Subadditivity and factorization of the relative entropy in spin systems and random permutations

Pietro Caputo

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Thank you Sergej !

Plan of the talk

- Gibbs measures: spin systems and permutations
- Relative entropy, subadditivity and factorizations
- Approximate Shearer inequalities
- A general class of Gibbs samplers (heat bath dynamics)
- Recent results for spin systems
- Entropy subadditivity for permutations and proof of a conjecture of Carlen, Lieb, Loss ('04) and Samorodnitsky ('08)

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[BC22] A. Bristiel, PC, *Entropy inequalities for random walks and permutations*, Annales I.H.P. 2022

[ALO21] N. Anari, K. Liu, and S. Oveis Gharan.
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[CLV21] Z. Chen, K. Liu, and E. Vigoda.
Optimal Mixing of Glauber Dynamics: Entropy Factorization via High-Dimensional Expansion, ACM STOC 2021

Spin systems on a graph and permutations

μ is a Gibbs measure: a probability on $\Omega = \times_{i=1}^{n} \Omega_i$ describing some interacting system:

For instance, a *spin system* on a graph $G = (V, E)$, with $|V| = n$, is a Gibbs measure μ on $\Omega = [q]^V$, $[q] = \{1, \ldots, q\}$ for some $q \in \mathbb{N}$, associated with some interaction along the edges of G.
[Some results for continuous spins as well]

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Some examples:

Potts Model: $\mu(\sigma) = \frac{\exp(\beta M(\sigma))}{Z(G,\beta)}$, $M(\sigma) = \sum_{xy \in E} \mathbf{1}(\sigma_x = \sigma_y)$

Here $q \geqslant 2$. When $q = 2$ it is known as the **Ising Model**. When $\beta \geqslant 0$ the Potts model is called ferromagnetic.

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q-Colorings: $\mu(\sigma) = \frac{\mathbf{1}(\sigma \in \Omega_{G,q})}{|\Omega_{G,q}|}$, $\Omega_{G,q} = \{\text{proper } q\text{-colorings of } G\}$

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Permutations: $\mu(\sigma) = \frac{1}{n!} \mathbf{1}(\sigma \in S_n)$, $S_n =$ permutations of $[n]$

Entropy subadditivity

The entropy of $f : \Omega \mapsto \mathbb{R}_+$ w.r.t. μ is defined by

$$\mathrm{Ent}(f) = \mu\left[f \log(f/\mu[f])\right] = \int f \log(\tfrac{f}{\mu[f]})\, d\mu\,.$$

Rel. entropy, KL-div. $\mathrm{Ent}(f) = H(f\mu \mid \mu)$ when $\mu[f] = 1$.

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Approximate Subadditivity with const. C : Let $f_x(\sigma) := \mu(f \,|\, \sigma_x)$,

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[Barthe '98; Carlen,Lieb,Loss '04; Carlen, Cordero Erausquin '09,...]

- $C = 1$ if μ is product
- $C = 2$ if μ is uniform over the sphere $\mathbb{S}^{n-1}$ (optimal)
- $C = 2$ if μ is uniform over permutations S_n (NOT optimal)
- Equivalent to B-L type ineq. for all $\varphi_x : \Omega_x \mapsto \mathbb{R}_+$,

$$\mu\Big[\textstyle\prod_{x\in[n]} \varphi_x(\sigma_x)\Big] \;\leqslant\; \prod_{x\in[n]} \mu\Big[\varphi_x(\sigma_x)^C\Big]^{\frac{1}{C}},$$

Questions: spin systems ? permutations ?

Entropy tensorization

Approximate Tensorization with const. C :

Let $\mathrm{Ent}_x f := \mathrm{Ent}(f \mid \sigma_y\,,\; y \neq x)$,

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Essentially equivalent to a (modified) log-Sobolev inequality for Glauber dynamics:

- $C = 1$ if μ is product
- Spin systems on $G \subset \mathbb{Z}^d$ under *Strong Spatial Mixing* (SSM): Stroock-Zegarlinski '92; Martinelli, Olivieri '94; Cesi '01.
- For general graphs at high temp: C,Menz,Tetali'14; Marton'14; Bauerschmidt, Bodineau'19,
- Negative dependence: Cryan,Guo,Mousa'19; Hermon,Salez'19
- Major recent progress: entropic independence by Anari et al.'21, Chen, Feng, Yin, Zhang'21, stochastic localization by Chen, Eldan'22

Natural problem: find unified framework for subadd. and tensoriz.

