Transport-Entropy forms of direct and reverse Santaló type inequalities

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Introduction

The starting point of this talk is the following celebrated paper by Sergey Bobkov and Friedrich Götze:

S.G. Bobkov, F. Götze, **Exponential integrability and transportation cost related to logarithmic Sobolev inequalities**, *J. Funct. Anal.* 163, No. 1, 1-28 (1999).

This paper contains several important results:

- a full characterization of the set of probability measures on $\mathbf{R}$ satisfying the log-Sobolev inequality,
- an improved Herbst type argument to deduce deviation bounds from the log-Sobolev inequality,
- a nice duality argument relating transport-entropy inequalities to infimum convolution inequalities.

Introduction

In all what follows E is a Polish space.

Transport cost

Given a cost function $c : E \times E \to \mathbf{R}_+ \cup \{+\infty\}$ and two probability measures μ, ν on E, the optimal transport cost $\mathcal{T}_c(\nu, \mu)$ is defined by

$$\mathcal{T}_c(\nu, \mu) = \inf \mathbb{E}[c(X, Y)],$$

where the infimum runs over all couplings (X, Y) between ν and μ.

Transport-entropy inequalities (Marton, Talagrand '90's)

A probability measure μ satisfies the transport-entropy inequality with the cost function c, if

$$(1) \qquad \mathcal{T}_c(\mu, \nu) \leq H(\nu|\mu) := \int \log \frac{d\nu}{d\mu}\, d\nu, \qquad \forall \nu \ll \mu$$

A probability measure μ satisfies the *symmetrized* transport-entropy inequality with the cost function c, if

$$(2) \qquad \mathcal{T}_c(\nu_1, \nu_2) \leq H(\nu_1|\mu) + H(\nu_2|\mu), \qquad \forall \nu_1, \nu_2 \ll \mu$$

- These inequalities have a deep connection to the concentration of measure phenomenon.
- Inequality (2) obviously implies (1).
- Conversely, as soon as $\mathcal{T}_c$ satisfies some approximate triangle inequality, then (1) implies (2) (up to constants).

Introduction

Theorem (Bobkov-Götze '99)

- A probability measure μ satisfies the transport-entropy inequality with the cost function c, if and only if for all bounded continuous function $f : E \to \mathbf{R}$

$$(1') \qquad \int e^{Q_c f}\, d\mu \leq e^{\int f\, d\mu},$$

where

$$Q_c f(x) = \inf_{y \in E} \{f(y) + c(x,y)\}, \qquad x \in E$$

- A probability measure μ satisfies the symmetrized transport-entropy inequality with the cost function c, if and only if for all bounded continuous function $f : E \to \mathbf{R}$

$$(2') \qquad \int e^{Q_c f}\, d\mu \int e^{-f}\, d\mu \leq 1$$

a.k.a the (τ) Property introduced by Maurey in '91.

(Here I quote a straightforward generalization of the result initially proved in BG'99)

Introduction

$$(1') \qquad \int e^{Q_c f}\, d\mu \leq e^{\int f\, d\mu}, \qquad \forall f \text{ bounded continuous}$$

Interest of this dual formulation:

- For p.m on $\mathbf{R}^d$ and (say) c being the squared Euclidean norm, (1') or (2') can be easily related to the Prekopa-Leindler inequality.
- Another advantage of (1') is that it can be established using Hamilton-Jacobi equations whose solutions can be expressed using inf-convolution operators.

$\rightsquigarrow$ See the 2001 paper by Sergey Bobkov, Ivan Gentil and Michel Ledoux about hypercontractivity properties of the Hamilton-Jacobi semigroup and their alternative proof of the Otto-Villani theorem.

Introduction

Bobkov and Götze result is based on the following classical duality relation between relative entropy and log-Laplace functionals ...

Proposition

Let m be a Borel measure on E.
If ν is a probability measure on E with $\nu = h.m$ such that $\log h \in L^1(\nu)$, then

$$H(\nu|m) = \sup\left\{\int f\, d\nu - \log\int e^f\, dm : f \text{ s.t } \int e^f\, dm < +\infty\right\}.$$

Conversely, if $\int e^f\, dm < +\infty$, then

$$\log\int e^f\, dm = \sup\left\{\int f\, d\nu - H(\nu|m) : \nu = hm \text{ with } \log h \in L^1(\nu)\right\}.$$

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...and on the Kantorovich dual formulation of the $\mathcal{T}_c$ transport cost:

Theorem

If c is l.s.c, then for all probability measures ν_1, ν_2 on E,

$$\mathcal{T}_c(\nu_1,\nu_2) = \sup_{f\in\mathcal{C}_b(E)}\left\{\int Q_c f\,d\nu_1 - \int f\,d\nu_2\right\}.$$

Aim of the talk: adapt this Bobkov-Götze duality argument to obtain transport-entropy versions of direct and converse Blaschke-Santaló inequalities.

