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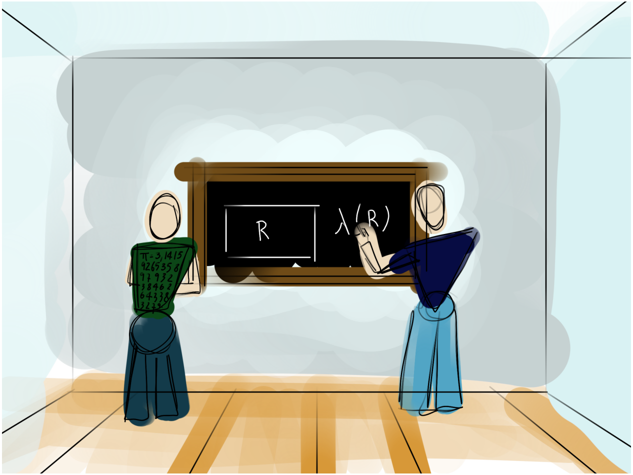
Around the log-Brunn-Minkowski inequality

61 probability encounters - In honour of Sergey Bobkov

Toulouse
May 29th - June 2nd, 2023



Paris, July 2006

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Here $|\cdot|$ denotes the Lebesgue measure.

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Remark. h_k is a 1-homogeneous convex function. Viceversa, every 1-homogeneous convex function $h\colon \mathbb{R}^n \to \mathbb{R}$ is the support function of a (unique) convex body.

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If K and L are convex bodies in $\mathbb{R}^n$, then

$$h_{K+L} = h_K + h_L \quad \text{and} \quad h_{\alpha K} = \alpha h_K$$

for every $\alpha \geq 0$.

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Define

$$\alpha \cdot K +_p \beta \cdot L,$$

through:

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- Ingredient 3: a standard argument based on homogeneity of the Lebesgue measure.

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- The main problem when passing to $p \leq 1$ (and $p \geq 0$), is that, given two convex bodies K and L,

$$(h_K^p + h_L^p)^{1/p}$$

is **not convex**, in general (and then it is not a support function).

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  $h_{K[f]}$ is the largest 1-homogeneous convex function smaller than f.

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- A similar inequality (we will call it (p-BM)) can be conjectured for $0 < p < 1$.

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- Unconditional convex bodies - Saroglou, 2015 (also based on previous results by Cordero-Erausquin, Fradelizi, Maurey, 2004).

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- $p \in [1 - \frac{c}{-0.75}, 1]$ under an inclusion assumption - Hosle, Kolesnikov, Livshyts, 2020 (with a unifying approach including other inequalities, like Gardner-Zvavitch).

See also:

- Kolesnikov-Livshyts, 2020: *On the local version of the log-Brunn-Minkowski conjecture and some related geometric inequalities*;
- E. Milman, 2021: *A sharp centro-affine isospectral inequality of Szegö-Weinberger type and the L^p-Minkowski problem*.

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See also an unpublished note of Dario Cordero-Erausquin & Galyna Livshyts.

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Remark. (*) is stronger than Prékopa-Leindler inequality.

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$$\begin{aligned} \int_{\mathbb{R}^n} e^{-u_t}\,dz = \int_{\mathbb{R}^n} e^{-h_{\mathrm{conv}(K_0\cup K_1)}} &= c(n)|(\mathrm{conv}(K_0 \cup K_1))^\circ| \\ &= c(n)|K_0^\circ \cap K_1^\circ| \\ \left(\int_{\mathbb{R}^n} e^{-u_0}\right)^{1-t}\left(\int_{\mathbb{R}^n} e^{-u_1}\right)^t &= c(n)|K_0^\circ|^{1-t}|K_1^\circ|^t. \end{aligned}$$

Counterexamples (support functions)

Let K be a convex body containing the origin.

$$\int_{\mathbb{R}^n} e^{-h_K}\,dx = c(n)|K^\circ|, \qquad \begin{aligned} K^\circ &= \{x \colon \langle x, y\rangle \le 1\, \forall y \in K\} \\ &= \text{polar body of } K. \end{aligned}$$

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$$|K_0^\circ \cap K_1^\circ| \ge |K_0^\circ|^{1-t}|K_1^\circ|^t \quad \text{is in general false.}$$

Back to the inequality

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- We compute the infinitesimal form of $(*)$.

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$$\mu_{u_0} = v_0 \mu_{u_0}^{ck}$$

where $\mu_{u_0}^{ck}$ is the *moment measure*, considered by Cordero-Erasquin and Klartag.

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Compare with the infinitesimal form of the Prékopa-Leindler inequality (Brascamp-Lieb, 1976):

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- Checked for $n = 1$.
- Checked for n large enough, with the crucial help of Yaozhong Qiu.

See also: S. Bobkov, M. Ledoux, *From Brunn-Minkowski to Brascamp-Lieb and to logarithmic Sobolev inequalities, 2000.*