Entropy factorizations: Approximate Shearer inequalities

Let μ_A^τ be the conditional distribution $\mu(\cdot|\sigma_{A^c} = \tau)$, $A \subset V$
τ is a boundary condition or a pinning. For $f : \Omega \mapsto \mathbb{R}$, $\mu_A f$ is conditional expectation $\mu_A f(\sigma) := \mu_A^{\sigma_{A^c}}[f]$ and
$\mathrm{Ent}_A(f) := \mu_A [f \log(f/\mu_A[f])]$ is conditional entropy :

$$\mu\,[\mathrm{Ent}_A(f)] = \mu\Big[\mu_A\,[f \log(f/\mu_A[f])]\Big] = \mathrm{Ent}\, f - \mathrm{Ent}\,(\mu_A f).$$

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Let $\alpha := \{\alpha_A,\, A \subset V\}$ a probability and $\gamma(\alpha) := \min_x \sum_{A \ni x} \alpha_A$.

Def: α-block factorization with const. $C(\alpha)$:

$$\gamma(\alpha)\,\mathrm{Ent}\, f \;\leqslant\; C(\alpha)\,\sum_{A \subset [n]} \alpha_A\, \mu\,[\mathrm{Ent}_A f]\,, \qquad f : \Omega \mapsto \mathbb{R}_+,$$

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Remarks:

- $C(\alpha) \equiv 1$ for all α if μ is product (Shearer inequality)
- Equivalent to subadditivity statement (by chain rule):

$$\sum_{A \subset [n]} \alpha_A\, \mathrm{Ent}\, \mu_A f \;\leqslant\; \left[1 - \frac{\gamma(\alpha)}{C(\alpha)}\right] \mathrm{Ent}\, f, \qquad f : \Omega \mapsto \mathbb{R}_+,$$

- $\alpha_A = \frac{1}{n}\mathbf{1}_{|A|=1} \Rightarrow$ App.Tens. and $\alpha_A = \frac{1}{n}\mathbf{1}_{|A|=n-1} \Rightarrow$ App.Sub.

Gibbs samplers, Mixing

Consider the Markov chain where at each step a subset $A \subset [n]$ is picked with probab. α_A and its spins $\sigma_A = \{\sigma_x,\ x \in A\}$ are updated according to $\mu_A^{\sigma_{A^c}}$. This chain has transition operator

$$P_\alpha f = \sum_{A \subset [n]} \alpha_A \, \mu_A f \,, \qquad f : \Omega \mapsto \mathbb{R},$$

call it the α- block dynamics. Note: $\alpha_A = \frac{1}{n} \mathbf{1}_{|A|=1} \Rightarrow$ Glauber dyn.

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P_α is reversible and has Dirichlet form:

$$\mathcal{D}_\alpha(f,g) = \langle f, (1 - P_\alpha) g\rangle = \sum_{A\subset[n]} \alpha_A \mu \left[\mathrm{Cov}_A(f,g)\right]$$

where $\mathrm{Cov}_A(f,g) = \mu_A[(f - \mu_A f)(g - \mu_A g)]$. Mixing time:

$$T_{\mathrm{mix}}(P_\alpha) := \inf\{t \in \mathbb{N} : \max_\sigma \| P_\alpha^t(\sigma,\cdot) - \mu\|_{TV} \ \leqslant \ 1/4\} \,.$$

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By Pinsker's inequality $\|\nu - \mu\|_{TV}^2 \leqslant \frac{1}{2} H(\nu|\mu)$,

$$\mathrm{Ent}(P_\alpha f) \leqslant (1-\delta)\mathrm{Ent}(f) \quad \Rightarrow \quad T_{\mathrm{mix}}(P_\alpha) \leqslant 4\,\delta^{-1} \log\log(1/\mu_*) \,,$$

where $\mu_* = \min_\sigma \mu(\sigma)$.

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where $\mu_* = \min_\sigma \mu(\sigma)$. The entropy contraction above is a discrete time analog of the *Modified log-Sobolev inequality*

$$\mathcal{D}(f, \log f) \geqslant \delta \, \mathrm{Ent}(f).$$

Block Factorization and Mixing

Lemma

If α-B.F. holds with constant $C(\alpha)$, then

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In particular, $T_{\mathrm{mix}}(P_\alpha) = O(\frac{C(\alpha)}{\gamma(\alpha)} \log n)$.

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Proof. α-B.F. means that

$$\sum_A \alpha_A \mu[\mathrm{Ent}_A(f)] \geqslant \frac{\gamma(\alpha)}{C(\alpha)}\mathrm{Ent}(f).$$

By convexity of $\mathrm{Ent}(\cdot)$:

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Note: the mixing time bound is tight up to $O(\log n)$ since the spectral gap always satisfies $\lambda(P_\alpha) \geqslant \gamma(\alpha)$. Often optimal mixing.
Ex: for Glauber dynamics $T_{\mathrm{mix}} = O(n\log n)$ if $C = O(1)$

How to establish Block Factorization ?

Three sets of results for spin systems:

in each of the following cases we prove α-Block Factorization of entropy with $C(\alpha) = O(1)$ for all α:

- Strong spatial Mixing (on $\mathbb{Z}^d$)
- Spectral Independence (general graphs)
- Contractive Coupling (general graphs, general coupling)

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For permutations : We prove exact α-BF for all homogeneous α, that is for all α of the form

$$\alpha_A = \sum_{\ell=1}^{n} w_\ell \, \mathbf{1}_{|A|=\ell} \,, \qquad w_\ell \geqslant 0.$$

Entropy factorization for $G \subset \mathbb{Z}^d$ under SSM

Theorem (CP21)

For $G \subset \mathbb{Z}^d$, under SSM , the α-BF holds with $C(\alpha) = O(1)$ for all α, uniformly in n and the boundary conditions.

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For $G \subset \mathbb{Z}^d$, under SSM , the α-BF holds with $C(\alpha) = O(1)$ for all α, uniformly in n and the boundary conditions.

1. Reduce to proving a bipartite factorization into even/odd vertices $\alpha_E = \alpha_O = 1/2$.
2. Use suitable recursive strategy to prove it for even/odd case (main difficulty: lack of a simple additive structure).

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1. Reduce to spin/edge factorization for Edwards-Sokal coupling ν:

$$\mathrm{Ent}_\nu(F) \leqslant C\left[\nu\left(\mathrm{Ent}_\nu(F|\text{spin}) + \mathrm{Ent}_\nu(F|\text{edge})\right)\right].$$

2. Lift the even/odd factorization to spin/edge factorization
3. Lower bound $T_{\mathrm{mix}}(P_{\mathrm{SW}})$ by disagreement percolation estimates.

Note: it covers the whole uniqueness region $\beta < \beta_c$ in $d = 2$.

General graphs: Spectral independence (SI)

[ALO20] introduced SI and used it to prove a poly(n) bound for the Glauber dynamics of the hard-core gas in the uniqueness regime.

$$J(x, a; y, b) = \mu(\sigma_y = b | \sigma_x = a) - \mu(\sigma_y = b) \quad \text{for } x \neq y.$$

J is a $\mathcal{X} \times \mathcal{X}$ matrix, $\mathcal{X} = [n] \times [q]$ with real eigenvalues $\lambda_i(J)$.

Definition

μ is η-spectrally independent if $\lambda_{\max}(J) \leqslant \eta$ for all possible pinnings. (Note: $\eta \geqslant 0$).

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Main idea: η-SI with $\eta = O(1)$ enables a powerful recursive scheme to prove spectral gap for the Glauber dynamics . This "local-to-global" approach was developed in the abstract setting of simplicial complexes: based on recent work of Oppenheim, Dinur–Kaufman, Alev–Lau on high dim. expanders.

Main result under Spectral Independence

Theorem (BCCPSV22)

If μ is η-SI for some $\eta = O(1)$ then the α- BF holds with $C(\alpha) = O(1)$ for all α, uniformly in n and the boundary conditions. Therefore, all α- block dynamics have optimal $T_{\text{mix}} = O(\gamma(\alpha)^{-1} \log n)$. Moreover, for ferromagnetic Ising/Potts, the SW dynamics has $T_{\text{mix}} = O(\log n)$.

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To prove it we extend the recursive approach of [ALO20,CLV20] and show a multi-partite factorization

$$\mathrm{Ent}(f) \leqslant C \sum_{i=1}^{k} \mu\left[\mathrm{Ent}_{V_i}(f)\right]$$

where V_i are independent sets with $V = \cup_{i=1}^{k} V_i$, and $k \leqslant \Delta + 1$.

The multi-partite factorization is then lifted to a general BF.

