On the intertwining approach for proving Poincaré type functional inequalities

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Based on a series of works with M. Bonnefont

- Consider on $\mathbb{R}^n$ the probability measure

$$d\mu(x) \propto e^{-V(x)} dx,$$

where V is some smooth potential on $\mathbb{R}^n$.

$\hookrightarrow$ Canonical diffusion operator:

$$Lf = \Delta f - \langle \nabla V, \nabla f \rangle,$$

which is (essentially) self-adjoint in $L^2(\mu)$ and non-positive:

$$\int f\, Lg\, d\mu = \int Lf\, g\, d\mu = -\int \langle \nabla f, \nabla g \rangle\, d\mu.$$

- Spectrum : $\sigma(-L) \subset [0, \infty)$ with $\lambda_0(-L) = 0$ (associated to const).

- Spectral gap: given $\lambda > 0$, we have

$$\sigma(-L) \subset \{0\} \cup [\lambda, \infty),$$

iif the Poincaré inequality holds with constant λ: for all $f \perp$ const,

$$\lambda \int f^2 \, d\mu \quad \leq \quad \int |\nabla f|^2 \, d\mu \quad \left(= \int f \, (-Lf) \, d\mu \right).$$

- Optimal constant $\lambda_1(-L)$, called the spectral gap (of $-L$).
- Describes the speed of convergence to equilibrium in $L^2(\mu)$ of the semigroup $P_t = e^{tL}$ related to the underlying Markov process solution to the SDE

$$dX_t = -\nabla V(X_t) \, dt + \sqrt{2} \, dB_t.$$

Theorem (Brascamp-Lieb '76)

If $\mathrm{Hess}\, V$ *is positive definite, then the Brascamp-Lieb inequality holds: for all* $f \perp$ *const,*

$$\int f^2 \, d\mu \leq \int \langle \nabla f, (\mathrm{Hess}\, V)^{-1} \nabla f \rangle \, d\mu.$$

In particular if V *is uniformly convex, i.e. there exists* $\rho > 0$ *such that*

$$\inf_{x \in \mathbb{R}^n} \mathrm{Hess}\, V(x) \geq \rho \, Id,$$

then

$$\lambda_1(-L) \geq \rho.$$

$\hookrightarrow$ An instance of the famous Bakry-Émery '85 curvature-dimension condition $CD(\rho, \infty)$, optimal in the Gaussian case.

Proof of Helffer '98 based on the L^2 method of Hörmander:

- Consider the Poisson equation

$$f = -Lg \quad \left(= -\Delta g + \langle \nabla V, \nabla g \rangle\right),$$

where the centered f is given and g is the unknown.

- Key point: the following intertwining between gradient and operators:

$$\nabla f = -\nabla L g = -(\mathcal{L} - \mathrm{Hess}\, V)(\nabla g),$$

where $\mathcal{L}$ is the (diagonal) matrix operator acting on vector fields:

$$\mathcal{L} = \mathrm{diag}\, L.$$

$\hookrightarrow$ Reminiscent of Weitzenböck formula for differential forms in Riemannian geometry and of Bakry-Émery theory.

- At the level of semigroups:

$$\nabla P_t f = \mathcal{P}_t^{\mathrm{Hess}\, V} (\nabla f),$$

with $(\mathcal{P}_t^{\mathrm{Hess}\, V})_{t\geq 0}$ the Feynman-Kac semigroup acting on vector fields with generator the Schrödinger type operator $\mathcal{L} - \mathrm{Hess}\, V$.

