# On Fenchel and Bézout type inequalities for the Lebesgue and the Gaussian measures.

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Recent work in collaboration with
Dylan Langharst, Mokshay Madiman, Mathieu Meyer and Artem Zvavitch:
F.-Madiman-Zvavitch: Sumset estimates in convex geometry. arXiv:2206.01565.
FMMZ: On the volume of the Minkowski sum of zonoids. arXiv:2206.02123.
Dylan Langharst, Mokshay Madiman and Artem Zvavitch:
Weighted Brunn-Minkowski Theory I: On Weighted Surface Area Measures
Weighted Brunn-Minkowski Theory II: On Inequalities for Mixed Measures

61 Probability Encounters, In honour of Sergey Bobkov
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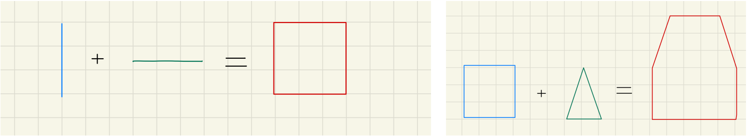

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From the arithmetic-geometric inequality, we deduce

$$\mathrm{vol}(A)V(A[n-2],B,C) \leq 2V(A[n-1],B)V(A[n-1],C).$$

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- Prehistory: Brunn-Minkowski, Minkowski and Fenchel
- Betke-Weil's conjecture for mixed volumes

Bézout type inequalities

- Bézout type inequality for mixed volumes
- Bézout type inequality for Minkowski sums
- Bézout inequality for sums with constant one
- Courtade's conjecture

Generalization to measures

- Minkowski and Fenchel's inequalities
- Bézout for rotation invariant measures

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$\mathrm{vol}(A) V(B[j], C[k], A[n-j-k]) \leq \min\left(\binom{n}{j}, \binom{n}{k}\right) V(B[j], A[n-j]) V(C[k], A[n-k])$.

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Brazitikos-Giannopoulos-Liakopoulos (2016): $b_m \leq 2^{2^{m-1}-1}$.
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Ndiaye (2023+): $b_m \leq \min_{1 \leq k \leq m} 2^{\frac{k(k-1)}{2}} \frac{n^{m-k}}{(m-k)!}$ and more generalizations of Xiao.

# Bézout type inequality for mixed volumes

Let us recall the definitions from Maud's talk: for a class $\mathcal{C}$ of convex bodies, $b_m^{\mathcal{C}}(A)$ is the smallest constant such that for any $K_1, \ldots, K_m \in \mathcal{C}$

$$\mathrm{vol}(A)^{m-1} V(K_1, \ldots, K_m, A[n-m]) \leq b_m^{\mathcal{C}}(A) \prod_{i=1}^{m} V(K_i, A[n-1]),$$

and $b_m^{\mathcal{C}} = \sup_{A \in \mathcal{C}} b_m^{\mathcal{C}}(A)$. Moreover, $b_m(A) = b_m^{\mathcal{K}}(A)$ and $b_m = b_m^{\mathcal{K}}$ where $\mathcal{K}$ is the class of all convex bodies. Let $\mathcal{Z}$ be the class of zonoids.
Hug-Schneider conjecture that $b_m^{\mathcal{Z}}(B_2^n) = b_m(B_2^n)$ and prove that

$$b_m^{\mathcal{Z}}(B_2^n) = v_n^{m-1} \frac{(n-m)!}{n!} v_{n-m} \left( \frac{n}{v_{n-1}} \right)^m, \quad b_2^{\mathcal{Z}}(B_2^n) = \frac{n}{n-1} \frac{v_n v_{n-2}}{v_{n-1}^2}.$$

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Ndiaye (2023+): $b_m \leq \min_{1 \leq k \leq m} 2^{\frac{k(k-1)}{2}} \frac{n^{m-k}}{(m-k)!}$ and more generalizations of Xiao.

Question: for any convex body $A$: $b_m(A) = b_m^{\mathcal{Z}}(A)$? It would imply that $b_m = \frac{m^{m-1}}{(m-1)!}$.

# Plan

# Bézout type inequality for Minkowski sums

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For a class $\mathcal{C}$ of convex bodies, $c^{\mathcal{C}}_{n,m}(A)$ is the smallest constant such that for any $K_1, \ldots, K_m \in \mathcal{C}$

$$\mathrm{vol}(A)^{m-1}\mathrm{vol}(A + K_1 + \ldots, K_m) \leq c^{\mathcal{C}}_{n,m}(A) \prod_{i=1}^{m} \mathrm{vol}(A + K_i),$$

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Ndiaye (2023+) used his new Bézout inequalities for mixed volumes to prove:

$$c_{n,2} \leq c_{n,m} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

# Plan

# Bézout inequality for sums with constant one

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We look for which $A$ and which class $\mathcal{C}$ of convex bodies, $c_{n,m}^{\mathcal{C}}(A) = 1$: for any $K_1, \ldots, K_m \in \mathcal{C}$

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3) We conjecture that (3B) holds for $A = B^n_2$ and $B, C$ be any compact convex sets. We proved it if $B$ is a zonoid.
4) From Böröczky-Hug's inequalities, (BS) holds for $A = B^n_2$ and $K_2, \dots, K_m$ zonoids. In the same way, the generalized Betke-Weil conjecture implies (BS) for $A = B^n_2$.

# Plan

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1) Statement of the conjecture. Let $B, C$ be compact convex sets in $\mathbb{R}^n$. Is it true that

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## Theorem (F.-Madiman-Meyer-Zvavitch 2022+)

*Let $A, B, C$ be convex compact sets in $\mathbb{R}^2$. Then*

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The main tool is Fenchel's inequality:

$$(|A|V(B, C) - V(A, B)V(A, C))^2 \leq \left(V(A, B)^2 - |A||B|\right)\left(V(A, C)^2 - |A||C|\right).$$

# Plan

# F-concavity

## Definition

A Borel measure $\mu$ is $F$-concave on a class $\mathcal{C}$ of compact Borel subsets of $\mathbb{R}^n$ if there exists a continuous, strictly monotone function $F : (0, \mu(\mathbb{R}^n)) \to (-\infty, \infty)$ such that, for every pair $K, L \in \mathcal{C}$ and every $\lambda \in [0, 1]$, one has

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- Let $\Phi(x) = \gamma_1((-\infty, x))$. Then, $\gamma_n$ is $\Phi^{-1}$ concave on the set of compact Borel subsets of $\mathbb{R}^n$ (Ehrhard, Borell)

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When $F(x) = x^s, s > 0$ this can be written as

$$\mu((1-\lambda)K + \lambda L)^s \geq (1-\lambda)\mu(K)^s + \lambda\mu(L)^s,$$

and we say $\mu$ is $s$-concave. In the limit as $s \to 0$, we obtain the case of log-concavity:

$$\mu((1-\lambda)K + \lambda L) \geq \mu(K)^{1-\lambda}\mu(L)^{\lambda}.$$

The Gaussian Measure on $\mathbb{R}^n$ is given by $d\gamma_n(x) := (2\pi)^{-n/2}e^{-|x|^2/2}dx$.

## Concavity of the Gaussian Measure

- log-concave over compact Borel subsets of $\mathbb{R}^n$
- Let $\Phi(x) = \gamma_1((-\infty, x))$. Then, $\gamma_n$ is $\Phi^{-1}$ concave on the set of compact Borel subsets of $\mathbb{R}^n$ (Ehrhard, Borell)
- $1/n$ concave over the set of symmetric convex bodies (Gardner and Zvavitch, Kolesnikov and Livshyts, Eskenazis and Moschidis)

# Mixed measures of Bodies

Definitions of mixed measures Let $\mu$ be a Borel measure on $\mathbb{R}^n$ supported on a class $\mathcal{C}$ of compact Borel sets with non-empty interior closed under Minkowski addition. Then, for $A, B, C \in \mathcal{C}$:

-the mixed measure of $(n-1)$ copies of $A$, one copy of $B$ is

$$\mu(A; B) = \frac{\partial}{\partial t}\mu(A + tB)(0).$$

- the mixed measure of $(n-2)$ copies of $A$, one copy of $B$ and one copy of $C$ is given by

$$\mu(A; B, C) = \frac{\partial^2}{\partial s \partial t}\mu(A + sB + tC)(0, 0).$$

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## Theorem (Integral representation of mixed measure)

*Let $\mu$ be a Borel measure on a class $\mathcal{C}$ of compacts sets closed under Minkowski addition. Suppose $\mu$ has differentiable density $\phi$. For $A, B, C \in \mathcal{C}$ with $A$ being $C^2_+$, then one has*

$$\mu(A; B, C) := (n-1)\int_{\mathbb{S}^{n-1}} \phi(n_A^{-1}(u))h_C(u)dS_{A[n-2],B[1]}(u)$$
$$+ \int_{\mathbb{S}^{n-1}} \langle \nabla\phi(n_A^{-1}(u)), \nabla h_B(u)\rangle h_C(u)dS_A(u).$$

# Gaussian Measure

- Denote by $\varphi(x) = \frac{e^{-|x|^2/2}}{(2\pi)^{n/2}}$ the density of the standard Gaussian measure.

$$\gamma_n(A; B, C) = (n-1)\int_{\mathbb{S}^{n-1}} \varphi(|\nabla h_A(u)|) h_C(u) dS_{A[n-2],B[1]}(u) - \int_{\mathbb{S}^{n-1}} \langle \nabla h_A(u), \nabla h_B(u)\rangle h_C(u) \varphi(|\nabla h_A(u)|) dS_A(u) du.$$

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- **Example:** Let $B = [-\xi, \xi]$ for some $\xi \in \mathbb{S}^{n-1}$. Then, $V(A[n-2], B, B) = 0$. But,

$$\gamma_n(A; [-\xi, \xi], [-\xi, \xi]) = \int_{\mathbb{S}^{n-1}} \langle \nabla h_A(u), \xi\rangle \langle u, \xi\rangle \varphi(|\nabla h_A(u)|)\, dS_A(u) du.$$

# Minkowski's First, Second and Quadratic Inequalities

Minkowski's First and Second Inequalities for $F$-concave measures:

Livshyts: Let $\mu$ be $F$-concave on a class of compact Borel sets $\mathcal{C}$. Assume that $F$ increases. Then, for $K, L \in \mathcal{C}$, the function $f(\lambda) = F\left(\mu\left((1-\lambda)K + \lambda L\right)\right) - (1-\lambda)F(\mu(K)) - \lambda F(\mu(L))$ is concave, non-negative and $f(0) = f(1) = 0$ so $f'(0) \geq 0$ and $f''(0) \leq 0$.

$$\mu(K, L) \geq \mu(K, K) + \frac{F(\mu(L)) - F(\mu(K))}{F'(\mu(K))}.$$

(FLMZ): Furthermore, if $\mu$ also has differentiable density, then

$$-\frac{F''(\mu(K))}{F'(\mu(K))}\mu(K; L)^2 \geq \mu(K; L, L)$$

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Fenchel inequality for mixed measures, FLMZ Let $f(s,t) = F(\mu(A + sB + tC))$. Then, if $F$ increases, $f$ is concave and so

$$\left(\frac{\partial^2 f}{\partial s \partial t}(0,0))\right)^2 \leq \frac{\partial^2 f}{\partial s^2}(0,0)\frac{\partial^2 f}{\partial t^2}(0,0). \quad (2)$$

One has

$$\frac{\partial^2 f}{\partial s^2}(0,0) = F''(\mu(A))\,\mu(A; B)^2 + F'(\mu(A))\,\mu(A; B, B),$$

and similarly for $\frac{\partial^2 f}{\partial s^2}(0,0)$. But also, from the definition of $\mu(A; B, C)$, one has

$$\frac{\partial^2 f}{\partial s \partial t}(0,0)) = F''(\mu(A))\,\mu(A; B)\mu(A; C) + F'(\mu(A))\,\mu(A; B, C).$$

# Plan

History
- Prehistory: Brunn-Minkowski, Minkowski and Fenchel
- Betke-Weil's conjecture for mixed volumes

Bézout type inequalities
- Bézout type inequality for mixed volumes
- Bézout type inequality for Minkowski sums
- Bézout inequality for sums with constant one
- Courtade's conjecture

Generalization to measures
- Minkowski and Fenchel's inequalities
- Bézout for rotation invariant measures

# Bézout for Measures

Generalization of Arstein-Avidan-Florentin-Ostrover's inequality:

## Theorem

*Let $\mu$ be a rotationally invariant log-concave measure with density $e^{-\varphi(|x|)}$. Then, for every $R > 0$, $Z \in \mathcal{Z}^n$ and $C$, one has*

$$\mu(RB_2^n; Z)\mu(RB_2^n; C) \geq A_{\mu,R} \frac{\kappa_{n-1}^2}{\kappa_{n-2}\kappa_n} \mu(RB_2^n)\mu(RB_2^n; Z, C),$$

*where*

$$A_{\mu,R} := \frac{n}{n+1}\left(1 + \frac{1}{n - \varphi'(R)R}\right) \geq 1; \quad \lim_{R\to 0} A_{\mu,R} = 1.$$

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## Gaussian Measure and $R = 1$

Fix $n \geq 2$. Let $Z \in \mathcal{Z}^n$, and $C \ni 0$. Then, one has

$$\gamma_n(B_2^n; Z)\gamma_n(B_2^n; C) \geq e^{-\frac{(2n+1)}{2(n+1)^2}} \frac{n}{n-1} \frac{\kappa_{n-1}^2}{\kappa_{n-2}\kappa_n} \gamma_n(B_2^n)\gamma_n(B_2^n; Z, C).$$

Furthermore, this is sharper than in the above.

# Open questions

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1) Hug-Schneider's conjecture: for any convex bodies:

$$V(K_1,\dots,K_m,B_2^n[n-m]) \le \frac{(n-m)!}{n!} v_{n-m} \left(\frac{n}{v_{n-1}}\right)^m \prod_{i=1}^m V(K_i,B_2^n[n-1]), \quad (HS)$$

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2) Courtade's conjecture: Let $n \geq 3$ and $B, C$ be convex compact sets in $\mathbb{R}^n$. Then

$$(|B||C|)^{1/n} + (|B_2^n||B_2^n+B+C|)^{1/n} \leq (|B_2^n+B||B_2^n+C|)^{1/n}\,? \quad (CC)$$

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3) 3 zonoids' conjecture: Let $n \ge 4$. For any zonoids $A, B, C$ in $\mathbb{R}^n$ do we have

$$|A+B+C||A| \le |A+B||A+C|? \quad (3B)$$

4) Strong 3 zonoids' conjecture: Let $n \ge 3$. For any zonoids $A, B, C$ in $\mathbb{R}^n$ do we have

$$(|B||C|)^{1/n} + (|A||A+B+C|)^{1/n} \le (|A+B||A+C|)^{1/n}?$$

# End

Thank you!