Some remarks on the SI approach

Strength :

- It allows us to prove tight bounds in some cases up to the tree uniqueness threshold. For instance, for ferro-Ising, our results on arbitrary block dynamics and SW dynamics hold for all $\beta < \beta_c(\Delta) = \log(\frac{\Delta}{\Delta-2})$. Previously known only for Glauber dynamics from Mossel, Sly (2013).
- SI is very flexible: we show that it covers all standard spatial mixing notions such as Dobrushin-uniqueness condition or SSM, and holds as soon as μ admits some form of positive curvature, that is the existence of a contractive coupling. See below for more precise statements

Restrictions:

- our results for BF require bounded degree $\Delta = O(1)$. [Not for subadditivity]
- they do not apply to unbounded or continuous spins (need b-marginal bound $\min_{x,a} \mu(\sigma_x = a) \geqslant b$ with $1/b = O(1)$).

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$$\frac{\ell}{n} \operatorname{Ent} f \leqslant \frac{K(n,\ell)}{\binom{n}{\ell}} \sum_{|A|=\ell} \mu \left[\operatorname{Ent}_A f\right], \qquad K(n,\ell) = \frac{\ell \log(n!)}{n \log(\ell!)}.$$

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The inequality is saturated uniquely at multiples of a Dirac mass.
In particular ($\ell = n-1$): subadd. with $C_n = \frac{n \log n}{\log(n!)} = 1 + O(\frac{1}{\log n})$,

$$\sum_{x\in[n]} \mathrm{Ent}\, f_x, \ \leqslant \ \frac{n \log n}{\log(n!)} \ \mathrm{Ent}\, f .$$

Note that $\ell = 1$ is trivial since fixing all labels except x determines the label at x. Similarly, the case $\ell = n$ is trivial with $K(n,n) = 1$.
Proof uses martingale recursive approach as in the proof of Log-Sob and modified Log-Sob for Random Transpositions, see [Lee,Yau '00] and [Goel '05], [Guo,Quastel '05]. Note: we compute optimal constants exactly (an advantage of BF over LSI or MLSI).

A combinatorial application

The following sharp upper bound on the permanent of a matrix with arbitrary nonnegative entries was independently conjectured by [Carlen, Lieb, Loss '04] and by [Samorodnitsky '08] . Let $A = (a_{i,j})$ denote an $n \times n$ matrix, and consider its permanent

$$\mathrm{perm}(A) = \sum_{\sigma \in \mathcal{S}_n} \prod_{i=1}^{n} a_{i,\sigma_i}.$$

Theorem

For any $p \geqslant 1$, for any $n \times n$ matrix A with nonnegative entries,

$$\mathrm{perm}(A) \leqslant \max\left\{1, \frac{n!}{n^{n/p}}\right\} \prod_{i=1}^{n} \|R_i\|_p,$$

where R_i is the $i-$th row of A and $\|\cdot\|_p$ is the ℓ_p-norm, with equality uniquely achieved at either the identity or the all 1 matrix.

Permanent upper bound

Note that 1 and $\frac{n!}{n^{n/p}}$ correspond to the case where A is the identity matrix or A is the all-1 matrix respectively.
The proof uses the subadditivity from previous theorem,

$$\sum_{x\in[n]} \mathrm{Ent}\, f_x, \leqslant \frac{n\log n}{\log(n!)} \mathrm{Ent} f, \qquad f : S_n \mapsto \mathbb{R}_+,$$

Setting $p_c := \frac{n\log n}{\log(n!)}$, this is equivalent to: $\forall \varphi_x : [n] \mapsto \mathbb{R}_+$,

$$\mu\Big[\prod_{x\in[n]} \varphi_x(\sigma_x)\Big] \leqslant \prod_{x\in[n]} \mu\Big[\varphi_x(\sigma_x)^{p_c}\Big]^{1/p_c},$$

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If $a_{x,y} = \varphi_x(y)$, the L.H.S. is $(1/n!)\mathrm{perm}(A)$, while R.H.S. is $n^{-n/p_c} \prod_{x\in[n]} \|R_x\|_{p_c} = (1/n!) \prod_{x\in[n]} \|R_x\|_{p_c}$, where we use that $u(p) = \frac{n!}{n^{n/p}}$ satisfies $u(p_c) = 1$.