- In dimension 1, the Feynman-Kac semigroup $(\mathcal{P}_t^{\mathrm{Hess}\, V})_{t\geq 0}$ admits a simple probabilistic representation:

$$P_t^{\partial_x^2 V} f = \mathbb{E}\left[f(X_t) \, \exp\left(- \int_0^t \partial_x^2 V(X_s)\, ds \right) \right].$$

The tangent process satisfies:

$$\partial_x X_t = \partial_x \left(x - \int_0^t \partial_x V(X_s)\, ds + \sqrt{2}\, B_t \right) = 1 - \int_0^t \partial_x^2 V(X_s)\, \partial_x X_s\, ds,$$

so that

$$\partial_x X_t = \exp\left(- \int_0^t \partial_x^2 V(X_s)\, ds \right),$$

and thus,

$$\begin{aligned} \partial_x P_t f &= \partial_x \mathbb{E}[f(X_t)] = \mathbb{E}[\partial_x f(X_t)\, \partial_x X_t] \\ &= \mathbb{E}\left[\partial_x f(X_t) \exp\left(- \int_0^t \partial_x^2 V(X_s)\, ds \right) \right] = P_t^{\partial_x^2 V} \partial_x f. \end{aligned}$$

$\hookrightarrow$ In the multidimensional case, the situation is somewhat similar and $(\mathcal{P}_t^{\mathrm{Hess}\, V})_{t \geq 0}$ admits an explicit expression.

Coming back to Helffer's proof of BL's inequality, we have:

$$\begin{aligned}
\int f^2 \, d\mu &= \int f\,(-Lg)\, d\mu \\
&= \int \langle \nabla f, \nabla g \rangle \, d\mu \\
&\overset{(intert.)}{=} \int \langle \nabla f, (-\mathcal{L} + \mathrm{Hess}\, V)^{-1} (\nabla f) \rangle \, d\mu \\
&\overset{(-\mathcal{L} \geq 0)}{\leq} \int \langle \nabla f, (\mathrm{Hess}\, V)^{-1} \, \nabla f \rangle \, d\mu.
\end{aligned}$$

- Does this proof give an information on extremal functions ? Yes. Equality in BL's inequality holds iif

$$\mathcal{L}(\nabla g) = 0,$$

i.e., $\nabla g = c \in \mathbb{R}^n$, thus

$$g(x) = \langle x, c \rangle + m,$$

which implies that

$$f(x) = -Lg(x) = \langle -\mathcal{L}(x), c \rangle = \langle \nabla V(x), c \rangle.$$

Spectral interpretation.

Theorem (Johnsen '00)

The operators $(-L)_{|const^\perp}$ *and* $(-\mathcal{L} + \mathrm{Hess}\, V)_{|\nabla}$ *are unitarily equivalent, the unitary transformation being the Riesz transform* $\nabla(-L)^{-1/2}$*. Consequently, we have*

$$\sigma((-L)_{|const^\perp}) = \sigma((-\mathcal{L} + \mathrm{Hess}\, V)_{|\nabla}).$$

Thus

$$\begin{aligned} \lambda_1(-L) &= \lambda_0((-L)_{|\mathrm{const}^\perp}) \\ &= \lambda_0((-\mathcal{L} + \mathrm{Hess}\, V)_{|\nabla}) \\ &\geq \lambda_0(-\mathcal{L}) + \inf_{x\in\mathbb{R}^n} \rho\,(\mathrm{Hess}\, V(x)) \\ &= \inf_{x\in\mathbb{R}^n} \rho\,(\mathrm{Hess}\, V(x)), \end{aligned}$$

where $\rho(M)$ is the smallest eigenvalue of a given M.

- How to obtain BL inequalities involving a convenient estimate on $\lambda_1(-L)$ when V is not uniformly convex (and even not convex) ?
- How to obtain BL inequalities leading to convenient weighted Poincaré type inequalities of the form: for all $f \perp$ const,

$$\int f^2 \, d\mu \leq \int \sigma^2 \, |\nabla f|^2 \, d\mu \quad ?$$

$\hookrightarrow$ **Idea:** to introduce in the previous intertwining a smooth weight $x \in \mathbb{R}^d \mapsto A(x) \in GL_n(\mathbb{R})$:

$$A \nabla L = A(\mathcal{L} - \mathrm{Hess}\, V)(A^{-1} A \nabla) = (\mathcal{L}_A - \mathcal{M}_A)(A\nabla),$$

where $\mathcal{L}_A$ is a (non-diagonal) matrix operator acting on vectors fields as

$$\mathcal{L}_A F = \mathcal{L} F + 2\, A \nabla A^{-1} \cdot \nabla F,$$

and $\mathcal{M}_A$ is the matrix acting as a 0-order operator:

$$\mathcal{M}_A = A\, \left(\mathrm{Hess}\, V - \mathcal{L}(A^{-1})\, A\right)\, A^{-1}.$$

- If A is diagonal then so is $\mathcal{L}_A$.

- Above, the contraction $\nabla A^{-1} \cdot \nabla F$ is a vector: if $A^{-1} = (a^{i,j})_{i,j}$ then

$$(\nabla A^{-1} \cdot \nabla F)_i = \sum_j \langle \nabla a^{i,j}, \nabla F_j \rangle.$$

- What about symmetry and non-positivity of those operators ?
  $\hookrightarrow$ Given the symmetric and positive definite matrix $S = (AA^T)^{-1}$, denote $L^2(S, \mu)$ the space of vector fields F such that

$$\int \langle F, S\, F \rangle \, d\mu = \int |A^{-1}\, F|^2 \, d\mu < \infty.$$

- Since $(\mathcal{L} - \mathrm{Hess}\, V)_{|\nabla}$ and $(\mathcal{L}_A - \mathcal{M}_A)_{|\nabla_A}$ are conjugate operators, the second inherits from the first one the properties of symmetry and non-positivity on $L^2(S, \mu)$.
- Not so clear for the operator $\mathcal{L}_A$.
  - Assumption (A_{sym}) the matrix $(A^{-1})^T \nabla A^{-1}$ is symmetric.

Lemma

Under (A_{sym}), the operator $\mathcal{L}_A$ is (essentially) self-adjoint in $L^2(S,\mu)$ and non-positive. In particular for all F, G,

$$\int \langle \mathcal{L}_A F, S\, G\rangle \, d\mu = - \int \nabla F \, S \, \nabla G \, d\mu,$$

where

$$\int \nabla F \, S \, \nabla G \, d\mu = \sum_k \int \langle \partial_k F, S \, \partial_k G\rangle \, d\mu.$$

- In dimension 1, the intertwining with weight a is a composition of the first intertwining with Doob's h-transform (with $h = 1/a$):

$$\partial_x P_t f \overset{(intert.)}{=} \mathbb{E}\left[\partial_x f(X_t)\, \exp\left(-\int_0^t \partial_x^2 V(X_s)\, ds\right)\right]$$
$$\overset{(Girsanov)}{=} \mathbb{E}\left[\partial_x f(X_{a,t})\, \exp\left(-\int_0^t \partial_x^2 V(X_{a,s})\, ds\right)\, M_t^{(a)}\right],$$

where $(X_t^{(a)})_{t\geq 0}$ has generator

$$\begin{aligned} L_a f &= Lf + 2a\partial_x a^{-1}\,\partial_x f \\ &= \partial_x^2 f - \partial_x\left(V + \log(a^2)\right)\,\partial_x f, \end{aligned}$$

and $(M_t^{(a)})_{t\geq 0}$ is the Girsanov martingale

$$\begin{aligned} M_t^{(a)} &= \frac{a(X_{a,t})}{a}\,\exp\left(-\int_0^t \frac{L_a(a)}{a}(X_{a,s})\, ds\right) \\ &= \frac{a(X_{a,t})}{a}\,\exp\left(+\int_0^t a\, L(1/a)\,(X_{a,s})\, ds\right), \end{aligned}$$

Hence the intertwining with weight a rewrites as

$$\begin{aligned} a\partial_x P_t f &= \mathbb{E}\left[(a\partial_x f)(X_{a,t}) \exp\left(-\int_0^t \left(\partial_x^2 V - a\, L(1/a)\right)(X_{a,s})\, ds\right)\right] \\ &= P_{a,t}^{\mathcal{M}_a}(a\partial_x f). \end{aligned}$$

Recall the Poisson equation $f = -Lg$. Using the intertwining with weight in Helffer's proof gives (recall that $S = (AA^T)^{-1}$)

$$\begin{aligned}
\int f^2 \, d\mu &= \int \langle \nabla f, \nabla g \rangle \, d\mu \\
&= \int \langle A\nabla f, S\, A\nabla g \rangle \, d\mu \\
&\overset{(intert.)}{=} \int \langle A\nabla f, S\, (-\mathcal{L}_A + \mathcal{M}_A)^{-1} (A\nabla f) \rangle \, d\mu \\
&\overset{(-\mathcal{L}_A \geq 0)}{\leq} \int \langle A\nabla f, S\, \mathcal{M}_A^{-1} \, A\nabla f \rangle \, d\mu \\
&= \int \langle \nabla f, \left(\mathrm{Hess}\, V - \mathcal{L}(A^{-1})\, A\right)^{-1} \nabla f \rangle \, d\mu,
\end{aligned}$$

which looks like BL's inequality (the original one recovered with $A = Id$).

Summarizing, we obtain the following Generalized BL inequality.

Theorem (Arnaudon, Bonnefont, J. '18)

Assume (A_{sym}) *and that the matrix* $\mathrm{Hess}\, V - \mathcal{L}(A^{-1})\, A$ *is positive definite. Then for all* $f \perp \mathrm{const}$,

$$\int f^2 \, d\mu \leq \int \langle \nabla f, \left(\mathrm{Hess}\, V - \mathcal{L}(A^{-1})\, A\right)^{-1} \nabla f \rangle \, d\mu.$$

Moreover the spectral gap satisfies

$$\lambda_1(-L) \geq \inf_{\mathbb{R}^n} \rho \left(\mathrm{Hess}\, V - \mathcal{L}(A^{-1})\, A\right).$$

$\hookrightarrow$ Authors dealing recently with BL type inequalities: Hargé '08, Helffer '98, Barthe-Cordero Erausquin '13, Nguyen '14, Kolesnikov-Milman '17, Cordero Erausquin '17, Bolley-Gentil-Guillin '18, and others...

- How to choose weight A in the GBL inequality ?
  Choose $A = (Jac\, H^T)^{-1}$ for H diffeomorphism on $\mathbb{R}^n$, so that

$$\begin{aligned}
\mathrm{Hess}\, V - \mathcal{L}(A^{-1})\, A &= \mathrm{Hess}\, V - \mathcal{L}(Jac\, H^T)\,(Jac\, H^T)^{-1} \\
&= \left(\mathrm{Hess}\, V\, Jac\, H^T - \mathcal{L}(Jac\, H^T)\right)\,(Jac\, H^T)^{-1} \\
&= \left((-\mathcal{L} + \mathrm{Hess}\, V)\, Jac\, H^T\right)\,(Jac\, H^T)^{-1} \\
&\overset{(intert.)}{=} -Jac\, \mathcal{L} H^T\,(Jac\, H^T)^{-1}.
\end{aligned}$$

$\hookrightarrow$ The previous spectral gap estimate becomes

$$\lambda_1(-L) \geq \inf_{\mathbb{R}^n} \rho\left(-Jac\, \mathcal{L} H^T\,(Jac\, H^T)^{-1}\right),$$

which generalizes the famous one-dimensional Chen-Wang '97 estimate.

- Equality case in the GBL inequality holds iif

$$\mathcal{L}_A(A\nabla g) = 0,$$

i.e., $A\nabla g = c \in \mathbb{R}^n$.

Assuming moreover $A = (Jac\, H^T)^{-1}$ entails $\nabla g = Jac\, H^T c$, hence

$$g = \langle H, c\rangle + m,$$

which implies that

$$f(x) = -Lg(x) = \langle -\mathcal{L}H, c\rangle.$$

$\hookrightarrow$ BL's inequality is recovered with $H(x) = x$.

- Equality case reaching $\lambda_1(-L)$?

$\hookrightarrow$ Depends on the structure of the eigenspace $E_{\lambda_1(-L)}$ (required of full dimension n, cf. Barthe-Klartag '20 for measures with enough symmetries).

- Perturbed product measures of the type

$$V(x) = \sum_{i=1}^{n} U_i(x_i) + \varphi(x),$$

arising in statistical mechanics, cf. Helffer '98, Bodineau-Helffer '99, Ledoux '01, Gentil-Roberto '01, Chen '08, Barthe-Cordero Erausquin '13, Barthe-Klartag '20, and others...

$\hookrightarrow$ Convenient spectral estimate for those models: Helffer's criterion:

$$\lambda_1(-L) \quad \geq \quad \inf_{x\in\mathbb{R}^n} \rho \left(\widetilde{\mathrm{Hess}\, V}(x) + \mathrm{diag}\, \lambda_1^{x_{-i}} \right),$$

where $\widetilde{M} = M - \mathrm{diag}\, M$ and $\lambda_1^{x_{-i}}$ is the spectral gap of the 1D conditional distribution of x_i knowing x_{-i}.

- **A curious non-convex example:** a Gaussian model perturbed by quartic interaction: for $\beta > 0$,

$$V(x) = \sum_{i=1}^{n} \frac{x_i^2}{2} + \beta \sum_{i=1}^{n} x_i^2 \, x_{i+1}^2, \quad x \in \mathbb{R}^n,$$

studied by Helffer-Nier '03 for discreteness of the spectrum.

Since the 1D conditional distributions are Gaussian, we have

$$\widetilde{\mathrm{Hess}\, V}(x) + \mathrm{diag}\, \lambda_1^{x_{-i}} = \mathrm{Hess}\, V(x), \quad x \in \mathbb{R}^n.$$

$\hookrightarrow$ Helffer's result does not apply !

- What about the intertwining approach ?

Coming to the general perturbed product measure case, the choice

$$H(x) = (h_1(x_1), \ldots, h_n(x_n))^T, \quad x \in \mathbb{R}^n,$$

for convenient 1D strictly monotone functions h_i entails that the matrix weight A in the intertwining is diagonal.

$\hookrightarrow$ Important quantities to control for a relevant compensation of terms in the matrix $-Jac\, \mathcal{L}H(x)^T\, (Jac\, H(x)^T)^{-1}$:

$$\underbrace{\mathrm{Hess}\, \varphi(x)}_{\text{interaction term}}, \quad \underbrace{\partial^2_{x_i} U_i(x_i), \quad |\partial_{x_i} U_i(x_i)|^2}_{\text{product measure part}}, \quad \underbrace{\partial_{x_i}\varphi(x)\partial_{x_i} U_i(x_i)}_{\text{mix contribution}}.$$

Proposition (Bonnefont, J. '21)

For the Gaussian model perturbed by quartic interaction, there exists two explicit dimension-free constants $\beta_0 > 0$ and $C > 0$ such that for all $\beta \in [0, \beta_0]$,

$$\lambda_1(-L) \geq C.$$

- Coming back to the second question, does the GBL inequality allow to obtain convenient weighted Poincaré type inequalities of the form: for all $f \perp$ const,

$$\int f^2 \, d\mu \leq \int \sigma^2 \, |\nabla f|^2 \, d\mu \quad ?$$

The simple Gaussian case:

$\hookrightarrow$ Taking $A = a \, I_d$ with some radial function a leads by the GBL inequality to the following weighted Poincaré inequality: for all $f \perp$ const,

$$\int f^2 \, d\mu \leq \int \frac{|\nabla f|^2}{1 - a \, L(1/a)} \, d\mu,$$

and choosing conveniently the function a leads to

$$C_n \int f^2 \, d\mu \leq \int \frac{|\nabla f|^2}{1 + |x|^2} \, d\mu,$$

with $C_n \simeq 1/n$ for large dimension n.

- Second-order GBL inequalities related to second smallest positive eigenvalue (Bonnefont-J. '21).
- Iteration of intertwinings for recovering spectral comparison proposed by Milman '18 (Bonnefont, J., Steiner, in progress).
- Extension to the Riemannian case (Huguet '22).
- Exploiting the spectrum identity:

$$\sigma(-L)\backslash\{0\} \quad = \quad \sigma((-\mathcal{L}_A + \mathcal{M}_A)|_{A\nabla}).$$

  $\hookrightarrow$ Higher eigenvalues estimates in dimension 1 and gap estimates between consecutive first eigenvalues (Bonnefont-J. '22).
- Other functional inequalities (*e.g.*, log-Sobolev, cf. Steiner '21).
- