

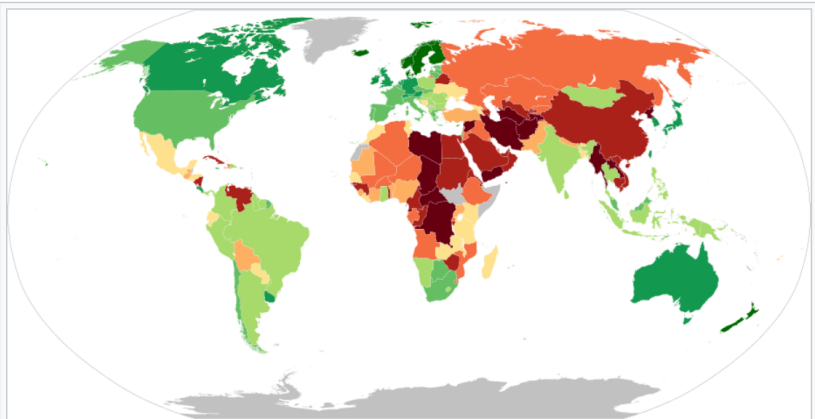
The Mathematics of Voting Systems

Niels Feld











IMT

2022-11-24

Democracy



The 2021 [Economist Intelligence Unit Democracy Index](#) map

Full democracies	Flawed democracies	Hybrid regimes	Authoritarian regimes	No data
 9.01-10	 7.01-8	 5.01-6	 3.01-4	
 8.01-9	 6.01-7	 4.01-5	 2.01-3	
			 0-2.00	

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Voters	A	B	C	D	E	F
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Rank 2	y	z	y	y	y	z
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(1)

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- A **voting system** is a function

$$\mathcal{P}^{\mathcal{V}} \rightarrow \mathcal{P}$$

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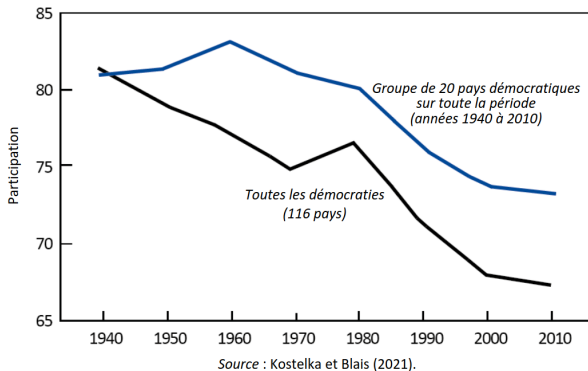
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Theorem (May's theorem)

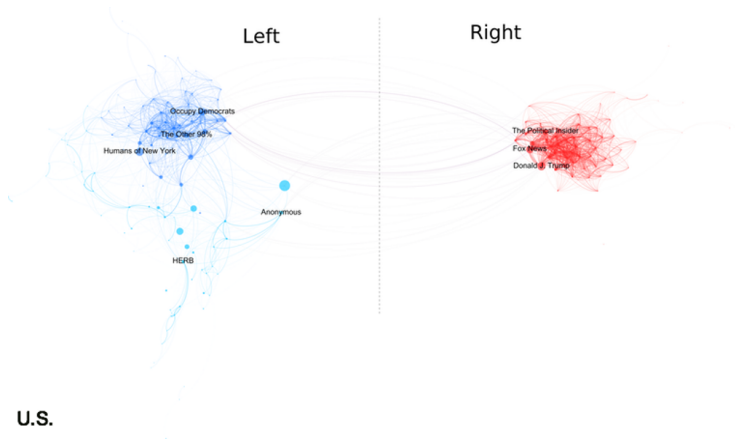
*A group decision function with an odd number of voters meets conditions 1, 2, 3, and 4 if and only if it is the simple **majority method** (i.e. a **referendum**).*

Abstention on the rise



Evolution of participation in national elections.

Political Polarization in the United States



Online behavior of Facebook users in the United States. Sharply different interests exist between left and right wing users. One year after the bitterly divisive election of Donald Trump as U.S. president, American Facebook users on the political right shared virtually no interests with those on the political left.

No Condorcet winner



2007 French presidential election

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2007 French presidential election

Bayrou-Sarkozy [edit]

Polling firm	Fieldwork date	Sample size	Bayrou UDF	Sarkozy UMP
Ipsos	19-20 Apr 2007	1,598	52.5%	47.5%
Ipsos	18-19 Apr 2007	1,209	52%	48%
TNS Sofres	18-19 Apr 2007	1,000	53%	47%
Ifop-Fiducial	17-19 Apr 2007	952	55%	45%
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Royal-Bayrou [edit]

Polling firm	Fieldwork date	Sample size	Royal PS	Bayrou UDF
Ifop-Fiducial	17-19 Apr 2007	952	42%	58%
Ifop-Fiducial	26-28 Mar 2007	954	43%	57%
TNS Sofres	14-15 Mar 2007	1,000	40%	60%
BVA	26-27 Feb 2007	869	45%	55%

Bayrou wins against Royal

Dependence of irrelevant alternatives

M. CHIRAC Jacques	19,88%
M. LE PEN Jean-Marie	16,86%
M. JOSPIN Lionel	16,18%
M. BAYROU François	6,84%
Mme LAGUILLER Arlette	5,72%
M. CHEVENEMENT Jean-Pierre	5,33%
M. MAMERE Noël	5,25%
M. BESANCENOT Olivier	4,25%
M. SAINT-JOSSE Jean	4,23%
M. MADELIN Alain	3,91%
M. HUE Robert	3,37%
M. MEGRET Bruno	2,34%
Mme TAUBIRA Christiane	2,32%
Mme LEPAGE Corinne	1,88%
Mme BOUTIN Christine	1,19%
M. GLUCKSTEIN Daniel	0,47%

2002 French presidential election.



Anti-Macron stencil message on a sidewalk in Paris in 2017.

Many different voting systems

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Condorcet winner criterion

The candidate who wins a majority of the vote in every head-to-head election against each of the other candidates – that is, a candidate preferred by more voters than any others – is the Condorcet winner.

Choosing of a voting system

Number of voters	5	4	2	6	8	2
Rank 1	x	x	t	t	z	t
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- Condorcet winner : z wins

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Theorem (Arrow's impossibility theorem)

*If a rank-order voting system satisfies conditions 1,2,3, then there is a **dictator** (i.e. there exists a single voter with the power to always determine the group's preference).*

Gibbard–Satterthwaite theorem

- **Strategy-proofness** : No voter should vote for another candidate or party than their sincere preference to prevent an undesirable outcome.





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Theorem (Gibbard–Satterthwaite theorem (1973-1975))

*Any voting system satisfying **unanimity** and **strategy-proofness** is **dictatorial**.*

Approval voting

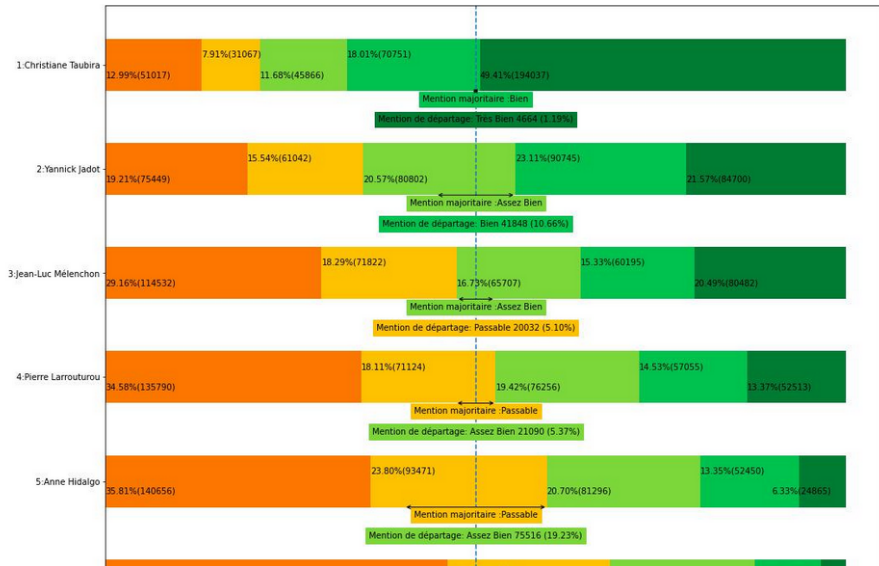
	Apr 25 THU	Apr 26 FRI	Apr 28 SUN	May 1 WED	May 5 SUN	May 10 FRI
 Tom		✓	✓			
 Paula	✓	✓	✓		✓	
 John		✓		✓		
 Emma		✓				