Based on joint works with M. Fradelizi, S. Sadowsky and S. Zugmeyer

Plan

1. Introduction

2. Transport-entropy forms of direct Blaschke-Santaló inequalities
   - Blaschke-Santaló inequality
   - Fathi's improved symmetrized Talagrand inequality
   - Other transport-entropy forms of Blaschke-Santaló on other model spaces

3. Transport-entropy forms of reverse Blaschke-Santaló inequalities
   - Inverse Santaló inequality: the Mahler conjecture

Blaschke-Santaló Inequality

Theorem

If $K \subset \mathbf{R}^n$ is a centrally symmetric convex body, then

$$\mathrm{Vol}(K)\mathrm{Vol}(K^\circ) \leq \mathrm{Vol}(B_2^n)^2,$$

where Vol denotes the Lebesgue measure on $\mathbf{R}^n$ and the polar of K is defined by

$$K^\circ := \{y \in \mathbf{R}^n : x \cdot y \leq 1, \forall x \in K\}.$$

As proved by Ball (1986) in the case of even functions and then extended by Artstein-Avidan, Klartag and Milman (2004) and Fradelizi and Meyer (2007), the Santaló inequality admits the following functional form:

Theorem

For any measurable function $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ such that $0 < \int e^{-f}\, dx < +\infty$, there exists $z \in \mathbf{R}^n$ such that

$$\int e^{-f_z}\, dx \int e^{-(f_z)^*}\, dx \leq (2\pi)^n,$$

where $f_z(x) = f(x - z)$, $x \in \mathbf{R}^n$. When f is even, z can be chosen to be 0.

More generally, if $\int x e^{-f(x)}\, dx = 0$, then z can be chosen to be 0 (Lehec 2009).

Plan

Talagrand transport-entropy inequality

Recall that the quadratic optimal transport cost W_2^2 is the optimal transport cost for the cost function

$$c(x,y) = |x-y|^2, \qquad x,y \in \mathbf{R}^n$$

with $|\cdot|$ the standard Euclidean norm.

Theorem (Talagrand (1996))

The standard Gaussian measure γ_n on $\mathbf{R}^n$ satisfies:

$$W_2^2(\nu,\gamma_n) \leq 2H(\nu|\gamma_n)$$

for all probability measures ν on $\mathbf{R}^n$.
The constant 2 is sharp. Equality cases for $n=1$: $\nu = \mathcal{N}(a,1)$.

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Theorem (Symmetric form of Talagrand inequality)

For all probability measures ν_1, ν_2 on $\mathbf{R}^n$, it holds

$$W_2^2(\nu_1, \nu_2) \leq 4H(\nu_1|\gamma_n) + 4H(\nu_2|\gamma_n),$$

and the constant 4 is sharp.
Equality cases for $n = 1$: $\nu_1 = \mathcal{N}(-a, 1)$ and $\nu_2 = \mathcal{N}(a, 1)$, $a > 0$.

Fathi's improved symmetrized Talagrand inequality

The following improvement of the symmetric Talagrand inequality is due to Max Fathi :

Theorem (Fathi (2018))

For all probability measures ν_1, ν_2 on $\mathbf{R}^n$, one of which is centered, it holds

$$W_2^2(\nu_1, \nu_2) \leq 2H(\nu_1|\gamma_n) + 2H(\nu_2|\gamma_n).$$

This result can be derived from Santaló inequality (and implies it back). So it can be considered as a transport-entropy form of the Santaló inequality.

Sketch of proof of Fathi's result

According to Lehec's functional version Santaló inequality, for all measurable function $f : \mathbf{R}^n \to \mathbf{R}$ such that $\int x e^{-f(x)}\, dx = 0$

$$\int e^{-f}\, dx \int e^{-f^*(y)}\, dy \leq (2\pi)^n.$$

Do the change of function:

$$f(x) = \frac{|x|^2}{2} + g(x) \qquad x \in \mathbf{R}^n.$$

One gets

$$\int e^{Q_2 g}\, d\gamma_n \int e^{-g}\, d\gamma_n \leq 1, \qquad \forall g \text{ such that } \int x e^{-g(x)}\, \gamma_n(dx) = 0,$$

with $Q_2 g(x) = \inf_{y \in \mathbf{R}^n} \{g(y) + \frac{|x-y|^2}{2}\}$, $x \in \mathbf{R}^n$.

