Comparison problems for bodies, measures and functions.

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(based on a joint work with Alexander Koldobsky and Michael Roysdon)

"61 Probability Encounters, In honor of Sergey Bobkov",
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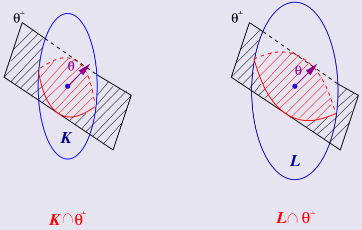
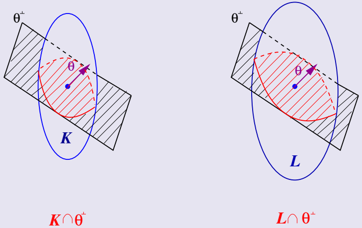


K, L origin symmetric convex bodies in $\mathbb{R}^n$. Let

$$\mathrm{vol}_{n-1}\left(K\cap\boldsymbol{\theta}^{\perp}\right)\leq \mathrm{vol}_{n-1}\left(L\cap\boldsymbol{\theta}^{\perp}\right), \forall\boldsymbol{\theta}\in\mathbb{S}^{n-1}.$$

Does it follow that

$$\mathrm{vol}_n(K)\leq\mathrm{vol}_n(L)?$$



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Yes, $n\leq 4$; No, $n\geq 5$.

K. Ball, J. Bourgain, R. Gardner, A.Giannopoulos, A. Koldobsky, D. Larman, E. Lutwak, M. Papadimitrakis, C. Rogers, T. Schlumprecht, G. Zhang.

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General measures

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Fix $n \geq 2$. Given two convex origin-symmetric bodies K and L in $\mathbb{R}^n$ such that

$$\mu(K \cap \theta^\perp) \leq \mu(L \cap \theta^\perp)$$

for every $\theta \in \mathbb{S}^{n-1}$, does it follow that

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(A.Z., 2005):

The answer to the above problem is independent from the "choice" of measure and depends only on the dimension n (i.e. YES for $n \leq 4$ and NO for $n \geq 5$).

Does there exist a constant $\mathcal{L} > 0$, so that if

$$\text{vol}_{n-1}\left(K \cap \theta^{\perp}\right) \leq \text{vol}_{n-1}\left(L \cap \theta^{\perp}\right), \forall \theta \in \mathbb{S}^{d-1}$$

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Best known result follows from direct connection to the slicing problem of Bourgain and works of V. Milman, A. Pajor/J. Brogain/ B. Klartag / Y. Chen/ B. Klartag and J. Lehec / A. Jambulapati, Y. T. Lee and S. Vempala / B. Klartag:

$$\mathcal{L} \leq C\sqrt{\log n}$$

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$$\mathcal{L} \leq C\sqrt{\log n}$$

As we just noted his question is equivalent to the slicing problem of J. Bourgain:

Slicing Problem:

Does there exist a constant $\mathcal{L}_1$ such that for any convex symmetric body $K \subset \mathbb{R}^d$

$$\mathrm{vol}_n(K)^{\frac{n-1}{n}} \leq \mathcal{L}_1 \max_{\theta \in \mathbb{S}^{n-1}} \mathrm{vol}_{n-1}(K \cap \theta^{\perp})?$$

$$\mathcal{L} \approx \mathcal{L}_1$$

Isomorphic version Busemann-Petty problem for arbitrary measures.

Problem:

Does there exist a constant $\mathcal{L}_2$, so that if $\mu(K \cap \boldsymbol{\theta}^{\perp}) \leq \mu(L \cap \boldsymbol{\theta}^{\perp})$ for every $\boldsymbol{\theta} \in \mathbb{S}^{n-1}$, then $\mu(K) \leq \mathcal{L}_2 \mu(L)$?

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(A. Koldobsky - A.Z, 2014 / Denghui Wu, 2020)

- $\mathcal{L}_2 \le \sqrt{n}$ - Independent of μ!

