension shifting, see e.g. Brown's book, Chap III.7.]

Changing the ground ring. If $\Bbbk$ is a commutative ring and M is a $\Bbbk$-linear representation, then $H^*(G, M)$ is a graded $\Bbbk$-module, and there is a $\Bbbk$-linear isomorphism:

$$H^*(G, M) \simeq \mathrm{Ext}^*_{\Bbbk G}(\Bbbk, M) . \tag{10}$$

Functoriality. Cohomology $H^*(G, M)$ is covariant with respect to M, and contravariant with respect to G. To be more specific, every $\mathbb{Z}G$-linear morphism $f : M \to M'$ yields a morphism

$$H^*(G, f) : H^*(G, M) \to H^*(G, M') \tag{11}$$

which is induced by the cochain map $\mathrm{Hom}_{\mathbb{Z}G}(P_*, M) \to \mathrm{Hom}_{\mathbb{Z}G}(P_*, M')$ given by composition with f. Every morphism of groups $\alpha : H \to G$ yields a restriction morphism in cohomology:

$$\alpha^* : H^*(G, M) \to H^*(H, \alpha^* M) . \tag{12}$$

[3] This way of proving that the cardinal of a group G kills the cohomology is taken from the article of van der Kallen, A Friedlander-Suslin theorem over a noetherian base ring, arXiv:2212.14600. It carries out to the cohomology of arbitrary finite group schemes.

If P_* is a projective resolution of $\mathbb{Z}$ as a trivial $\mathbb{Z}G$-module and Q_* is a projective resolution of $\mathbb{Z}$ as a trivial $\mathbb{Z}H$-module then (**??**) is induced by the cochain map defined as the composition:

$$\mathrm{Hom}_{\mathbb{Z}G}(P_*, M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}H}(\alpha^* P_*, \alpha^* M) \to \mathrm{Hom}_{\mathbb{Z}H}(Q_*, \alpha^* M)$$

where the first cochain map is the canonical inclusion (every $\mathbb{Z}G$-linear map is $\mathbb{Z}H$-linear), and the second one is induced by composition with a chain map $\overline{\mathrm{id}} : \alpha^* P \to Q^*$ which lifts $\mathrm{id} : \mathbb{Z} \to \mathbb{Z}$.

Functoriality with respect to G has a nice description on the level of standard complexes. Namely, α^* is induced by the chain map $C^*(\alpha, \mathrm{id}) : C^*(G, M) \to C^*(H, \alpha^* M)$ which sends an function $f : G^{\times n} \to M$ to the function $f \circ \alpha^{\times n}$. (The proof of this fact has the same flavour as exercise **??**.)

Schapiro's lemma. Let H be a subgroup of G and let M be a representation of H. The coinduced representation $\mathrm{coind}_H^G M$ is the representation of G defined by:

$$\mathrm{coind}_H^G M = \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M)$$

with action of $g \in G$ sending and H-equivariant map f to the map $x \mapsto f(xg)$. Coinduction is a right adjoint to restriction $\mathbb{Z}G{-}\mathrm{Mod} \to \mathbb{Z}H{-}\mathrm{Mod}$. Schapiro's lemma asserts that the adjonction isomorphism extends to an isomorphism on the Ext-level:

$$H^*(H, M) \simeq H^*(G, \mathrm{coind}_H^G M)\ . \tag{13}$$

Action of G on the cohomology of its normal subgroups. Let H be a normal subgroup of G and let M be a representation of G. Then M^H is a subrepresentation of M, and the action of G on M^H extends to an action on the whole cohomology $H^*(H, M)$.

To be more specific, an element $g \in G$ acts as the composition of restriction along the conjugation morphism together with the morphism induced by the H-equivariant map $c_g^* M \to M$, $m \mapsto g^{-1}m$:

$$H^*(H, M) \xrightarrow{c_g^*} H^*(H, c_g^* M) \to H^*(H, M). \tag{14}$$

One can prove that G acts trivially on $H^*(G, M)$ in the same fashion as in exercise **??**. Here is an alternative proof (which can be also adapted to homology).