This is Bobkov-Götze dual formulation (2') with an extra centering condition.
Since one of the measures is assumed to be centered, this extra condition is not a problem.

Plan

A general functional formulation of the Blaschke-Santaló inequality

The following result was first proved by Ball '86 for even functions, then extended to arbitrary functions by successive works by Artstein-Avidan, Klartag and Milman '04, Fradelizi and Meyer '07 and Lehec '09:

Theorem

If $f : \mathbf{R}^n \to \mathbf{R}_+$ is integrable, then there exists a point $z \in \mathbf{R}^n$ such that for any measurable function $g : \mathbf{R}^n \to \mathbf{R}_+$ satisfying

$$f(x+z)g(y) \leq \rho(\langle x, y\rangle)^2, \qquad \forall x, y \in \mathbf{R}^n \text{ such that } \langle x, y\rangle > 0,$$

it holds

$$\int f(x)\,dx \int g(y)\,dy \leq \left(\int \rho(|x|^2)\,dx\right)^2,$$

where $\rho : \mathbf{R}_+ \to \mathbf{R}_+$ is some weight function such that $\int \rho(|x|^2)\,dx < +\infty$.
If f is even, then z can be chosen to be 0.

The previously stated functional Blaschke-Santaló inequality corresponds to the weight function

$$\rho_0(t) = e^{-t/2}, \qquad t \geq 0$$

A general transport-entropy inequality

In the spirit of Fathi's improvement of Talagrand's, one can easily get:

Theorem (Fradelizi-G.-Sadowsky-Zugmeyer '23)

If $\rho : \mathbf{R}_+ \to (0, \infty)$ is a continuous non-increasing function such that $\int \rho(|x|^2)\,dx < +\infty$, and $t \mapsto -\log \rho(e^t)$ is convex on $\mathbf{R}$, then the probability measure

$$\mu_\rho(dx) = \frac{\rho(|x|^2)}{\int \rho(|y|^2)\,dy}\,dx$$

satisfies the following inequality: for all $\nu_1, \nu_2 \in \mathcal{P}(\mathbf{R}^n)$ with ν_1 and ν_2 even,

$$\mathcal{T}_\omega(\nu_1, \nu_2) \leq H(\nu_1|\mu_\rho) + H(\nu_2|\mu_\rho),$$

where $\mathcal{T}_\omega$ is the optimal transport cost associated to the cost function ω defined by

$$\omega(x,y) = \begin{cases} \log\left(\frac{\rho(x\cdot y)^2}{\rho(|x|^2)\rho(|y|^2)}\right) & \text{if } x\cdot y \geq 0 \\ +\infty & \text{otherwise} \end{cases}, \qquad x, y \in \mathbf{R}^n.$$

Remarks:

- This is a transport-entropy form of Ball's functional Blaschke-Santaló inequality.
- If one linearizes the transport-entropy inequality around μ_ρ one recovers the sharp Brascamp-Lieb type inequality recently used by Cordero-Erausquin and Rotem in their study of the (B) inequality for rotationally invariant models.

Particular case of Barenblatt profiles

For $s > 0$, consider

$$\rho_s(t) = (1 - st)_+^{\frac{1}{2s}}, \qquad t \geq 0 \qquad \text{and} \qquad \gamma_s(dx) := \mu_{\rho_s}(dx) = \frac{1}{Z_s}\left[1 - s|x|^2\right]_+^{1/(2s)} dx.$$

Theorem (Fradelizi-G.-Sadowsky-Zugmeyer '23)

For any $s > 0$, the probability γ_s satisfies the following inequality: for any ν_1, ν_2 with ν_2 centered

$$\mathcal{T}_{c_s}(\nu_1, \nu_2) \leq H(\nu_1|\gamma_s) + H(\nu_2|\gamma_s),$$

where $c_s : B(0, 1/\sqrt{s}) \times B(0, 1/\sqrt{s}) \to \mathbf{R}$

$$c_s(x, y) = \frac{1}{s}\log\left(\frac{1 - sx \cdot y}{(1 - s|x|^2)^{1/2}(1 - s|y|^2)^{1/2}}\right), \qquad x, y \in B(0, 1/\sqrt{s}).$$

Remarks:

- This gives back Fathi's result in the Gaussian case by sending $s \to 0$.
- This result implies an analog of Lehec's functional Blaschke-Santaló inequality for the cost ρ_s, $s > 0$.
- The case $s < 0$ can also be considered and corresponds to Cauchy type distributions. In this case, one cannot allow couples ν_1, ν_2 with one centered.