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Does there exist a constant $\mathcal{L}_3 > 0$ such that for any convex symmetric body $K \subset \mathbb{R}^n$

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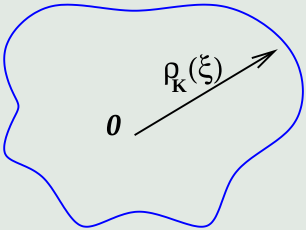
- No idea how $\mathcal{L}_2$ is connected to $\mathcal{L}_3$ (or how any of them really connected to $\mathcal{L}$).

Standard definition.

Radial function: $\rho_K(\xi) = \sup\{a : a\xi \in K\}$, for $\xi \in S^{n-1}$.



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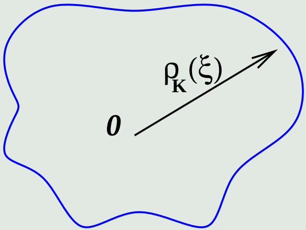

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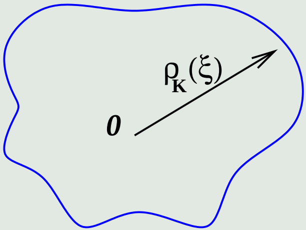

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- K is a star body if $\rho_K(\xi)$ is positive and continuous function on S^{n-1}.

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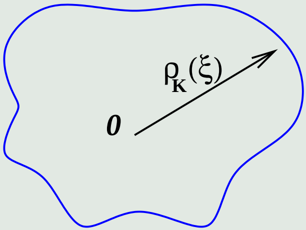

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Spherical coordinates and Volume:

$$\mathrm{vol}_n(K) = \int_{S^{n-1}} \int_0^{\rho_K(\theta)} r^{n-1} dr d\theta = \frac{1}{n} \int_{S^{n-1}} \rho_K^n(\theta) d\theta.$$

Can we further generalize this question and WHY?

Using spherical coordinates in $\xi^\perp$ we get

$$\mathrm{vol}_{n-1}(K\cap\xi^\perp)=\frac{1}{n-1}\int\limits_{S^{n-1}\cap\xi^\perp}\rho_K^{n-1}(\theta)d\theta=\frac{1}{n-1}R\rho_K^{n-1}(\xi).$$

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$$R\rho_K^{n-1}(\xi) \leq R\rho_L^{n-1}(\xi), \forall \xi \in \mathbb{S}^{n-1}.$$

Does it follow that $\int_{S^{n-1}} \rho_K^n(\xi) d\xi \leq \int_{S^{n-1}} \rho_L^n(\xi) d\xi$?

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Now let $f(\xi) = \rho_K^{n-1}(\xi)$ and $g(\xi) = \rho_L^{n-1}(\xi)$ the above becomes

Consider two positive, even functions $f, g : S^{n-1} \to \mathbb{R}$. Let

$$Rf(\xi) \le Rg(\xi), \forall \xi \in \mathbb{S}^{n-1}.$$

Does it follow that $\|f\|_{L_{\frac{n}{n-1}}(S^{n-1})} \le \|g\|_{L_{\frac{n}{n-1}}(S^{n-1})}$

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Spherical Comparison Problem

Fix $p > 0$. Consider two positive, even functions $f, g : S^{n-1} \to \mathbb{R}$. Let

$$Rf(\xi) \leq Rg(\xi), \forall \xi \in \mathbb{S}^{n-1}.$$

Does it follow that $\|f\|_{L_p(S^{n-1})} \leq \|g\|_{L_p(S^{n-1})}$?

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Case $p = 1$ is "easy" (it is a Fubini theorem on S^{n-1}). Case $p = n/(n-1)$ is a Busemann-Petty problem.

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Fix $p>0$. Consider two positive, even functions $f,g : S^{n-1} \to \mathbb{R}$. Let

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Take A - convex and symmetric apply Brunn concavity principle you get Busemann intersection inequality.

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Our "dream" to prove "reverse" inequalities, may be for some classes of functions.

Idea of the solution of the Busemann-Petty Problem in $\mathbb{R}^n$

$\text{vol}_{n-1}\left(K \cap \xi^{\perp}\right) \leq \text{vol}_{n-1}\left(L \cap \xi^{\perp}\right), \forall \xi \in \mathbb{S}^{n-1} \Longrightarrow \text{vol}_n(K) \leq \text{vol}_n(L)$?