Exercice 15. Another proof that G acts trivially on its cohomology. Show that the morphism $H^*(H, M) \to H^*(H, M)$ given by the action of $g \in G$ is the unique morphism of δ-functors which is equal to the map $M^H \to M^H$, $m \mapsto gm$, in degree 0. Deduce that for $H = G$, the action of G on its cohomology is trivial.

[Hint: see Brown Chap III.7, or Weibel Chap 2, for the notion of a δ-functor. One needs here to prove here that $H^*(H,-) : \mathbb{Z}G\mathrm{-Mod} \to \mathbb{Z}\mathrm{-Mod}$ is a *universal* δ-functor, which requires to check that the restriction to H of an injective representation of G remains injective.]

2.2 Homology versus cohomology

We have seen in the previous section that homology and cohomology have analogous properties. We shall see in this section that cohomology and homology are related by a universal coefficient exact sequence (**??**), thus the groups $H_*(G,M)$ and $H^*(G,M)$ contain closely related information. However, as we briefly explain it below, homology and cohomology both have their own advantages (that one should keep in mind before choosing to work with homology or cohomology).

The universal coefficient exact sequence. Let $\Bbbk$ be a commutative ring and let M be a $\Bbbk G$-module and let N be a $\Bbbk$-module. Then $g \in G$ acts on $\mathrm{Hom}_\Bbbk(M,N)$ by sending a $\Bbbk$-linear map f to the $\Bbbk$-linear map $gf : m \mapsto f(g^{-1}m)$. There is an isomorphism, natural with respect to M, N and the right $\Bbbk G$-module P:

$$\mathrm{Hom}_\Bbbk(P \otimes_{\Bbbk G} M, N) \simeq \mathrm{Hom}_{\Bbbk G}(P, \mathrm{Hom}_\Bbbk(M,N)) \;. \tag{15}$$

Replacing the $\Bbbk G$-module P by a projective resolution of the trivial $\Bbbk G$-module $\Bbbk$, the right-hand side of (**??**) becomes a cochain complex whose homology computes $H^*(G, \mathrm{Hom}_\Bbbk(M,N))$. Thus, if $\Bbbk$ is a PID, by applying the universal coefficient theorem to the left-hand side of (**??**), one obtains the universal coefficient exact sequence for group homology. Namely, for all i we have an exact sequence (which is natural with respect to M and N, and which splits non naturally with respect to M and N):

$$0 \to \mathrm{Ext}^1_\Bbbk(H_{i-1}(G,M),N) \to H^i(G,\mathrm{Hom}_\Bbbk(M,N)) \to \mathrm{Hom}_\Bbbk(H_i(G,M),N) \to 0 \quad . \tag{16}$$

Exercice 16. Let G be a finite group of cardinal n. Deduce from exercise **??** and the universal coefficient exact sequence (**??**) that $H_*(G,M)$ is killed by n for all representations M.

[Hint: consider the exact sequence with $\Bbbk = \mathbb{Z}$ and $N = \mathbb{Q}/\mathbb{Z}$.]

Exercice 17. Let $\alpha : H \to G$ be a morphism of groups, let M be a $\Bbbk$-linear representation of G ($\Bbbk$ a commutative ring), and let N be a $\Bbbk$-module.

1. Show that the following square commutes (the vertical arrow on the the right hand-side is given by restriction along α and the one on the left hand-side is induced by the canonical epimorphism $\alpha^*P\otimes_{\Bbbk H}\alpha^*M\twoheadrightarrow P\otimes_{\Bbbk G}M$):

$$\begin{array}{ccc} \mathrm{Hom}_{\Bbbk}(P\otimes_{\Bbbk G}M,N) & \xrightarrow[(??)]{\simeq} & \mathrm{Hom}_{\Bbbk G}(P,\mathrm{Hom}_{\Bbbk}(M,N)) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\Bbbk}(\alpha^*P\otimes_{\Bbbk H}\alpha^*M,N) & \xrightarrow[(??)]{\simeq} & \mathrm{Hom}_{\Bbbk H}(\alpha^*P,\mathrm{Hom}_{\Bbbk}(\alpha^*M,N)) \end{array}.$$