A symmetrized version of Kolesnikov's inequality

Let $\mathbb{S}^n \subset \mathbf{R}^{n+1}$ be the unit Euclidean sphere and let σ the uniform probability measure on $\mathbb{S}^n$.
Let $\alpha : \mathbb{S}^n \times \mathbb{S}^n \to \mathbf{R}_+ \cup \{+\infty\}$ be the cost function defined by

$$\alpha(u,v) = \begin{cases} \log\left(\frac{1}{u\cdot v}\right) & \text{if } u\cdot v > 0 \\ +\infty & \text{otherwise} \end{cases}, \qquad u,v \in \mathbb{S}^n$$

and denote by $\mathcal{T}_\alpha$ the corresponding transport cost.

Theorem (Kolesnikov '20)

For all even probability measures ν on $\mathbb{S}^n$, it holds

$$(n+1)\mathcal{T}_\alpha(\nu,\sigma) \leq H(\nu|\sigma).$$

Remarks:

- The cost function α has been introduced by Oliker '07 (see also Bertrand '16) in connection with the so-called Aleksandrov problem in convex geometry.
- If ν is not assumed to be even, this inequality can be false.

A symmetrized version of Kolesnikov's inequality

Kolesnikov's inequality can be improved as follows (with a different proof)

Theorem (Fradelizi-G.-Sadowsky-Zugmeyer '23)

For all even probability measures ν_1, ν_2 on $\mathbb{S}^n$, it holds

$$(n+1)\mathcal{T}_\alpha(\nu_1, \nu_2) \leq H(\nu_1|\sigma) + H(\nu_2|\sigma).$$

Remarks:

- This inequality can be partially deduced from our transport-entropy inequality for the Cauchy distribution on $\mathbf{R}^{n+1}$ by projection.
- The constant $n+1$ is sharp. More precisely, if one linearizes the symmetrized Kolesnikov inequality around σ, one gets that for all even function $f : \mathbb{S}^n \to \mathbf{R}$, it holds

$$2(n+1)\mathrm{Var}_\sigma(f) \leq \int |\nabla_{\mathbb{S}^n} f|^2 \, d\sigma.$$

  The constant $2(n+1)$ is the second non-zero eigenvalue of the Laplace operator on $\mathbb{S}^n$. Equality is reached for instance for $f(u) = u_1^2$, which is even.
- If one applies the classical Marton's argument, one obtains a concentration inequality on $\mathbb{S}^n$ which is very close to be optimal: if $A, B \subset \mathbb{S}^n$ are two symmetric subsets of $\mathbb{S}^n$, then $d_{\mathbb{S}^n}(A, B) \leq \pi/2$ and it holds

$$\sigma(A)\sigma(B) \leq \cos^{n+1}(d_{\mathbb{S}^n}(A, B)).$$

Sketch of proof

According to the Blaschke-Santaló inequality, for any symmetric convex body C in $\mathbf{R}^{n+1}$, it holds

$$|C||C^\circ| \leq |B_2^{n+1}|^2.$$

Calculating the volume of C in polar coordinate yields

$$|C| = |B_2^{n+1}| \int_{\mathbb{S}^n} \rho_C(u)^{n+1} \sigma(du),$$

where $\rho_C(u) = \sup\{t \geq 0 : tu \in C\}$, $u \in \mathbb{S}^n$, denotes the radial function of C. Similarly,

$$|C^\circ| = |B_2^{n+1}| \int_{\mathbb{S}^n} \rho_{C^\circ}(u)^{n+1} \sigma(du) = |B_2^{n+1}| \int_{\mathbb{S}^n} \frac{1}{h_C(u)^{n+1}} \sigma(du),$$

using that $\rho_{C^\circ} = 1/h_C$, where $h_C(u) = \sup_{x \in C} x \cdot u$, $u \in \mathbb{S}^n$, is the support function of C.
So, for every symmetric convex C body in $\mathbf{R}^{n+1}$, it holds

$$\int_{\mathbb{S}^n} \rho_C(u)^{n+1} \sigma(du) \int_{\mathbb{S}^n} \frac{1}{h_C(u)^{n+1}} \sigma(du) \leq 1.$$

On the other hand, as observed by Oliker, if ν_1, ν_2 are two even probability measures on $\mathbb{S}^n$, the Kantorovich duality reads

$$(n+1)\mathcal{T}_\alpha(\nu_1, \nu_2) = \sup_C \int -\ln\left(h_C^{n+1}\right) \, d\nu_1 + \int \ln\left(\rho_C^{n+1}\right) \, d\nu_2,$$

where the supremum runs over the set of all symmetric convex bodies C.
We conclude by applying the Bobkov-Götze duality argument.

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Inverse Santaló inequality: the Mahler conjecture

Conjecture (Mahler 1939)

If K is centrally symmetric, then

$$\mathrm{Vol}(K)\mathrm{Vol}(K^\circ) \geq \mathrm{Vol}(B_1^n)\mathrm{Vol}(B_\infty^n) = \frac{4^n}{n!}$$

Known cases

(1) Mahler proved the conjecture in dimension 2.

(2) Iriyeh and Shibata (2020) recently proved the conjecture in dimension 3. Alternative proof by M. Fradelizi, A. Hubard, M. Meyer, E. Roldan-Pensado and A. Zvavitch.

(3) Saint-Raymond (1981) showed that the Mahler conjecture holds true for unconditional convex bodies, that is to say convex body K satisfying

$$x = (x_1, \ldots, x_n) \in K \Rightarrow (\varepsilon_1 x_1, \ldots, \varepsilon_n x_n) \in K,$$

for all $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 1\}^n$.

(4) Bourgain and Milman (1987) showed that Mahler conjecture is asymptotically true: there exists some absolute constant $\alpha > 0$ such that for all $n \geq 1$ and all symmetric convex body $K \subset \mathbf{R}^n$, it holds

$$\mathrm{Vol}(K)\mathrm{Vol}(K^\circ) \geq \frac{\alpha^n}{n!}.$$

Functional form

Theorem (Fradelizi-Meyer '08)

The Mahler conjecture holds for all $n \geq 1$ if and only if the inequality

$$\int e^{-f}\,dx \int e^{-f^*}\,dx \geq 4^n$$

holds for all $n \geq 1$ and all even, lower semicontinuous and convex functions $f : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ such that $\int e^{-f}\,dx > 0$ and $\int e^{-f^*}\,dx > 0$. Moreover, the inequality is true for unconditional f.

For all $n \geq 1$, denote by

$$c_n^F = \inf_f \int e^{-f}\,dx \int e^{-f^*}\,dx$$

where f runs over the set of admissible convex functions on $\mathbf{R}^n$.

Mahler conjecture and improvements of log-Sobolev

Theorem (G. '22)

For any $n \geq 1$, the constant c_n^F is the best constant $c > 0$ (that is the greatest) in the inequality

$$(A) \qquad H(\eta_1|\gamma) + H(\eta_2|\gamma) + \frac{1}{2}W_2^2(\nu_1, \nu_2) \leq \frac{1}{2}I(\eta_1|\gamma) + \frac{1}{2}I(\eta_2|\gamma) + \log\left(\frac{(2\pi)^n}{c}\right),$$

where $\eta_1 = e^{-V_1}\,dx, \eta_2 = e^{-V_2}\,dx$ are arbitrary even log-concave probability measures on $\mathbf{R}^n$ with full support and, for $i = 1, 2$, ν_i is the so-called moment measure of η_i defined by

$$\nu_i = \nabla(V_i)_{\#}\eta_i$$

and $I(\eta|\gamma)$ is the relative Fisher information of η with respect to γ defined by

$$I(\eta|\gamma) = \int \left|\nabla \log \frac{d\eta}{d\gamma}\right|^2 \eta(dx).$$

Moreover, inequality (A) holds with the $c = 4^n$ if η_1, η_2 are assumed to be unconditional.

Remarks:

- If $W_2(\nu_1, \nu_2)$ is large enough, then inequality (A) above improves Gross' log-Sobolev inequality for the standard Gaussian measure γ: for all η with smooth enough density

$$H(\eta|\gamma) \leq \frac{1}{2}I(\eta|\gamma).$$

- In the unconditional case, Inequality (A) with $c = 4^n$ is sharp.

Equivalent formulation for the Lebesgue measure

Theorem (G. '22)

For all $n \geq 1$, the constant c_n^F is the best constant $c > 0$ in the following inequality: for all symmetric log-concave probability measures η_1, η_2 on $\mathbf{R}^n$ such that, for $i = 1, 2$, $\eta_i(dx) = e^{-V_i}\, dx$ for some $V_i : \mathbf{R}^n \to \mathbf{R}$, it holds

$$H(\eta_1|\mathrm{Leb}) + H(\eta_2|\mathrm{Leb}) + \mathcal{T}_{c_1}(\nu_1, \nu_2) \leq -\log(e^{2n}c),$$

where ν_1, ν_2 are the moment measures of η_1 and η_2 and $c_1 : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ is defined by

$$c_1(x, y) = -x \cdot y.$$

In a subsequent work with Fradelizi and Zugmeyer, we have obtained a simple proof of this inequality in dimension 1, with sharp constant $c = 4$.

Ingredients of the proof

The proof is again an adaptation of BG '99.

We consider a twisted duality where the Log-Laplace functional

$$f \mapsto \Lambda(f) := \log \int e^f \, dx$$

is replaced by the functional L defined by

$$L(f) := -\log \int e^{-f^*} \, dx.$$

Theorem (Cordero-Erausquin-Klartag '15)

If m is a log-concave measure on $\mathbf{R}^n$, then for all measurable functions $f_0, f_1 : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$, it holds

$$\int e^{-((1-t)f_0 + tf_1)^*} \, dm \geq \left(\int e^{-f_0^*} \, dm \right)^{1-t} \left(\int e^{-f_1^*} \, dm \right)^t.$$

In particular, the functional L is convex.

Simple consequence of the Prekopa-Leindler inequality.

Ingredients of the proof

For $\nu \in \mathcal{P}_1(\mathbf{R}^n)$, consider the dual functional of L:

$$K(\nu) := \sup_f \left\{ \int (-f)\, d\nu + \log \int e^{-f^*}\, dx \right\},$$

$$= \sup_f \left\{ \int (-f)\, d\nu - L(f) \right\}.$$

Ingredients of the proof

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$$= \sup_f \left\{ \int (-f)\, d\nu - L(f) \right\}.$$

The functional K admits a more pleasant alternative expression :

Proposition

For any $\nu \in \mathcal{P}_1(\mathbf{R}^n)$, it holds

$$K(\nu) = -\inf_{\eta \in \mathcal{P}_1(\mathbf{R}^n)} \{H(\eta|\mathrm{Leb}) - \mathcal{T}_{c_1}(\nu, \eta)\}.$$

Ingredients of the proof

The proof of (1) $\Rightarrow$ (2) relies on the characterization of moment measures by Cordero-Erausquin and Klartag.

Theorem (Cordero-Erausquin-Klartag '15/Santambrogio '16)

(a) A probability measure $\nu \in \mathcal{P}(\mathbf{R}^n)$ is the moment measure of some log-concave probability measure η_o on $\mathbf{R}^n$ such that $\eta_o(dx) = e^{-V_o}\,dx$ for some essentially continuous convex function $V_o : \mathbf{R}^n \to \mathbf{R} \cup \{+\infty\}$ if and only if $\nu \in \mathcal{P}_1(\mathbf{R}^n)$, ν is centered and its support is not contained in an hyperplan. The function V_o is moreover unique up to translations.

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(b) If ν is centered and its support is not contained in an hyperplan, then the probability measure η_o is up to translations the unique minimizer of the functional $\eta \mapsto H(\eta|\mathrm{Leb}) - \mathcal{T}_{c_1}(\nu, \eta)$ on $\mathcal{P}_1(\mathbf{R}^n)$:

$$\inf_{\eta \in \mathcal{P}_1(\mathbf{R}^n)} \{H(\eta|\mathrm{Leb}) - \mathcal{T}_{c_1}(\nu, \eta)\} = H(\eta_o|\mathrm{Leb}) - \mathcal{T}_{c_1}(\nu, \eta_o).$$

Perspectives

In a work in progress with M. Fradelizi, S. Sadowsky and S. Zugmeyer, we obtain a similar characterization of

$$c_{n+1} = \inf_{K \text{symmetric convex body of } \mathbf{R}^{n+1}} |K||K^\circ|$$

on the sphere $\mathbb{S}^n$.

One has the following correspondence

$$\begin{array}{rcl} \mathbf{R}^n & \leftrightarrow & \mathbb{S}^n \\ \gamma/Lebesgue & \leftrightarrow & \sigma \\ W_2^2/\mathcal{T}_{c_1} & \leftrightarrow & \mathcal{T}_\alpha \\ \text{Moment measures} & \leftrightarrow & \text{Cone measures} \end{array}$$

The main difference is that there is no uniqueness for cone measures.

Thank you for your attention !