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(A. Koldobsky) An origin-symmetric star body K in $\mathbb{R}^n$ is an **intersection body** if, and only if, ρ_K is a positive definite distribution on $\mathbb{R}^n$, i.e. "$\widehat{\rho_K}>0$".

Theorem (A. Koldobsky, M. Roysdon and A.Z., 2023):

Let f, g be even continuous positive functions on the sphere S^{n-1}, and suppose that

$$Rf(\theta) \leq Rg(\theta), \qquad \text{for all } \theta \in S^{n-1}. \tag{1}$$

Then:

(a) Suppose that for some $p > 1$ the function $|x|_2^{-1} f^{p-1}\left(\frac{x}{|x|_2}\right)$ represents a positive definite distribution on $\mathbb{R}^n$. Then $\|f\|_{L^p(S^{n-1})} \leq \|g\|_{L^p(S^{n-1})}$.

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Spherical Radon Transform Case

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The following hold true:

(c) Let g be an infinitely smooth strictly positive even function on S^{n-1} and $p > 1$. Suppose that the distribution $|x|_2^{-1} g^{p-1}\left(\frac{x}{|x|_2}\right)$ is not positive definite on $\mathbb{R}^n$. Then there exists an infinitely smooth even function f on S^{n-1} so that the condition (1) holds, but $\|f\|_{L^p(S^{n-1})} > \|g\|_{L^p(S^{n-1})}$.

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"Slicing Theorem" **FOR A VERY SPECIAL CLASS OF FUNCTIONS !**

Corrolary (A. Koldobsky, M. Roysdon and A.Z., 2023):

Let f be a positive even, continuous function on the sphere S^{n-1} Assume $p > 1$ and if $|x|_2^{-1} f^{p-1}\left(\frac{x}{|x|_2}\right)$ represents a positive definite distribution on $\mathbb{R}^n$, then

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Radon Transform in $\mathbb{R}^n$: Intersection Functions

Definition (A. Koldobsky, M. Roysdon and A.Z., 2023):

A non-negative, even, continuous, integrable function f on $\mathbb{R}^n$ is called an *intersection function* if, for every direction $\theta \in S^{n-1}$, the function

$$r \in \mathbb{R} \mapsto |r|^{n-1}\hat{f}(r\theta)$$

is a positive definite function on $\mathbb{R}$ for each $\theta \in S^{n-1}$ (note where $\widehat{f}$ denotes the Fourier transforms of f on $\mathbb{R}^n$).

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Theorem

An even, continuous, non-negative, and integrable function f defined on $\mathbb{R}^n$ is an intersection function if, and only if, for every direction $\theta \in S^{n-1}$, there exists a non-negative, even, finite Borel measure μ_θ on $\mathbb{R}$ such that

- *the function*

  $$\theta \in S^{n-1} \mapsto \int_{\mathbb{R}} \mathcal{R}\varphi(t,\theta)d\mu_\theta(t)$$

  belongs to $L_1(S^{n-1})$ whenever $\varphi \in \mathcal{S}(\mathbb{R}^n)$ (Schwartz space of rapidly decreasing infinitely differentiable test functions on $\mathbb{R}^n$), and

- $$\int_{\mathbb{R}^n} f\varphi = \int_{S^{n-1}} \int_{\mathbb{R}} \mathcal{R}\varphi(t,\theta)d\mu_\theta(t)d\theta.$$

  holds for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

Comparison Problem in $\mathbb{R}^n$

Theorem (A. Koldobsky, M. Roysdon and A.Z., 2023):

Let $p > 0$ and consider a pair of continuous, non-negative even functions $\varphi, \psi \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ satisfying the condition

$$\mathcal{R}\varphi(t,\theta) \le \mathcal{R}\psi(t,\theta) \quad \text{for all } (t,\theta) \in \mathbb{R} \times S^{n-1}.$$

Then:

(a) if $p > 1$ and φ^{p-1} is an intersection function, then $\|\varphi\|_{L^p(\mathbb{R}^n)} \le \|\psi\|_{L^p(\mathbb{R}^n)}$, and

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The following hold:

(c) Fix $p > 1$ and let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be non-negative and even. If ψ^{p-1} is not an intersection function, then there exists an even, non-negative $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\mathcal{R}\varphi(t,\theta) \leq \mathcal{R}\psi(t,\theta) \quad \text{for all } (t,\theta) \in \mathbb{R} \times S^{n-1},$$

but with $\|\psi\|_{L^p(\mathbb{R}^n)} < \|\varphi\|_{L^p(\mathbb{R}^n)}$.