2. Deduce that for all injective $\Bbbk$-modules N, there is an isomorphism *natural with respect to G:*

$$H^i(G,\mathrm{Hom}_{\Bbbk}(M,N))\simeq\mathrm{Hom}_{\Bbbk}(H_i(G,M),N)$$

3. Assume that N is an injective cogenerator of $\Bbbk$-modules. Show that $\alpha_*:H_i(H,\alpha^*M)\to H_i(G,\alpha^*M)$ is an isomorphism if and only if $\alpha^*:H^i(G,\mathrm{Hom}_{\Bbbk}(M,N))\to H^i(H,\alpha^*\mathrm{Hom}_{\Bbbk}(M,N)))$ is an isomorphism.

The advantages of homology. As we saw it in exercise **??**, group homology preserves filtered colimits[4].

The good behavior with colimits is very useful to reduce the study of group homology to smaller groups or to smaller representations. This is well illustrated by the computation of homology of arbitrary abelian groups with coefficients in a prime field $\mathbb{F}_p$. For the sake of concreteness, we briefly review this computation, with p odd.

1. One proves that $H_1(A,\mathbb{F}_p)\simeq A/pA$, and that there is a natural map ${}_pA=\{a\in A\mid pa=0\}\hookrightarrow H_2(A,\mathbb{F}_p)$. Moreover, by inspecting the bar construction, one proves that $H_*(A,\mathbb{F}_p)$ has the structure of a graded commutative $\mathbb{F}_p$-algebra with divided powers. Thus, there is a morphism of graded rings, natural with respect to A:

$$\Gamma_{\mathbb{F}_p}({}_pA)\otimes\Lambda_{\mathbb{F}_p}(A/pA)\to H_*(A,\mathbb{F}_p)\ , \tag{17}$$

where the left hand side is the tensor product of the free divided power algebra[5] on a copy of ${}_pA$ placed in degree 2 with an exterior algebra on a copy of A/pA placed in degree 1.

[4] In exercise **??**, the diagram of subgroups is indexed by $\mathbb{N}$, but the proof carries without change for a diagram of subgroups indexed by an arbitrary filtered ordered set.

[5] If V is an $\mathbb{F}_p$-vector space, let the symmetric group $\mathfrak{S}_d$ act on $V^{\otimes d}$ by permuting the factors of the tensor product, and set $\Gamma^d(V)=(V^{\otimes d})^{\mathfrak{S}_d}$. Then $\Gamma_{\mathbb{F}_p}(V)=\bigoplus_{d\geq 0}\Gamma^d_{\mathbb{F}_p}(V)$ as a graded vector space, with $\Gamma^d_{\mathbb{F}_p}(V)$ placed in degree $2d$. Thus, $\Gamma^d_{\mathbb{F}_p}(V)$ is a graded subspace of $T(V)=\bigoplus_{d\geq d}V^{\otimes d}$. It is in fact a graded subalgebra of $T(V)$ if we consider the latter as equipped with the shuffle product.

2. Our good knowledge of cyclic subgroups shows that (**??**) is an isomorphism if A is cyclic. With the Künneth theorem one deduces that (**??**) is an isomorphism if A is a finite product of cyclic groups (= a finitely generated abelian group).

3. The world of abelian groups is vast and there are full of scary weird abelian groups. But every abelian group is the filtered colimit of its finitely generated abelian subgroups. Since both sides of the natural map (**??**) preserve filtered colimits, this implies without any further computation, that (**??**) is an isomorphism for all abelian groups A!

The interested reader may find further details relative to the homology of abelian groups in Brown, Chap V. A functorial description of the mod 2 homology of abelian groups was found recently by S. O. Ivanov and A. A. Zaikovskii, arXiv:1810.12728.

The advantages of cohomology. Cohomology does not behave well with colimits (nor with limits!), but has at least two advantages over homology.

1) The low dimensional cohomology groups H^1, H^2, H^3 are tightly connected to the concrete problem of constructing group extensions of G.