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Related result: Bregman-Minc theorem

For any $n \times n$ matrix with $0, 1$ entries, Bregman-Minc inequality:

$$\mathrm{perm}(A) \leqslant \prod_{i=1}^{n} (\|R_i\|_1!)^{1/\|R_i\|_1}.$$

Conjectured by Minc ('63), proven by Bregman ('73).
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Contractive coupling implies Spectral Independence

Hamming distance: $d_{\mathrm{H}}(\sigma, \sigma') = \sum_{x \in V} \mathbf{1}(\sigma_x \neq \sigma'_x)$.

W-1 distance: $W_1(\mu, \nu) = \inf\{\mathbb{E}_\pi[d_{\mathrm{H}}(\sigma, \sigma')],\ \pi \in \mathcal{C}(\mu, \nu)\}$.

A Markov chain P has (Ollivier-Ricci) curvature $\rho \in (0, 1)$ if

$$W_1(P(\sigma, \cdot), P(\sigma', \cdot)) \leqslant (1 - \rho) d_{\mathrm{H}}(\sigma, \sigma'), \qquad \forall \sigma, \sigma' \in \Omega$$

In other words, if there exists a $(1 - \rho)$-contractive coupling.

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Theorem (BCCPSV22)

If the Glauber dynamics has curvature $\rho \in (0, 1)$, then $\lambda_{\max}(J) \leqslant \frac{2}{\rho n}$. In particular, if $\rho \geqslant \varepsilon / n$ then μ is η-spectrally independent with $\eta = 2/\varepsilon$. Moreover, it has BF with $C = O(1)$.

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The theorem can be considerably extended by allowing other distances and much more general Markov chains (see below).
But even in the above setting this is quite a strong result:
If Glauber has a contractive coupling then our theorems show that all heat bath dynamics as well as SW dynamics have optimal entropy decay and optimal mixing. [$\Rightarrow$ Peres-Tetali conjecture ?]

Main ideas

Use $\lambda_{\max}(J) \leqslant \max_{(x,a)\in\mathcal{X}} S(x,a)$,
$S(x,a) = \sum_{(y,b)\in\mathcal{X}} |\mu(\sigma_y = b|\sigma_x = a) - \mu(\sigma_y = b)|$, and

$$S(x,a) = \nu[f] - \mu[f],$$

where $\nu = \mu(\cdot|\sigma_x = a)$, $f(\sigma) = \sum_{(y,b)} \mathrm{sgn}(J(x,a;y,b))\mathbf{1}(\sigma_y = b)$.

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Since f is 2-Lipschitz: $S(x,a) \leqslant 2W_1(\mu,\nu)$.

Lemma (BCCPSV22)

(Ω, d) finite metric space, μ, ν distr. on Ω, and P, Q two MCs with stationary distr. μ, ν resp. If (P, d) has curvature $\rho > 0$, then

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In our case: $W_1(P(\sigma,\cdot),Q(\sigma,\cdot)) \leq \frac{1}{n}$, and therefore $S(x,a) \leqslant \frac{2}{\rho n}$.
As in Bresler-Nagaraj '19, the proof uses Poisson eq.
$(1-P)h = f - \mu[f], \quad \nu[f] - \mu[f] = \nu[(Q-P)h],$
$(Q-P)h(\sigma) \leqslant \mathrm{L}(h)W_{1,d}(P(\sigma,\cdot),Q(\sigma,\cdot)), \quad \mathrm{L}(h) \leqslant \mathrm{L}(f)/\rho.$

Extenstions

Definition

A collection $\mathcal{P} = \{P_\tau,\, \tau \text{ pinning}\}$ of MCs associated with μ is Φ-local if for any two adjacent pinnings τ, τ' and $\tau' = \tau \cup (x, a)$,

$$W_1(P_\tau(\sigma, \cdot), P_{\tau'}(\sigma, \cdot)) \le \Phi.$$

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Theorem

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Proof: very similar to previous theorem. Moreover, it extends to non-Hamming distance $d \asymp d_{\mathrm{H}}$. This is very useful in applications.

Some applications

1. For general spin systems Dobrushin uniqueness implies spectral independence. (Extending results of Hayes '06, Dyer,Goldberg,Jerrum '09 who proved that DU and related conditions imply curvature bounds).

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3. Ferromagnetic Potts model has contractive coupling for $\beta < \beta_1$ (Bordewich, Greenhill, Patel '16 use heat bath block dynamics with bounded block size) where $\beta_1 \approx$ tree uniqueness as $q \to \infty$.