On an approval ballot, the voter can select any number of alternatives.

Approval voting

- Strategy-proofness : high.
- Independence of irrelevant alternatives : high.
- Condorcet criterion : low.

Majority Judgment



Christiane Taubira wins *La Primaire Populaire* in January 2022.

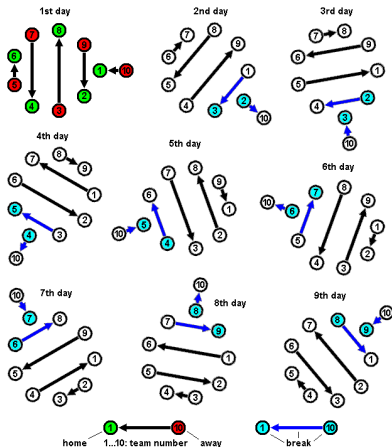
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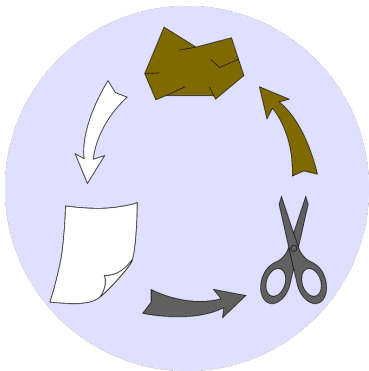
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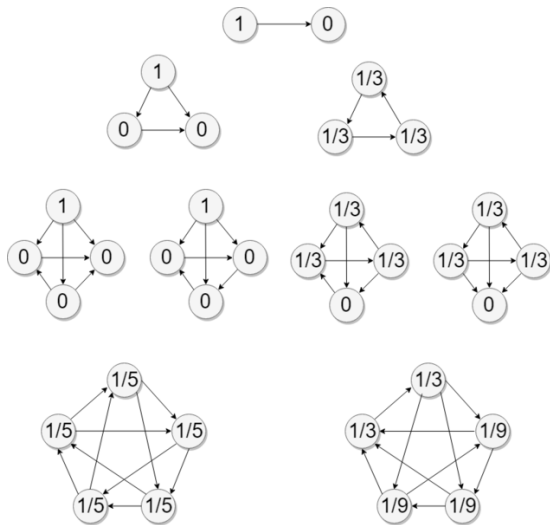


Condorcet paradox



There is no Condorcet winner.

Randomized Condorcet voting system



Examples of the probabilities assigned by the randomized Condorcet method for different graphs of the candidates' duels.

Randomized Condorcet voting system

- Strategy-proofness : high.
- Independence of irrelevant alternatives : high.
- Condorcet criterion : high.

Theorem

If a deterministic Condorcet winner exists, then the randomized Condorcet voting system is strategy-proof.

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Conclusion

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 - In 2021, an internal document leak from Facebook showed it was aware of harmful societal effects from its platforms (see *The Facebook Files*).
- **Incomplete set of preferences** : not everyone can give their opinion on everything (by lack of time, willingness, expertise, etc.).

Thank you !