2) More importantly, there are products in cohomology. Theses products are an important tool for concrete computations, and they also encodes important qualitative information relative to the representations of G.

We review these two features of cohomology in the next two subsections.

2.3 Low degree cohomology

A group extension of G by a group N is a short exact sequence of groups

$$1 \to N \to E \xrightarrow{\pi} G \to 1 \tag{18}$$

and two such extensions are isomorphic if there is a commutative diagram of groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & E & \xrightarrow{\pi} & G & \longrightarrow & 1 \\ & & \big\downarrow{\scriptstyle =} & & \big\downarrow{\scriptstyle \simeq} & & \big\downarrow{\scriptstyle =} & & \\ 1 & \longrightarrow & N & \longrightarrow & E' & \xrightarrow{\pi'} & G & \longrightarrow & 1 \end{array} .$$

A basic problem is to study the extensions of G by a given N. As we shall briefly explain it now, the cohomology groups H^1 and H^2 give information on this problem when N is abelian. We refer the reader to Brown, Chap IV for full details.

Consider an extension (**??**) with N abelian. Since N is abelian, the conjugation action of N on itself is trivial, hence the conjugation action of

E on N induces an action of $E/N = G$ on N. Therefore we may restrict the study to extensions (**??**) with a given action of G on N. Thus, from now on, we fix a group G and a representation N of G, and we investigate the extensions of G which correspond to our representation N.

The information contained in $H^1(G,N)$. By inspecting the standard complex $C^*(G,N)$, one computes:

$$H^1(G,N) = \{f : G \to N \mid f(gh) = f(g) + gf(h)\ \forall (g,h) \in G^2\}\ . \tag{19}$$

In particular if N is a trivial representation then $H^1(G,N)$ is isomorphic to the set of group morphisms $f : G \to N$.

The connection of the H^1 calculation (**??**) with the study of extensions of the form (**??**) is the following. Consider the group $N \rtimes G$: it is the set $N \times G$, endowed with the multiplication $(n,g)\cdot(n',g') = (n+gn', gg')$. This groups yields an extension

$$1 \to N \to N \rtimes G \xrightarrow{\pi} G \to 1$$

with $\pi(n,g) = g$. One proves that this extension is (up to isomorphism) the unique extension of the form (**??**) in which the morphism of group π has a section, i.e. the unique extension of the form (**??**) for which there is a morphism of groups $s : G \to N \rtimes G$ such that $\pi \circ s = \mathrm{id}$. Now one easily checks from the definition of the group structure of $N \rtimes G$ and the calculation (**??**) that there is a bijection:

$$\begin{array}{ccc} H^1(G,N) & \xrightarrow{\simeq} & \{\text{sections of } \pi\} \\ f & \mapsto & [g \mapsto (f(g), g)] \end{array} . \tag{20}$$

Exercice 18. Let $D_{2n} = \langle a, b \mid a^n = b^2 = 1, ab = ba^{-1}\rangle$ be the dihedral group of cardinal $2n$. This group contains the cyclic group $C_n = \langle a \mid a^n = 1\rangle$ as a normal subgroup. Assume that n is even. Compute $H^1(D_{2n}, \mathbb{F}_2)$ and show that the restriction map $H^1(D_{2n}, \mathbb{F}_2) \to H^1(C_n, \mathbb{F}_2)$ is surjective.

The information contained in $H^2(G,N)$. The subgroup of 2-cocycles of the standard complex $C^*(G,N)$ is:

$$Z^2(G,N) = \left\{ G^2 \xrightarrow{f} N \;\middle|\; \begin{array}{c} gf(h,k) - f(gh,k) + f(g,hk) - f(g,h) \\ \forall (g,h,k) \in G^3 \end{array} \right\} .$$

Extensions of G by N can be used to construct elements of $Z^2(G,N)$. To be more specific, if $\mathcal{E} = 1 \to N \to E \xrightarrow{\pi} G \to 1$ is such an extension, we can choose a *set theoretic* section of π, i.e. a function $s : G \to E$ such that $\pi \circ s = \mathrm{id}$. Then the function:

$$f_{\mathcal{E},s} : (g,h) \mapsto s(gh)s(h)^{-1}s(g)^{-1}$$

is a 2-cocycle. One can prove that the cohomology class of $f_{\mathcal{E},s}$ does not depend on the choice of the section s, and we denote this cohomology class by $c_{\mathcal{E}}$. Further verifications show that we obtain in this way a bijection:

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{extensions of } G \text{ by } N \end{array} \right\} & \xrightarrow{\simeq} & H^2(G,N) \\ [\![\mathcal{E}]\!] & \mapsto & c_{\mathcal{E}} \end{array} . \tag{21}$$

Observe that the extension $\mathcal{S} = 1 \to N \to N \rtimes G \xrightarrow{\pi} G \to 1$ is characterized (up to isomorphism) by the fact that it has a section which is a group morphism. The 2-cocycle associated to such a section is constant equal to zero, so that $c_{\mathcal{S}} = 0$. Hence, $c_{\mathcal{E}} = 0$ if and only if $\mathcal{E}$ is isomorphic to $\mathcal{S}$.

Exercice 19. Let n be an even positive integer.

1. Show that there is (up to isomorphism) only one non-split extension of C_n by C_2, namely the extension $1 \to C_2 \to C_{2n} \to C_n \to 1$. Compute a 2-cocycle associated with this extension.

2. Let $D_{4n} = \langle c, b \mid c^{2n} = b^2 = 1, cb = bc^{-1} \rangle$, and let $\pi : D_{4n} \to D_{2n}$ be the morphism such that $\pi(c) = a$ and $\pi(b) = b$ (with the same notations for the generators of D_{2n} as in exercise (**??**)). Compute a 2-cocycle associated with the extension $1 \to C_2 \to D_{4n} \xrightarrow{\pi} D_{2n} \to 1$.

3. Show that the restriction map $H^2(D_{2n}, \mathbb{F}_2) \to H^2(C_n, \mathbb{F}_2)$ is surjective.

2.4 Cup products

Let M and N be two $\Bbbk$-linear representations of a group G. The cup product is family of morphisms

$$\begin{array}{ccc} H^i(G,M) \otimes_{\Bbbk} H^j(G,N) & \to & H^{i+j}(G, M \otimes_{\Bbbk} N) \\ x \otimes y & \mapsto & xy \end{array} \tag{22}$$

which is natural with respect to G, M and N, which is graded commutative $xy = (-1)^{ij} yx$ and associative $x(yz) = (xy)z$. Moreover, if $i = j = 0$, the cup product coincides with the map

$$\begin{array}{ccc} M^G \otimes_{\Bbbk} N^G & \to & (M \otimes_{\Bbbk} N)^G \\ m \otimes n & \mapsto & m \otimes n \end{array} .$$

Finally, if $M = N = \Bbbk$ (with trivial action), then $M \otimes_{\Bbbk} N$ is canonically isomorphic to $\Bbbk$, so that the cup product (**??**) yields a graded commutative algebra structure on $H^*(G, \Bbbk)$.

The construction. The cup product (**??**) is defined as the composition

$$H^i(G,M)\otimes_{\Bbbk} H^j(G,N) \xrightarrow{\times} H^{i+j}(G\times G, M\otimes_{\Bbbk} N) \xrightarrow{\Delta^*} H^{i+j}(G, M\otimes_{\Bbbk} N)$$

where Δ^* is restriction along the morphism $\Delta : G \to G^2$, $g \mapsto (g,g)$, and $\times$ stands for the cross product, which is induced by the cochain map:

$$\mathrm{Hom}_{\Bbbk G}(P_*, M) \otimes \mathrm{Hom}_{\Bbbk G}(P_*, N) \xrightarrow{\otimes} Hom_{\Bbbk G\otimes_{\Bbbk}\Bbbk G}(P_*\otimes_{\Bbbk} P_*, M\otimes_{\Bbbk} N)\ .$$

Cyclic groups. For cyclic groups one can make concrete computations based on the nice small resolutions (**??**) and (**??**). We refer the reader to Evens, section 3.2 for details on these computations, and we bound ourselves to stating the results. If p is odd, there is an isomorphism of algebras:

$$H^*(C_{pk}, \mathbb{F}_p) \simeq \mathbb{F}_p[y]\otimes \Lambda(x) \tag{23}$$

where x is a class of degree 1 and y is a class of degree 2. If $p = 2$, there is a surprise. One has isomorphisms of algebras:

$$H^*(C_{2k}, \mathbb{F}_2) \simeq \begin{cases} \mathbb{F}_2[x] & \text{if } k \text{ is odd,} \\ \mathbb{F}_2[y]\otimes\Lambda(x) & \text{if } k \text{ is even,} \end{cases} \tag{24}$$

where x is a class of degree 1 and y is a class of degree 2. Thus, although the description of $H^*(C_{2k}, \mathbb{F}_2)$ as a graded $\mathbb{F}_2$-vector space is uniform for all k, the description of the products is not!

Exercice 20. Let n be an integer. Show that the restriction map $H^*(D_{2n}, \mathbb{F}_2) \to H^*(C_n, \mathbb{F}_2)$ is surjective.

[Hint: if n is odd, the statement is trivial. If n is even, use exercises **??** and **??**.]

3 Topology and group (co)homology

Everything presented so far is purely algebraic. But actions of groups on spaces provides strong connection of group (co)homology with topology. The simplest case of such an interaction is provided by the homology of the $K(G,1)$ spaces.

3.1 Homology of $K(G,1)$ spaces

A $K(G,1)$ *space* is a topological space X such that: (i) X is arcwise connected, with fundamental group isomorphic to G, and (ii) X admits a universal cover which is contractible.

Specifying a $K(G,1)$ space is equivalent to specifying a contractible space Y, equipped with an action of G which is (a) free, and[6] (b) such that every $y \in Y$ is contained in an open set $\mathcal{U}$ such that $g\mathcal{U} \cap \mathcal{U} = \emptyset$ if $g \neq 1$. Indeed, by the theory of coverings, the quotient map $Y \twoheadrightarrow Y/G =: X$ is then a covering map, and X is a $K(G,1)$ space.

If X is a $K(G,1)$ space, the singular (co)homology of X is isomorphic[7] to the (co)homology of G with trivial coefficients (in the case of cohomology it is an isomorphism of algebras):

$$H_*(G,\Bbbk) \simeq H_*(X,\Bbbk)\;, \qquad H^*(G,\Bbbk) \simeq H^*(X,\Bbbk)\;. \tag{25}$$

Exercice 21. Let G be a group acting freely on a contractible manifold. Show that G has no element of finite order.

Exercice 22. Let $G = \langle a_1, b_1, a_2, b_2, \dots a_g, b_g \mid [a_1,b_1]\cdots[a_n,b_n] = 1\rangle$. Compute $H_*(G,\Bbbk)$.

3.2 The proof of isomorphisms (??)

We concentrate on homology. The isomorphism is obtained by combining two independent arguments. We sketch these arguments here because they can be used or adapted to other situations.

A) If G acts on a space Y, then it acts on the set of singular simplices of Y, and the singular chain complex $C_*^{\mathrm{sing}}(Y,\Bbbk)$ is a complex of $\Bbbk G$-modules. Moreover:

[6] Note that condition (b) is automatically satisfied if Y is a normal space (e.g. a metric space or a CW complex) and if G is finite.

[7] This is actually the way that (co)homology was first defined historically, see Weibel, history of homological algebra. Chap 28 of History of topology, 797–836, North-Holland, Amsterdam, 1999.

- If the action of G on Y is free, then each $C_n^{\mathrm{sing}}(Y)$ is a projective module. This comes from the fact that is S is a set endowed with a free action of G, then $\Bbbk S$ (the free $\Bbbk$-module on S) is isomorphic to $(\Bbbk G)^{\oplus G/S}$ as a representation of G.
- If Y is contractible, then $C_*^{\mathrm{sing}}(Y,\Bbbk)$ has homology equal to $\Bbbk$ concentrated in degree zero.

To sum up: if Y is a contractible space with a free action of G, then $C_*^{\mathrm{sing}}(Y,\Bbbk)$ is a projective representation of the trivial representation $\Bbbk$.

As a consequence $H_*(G,\Bbbk)$ can be computed as the homology of the complex of $\Bbbk$-modules $C_*^{\mathrm{sing}}(Y,\Bbbk)_G$.

B) If Y is a space with a free action of G and $\pi : Y \twoheadrightarrow Y/G$ is a covering map, then π induces an isomorphism

$$C_*^{\mathrm{sing}}(Y,\Bbbk)_G \simeq C_*^{\mathrm{sing}}(Y/G,\Bbbk)\ .$$

This follows from the fact that if S is a set endowed with an action of G, then $(\Bbbk S)_G \simeq \Bbbk(S/G)$, and from the fact that π induces a bijection:

$$\{\text{ singular } n\text{-simplices of } Y\ \}/G \simeq \{\text{ singular } n\text{-simplices of } Y/G\ \}\ .$$

Exercice 23. Let X be an arcwise connected space with fundamental group G and n-connected universal cover. Show that $H_i(X,\Bbbk) \simeq H_i(G,\Bbbk)$ if $i \leq n$ and that there is a surjective map $\psi : H_{n+1}(X,\Bbbk) \twoheadrightarrow H_{n+1}(G,\Bbbk)$.

[Hint: modify A). You can also read Brown, Thm (5.2), where in addition $\operatorname{Ker}\psi$ is identified as the image of the Hurewicz map of X if $\Bbbk = \mathbb{Z}$.]

3.3 Groups acting on spaces and resolutions of $\Bbbk G$-modules

The argument **A)** above shows that free actions of G on contractible spaces provide projective resolutions of the trivial representation of G. However, the projective resolution $C_*^{\mathrm{sing}}(Y,\Bbbk)$ is not very useful for computational purposes because it is huge. In order to obtain smaller resolutions from topology, we can replace singular chains by *cellular chains*. This leads us to the following definition.

A *G-complex* is a CW-complex Y with an action of G such that the action permutes the cells of Y (i.e. if e is a cell of Y, then ge is a cell of Y).

Exercice 24. Consider S^n with the antipodal action of C_2. Define a CW structure on S^n which makes it a C_2-complex.

If Y is a contractible G-complex with free action, then the argument **A)** can be carried out with cellular chains, and the cellular complex $C_*^{\mathrm{cell}}(Y,\Bbbk)$ yields a resolution of the trivial representation $\Bbbk$ by projective $\Bbbk G$-modules.

Exercice 25. Work out the adaptation of the argument **A)** to cellular chain complexes.

In general, it is not easy to find nice small contractible G-complexes Y with free action. We can weaken our requirements on Y in two directions.

- First we can work with an n-connected G-complex Y with free action. Then $C_*^{\mathrm{cell}}(Y,\Bbbk)$ gives a projective resolution of $\Bbbk$ up to degree $n+1$.
- Second we can work with a G-complex Y on which the action of G is not free. In this case the $\Bbbk G$-modules $C_i^{\mathrm{cell}}(Y,\Bbbk)$ are no longer projective: we have an isomorphism of $\Bbbk G$-modules:

$$C_n^{\mathrm{cell}}(Y,\Bbbk) \simeq \bigoplus_{e\in\Sigma_n} \mathrm{ind}_{G_e}^G \Bbbk_\sigma \tag{26}$$

  where Σ_n is a set of representatives of the G-orbits of n-cells, G_e is the stabilizer of the cell e, and $\Bbbk_e$ is the $\Bbbk$-module $\Bbbk$ on which each $g\in G_e$ acts by multiplication by its topological degree. We refer the reader to Brown, Chap III, ex (5.5)(b) for an explanation of isomorphism (**??**).

  The complex $C_*^{\mathrm{cell}}(Y,\Bbbk)$ can then be used to link the homology of G with that of the stabilizers G_e through a spectral sequence (we will return to this in part II).