(d) Fix $0 < p < 1$ and let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be non-negative and even. If φ^{p-1} is not an intersection function, then there exists a non-negative, even $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\mathcal{R}\varphi \leq \mathcal{R}\psi$, but with $\|\psi\|_{L^p(\mathbb{R}^n)} < \|\varphi\|_{L^p(\mathbb{R}^n)}$.

Is the a humane way to understand "intersection functions"?

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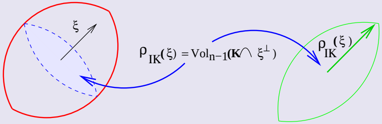
E. Lutwak: Intersection body, of a body K

$$K \longrightarrow IK$$



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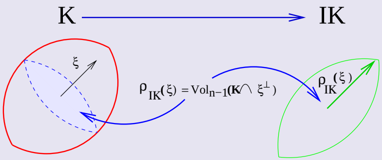

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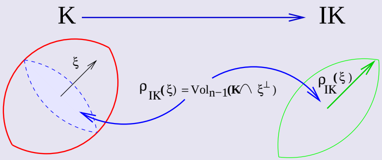

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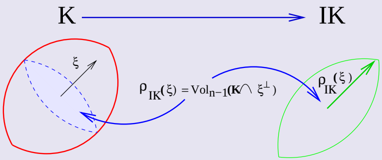

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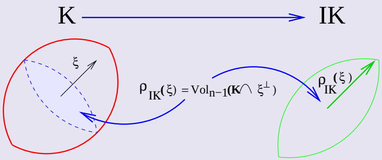

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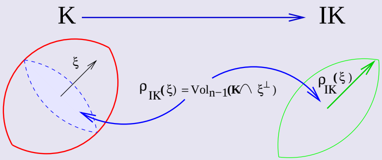

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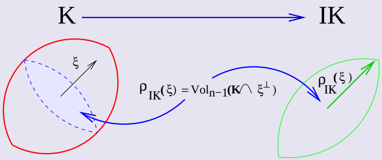

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Theorem

A function f on $\mathbb{R}^n$ is an intersection function of g if, and only if,

$$f = \frac{1}{\pi} \left(|x|_2^{-n+1} \left(g(t, \frac{x}{|x|_2}) \right)_t^{\wedge} (|x|_2) \right)_x^{\wedge},$$

where the interior Fourier transform is taken with respect to $t \in \mathbb{R}$, and the exterior Fourier transform is with respect to $x \in \mathbb{R}^n$.

Example (Exponentials)

Fix $q \in (0, 2]$, and let $\ell \in C(S^{n-1})$ be even and strictly positive. For each $\theta \in S^{n-1}$, set

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$$f_q(\xi) = \frac{1}{\pi}\left[|x|_2^{-n+1}\ell\left(\frac{x}{|x|^2}\right) e^{-|x|_2^q}\right]^\wedge_x (\xi)$$

is the intersection function of

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A non-example

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Again, take $\ell \in C(S^{n-1})$ even and strictly positive. To provide a non-example of an intersection function, for any $\theta \in S^{n-1}$ and $q > 2$, consider functions of the form $h_\theta(r) = \ell(\theta)\exp(-|r|^q)$, where $\ell \in C(S^{n-1})$ is strictly positive.

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But $\gamma_q(t)$ is not always non-negative, so the function f given by

$$f(x) = \frac{1}{\pi}\left[|x|_2^{-n+1}\ell\left(\frac{x}{|x|_2}\right)e^{-|x|_2^q}\right]_\xi^\wedge(x)$$

fails to be an intersection function.

Дорогой Серёжа,

Поздравляю вас с замечательным юбилеем! желаю здоровья, счастья и мира!

Ваш,

Артём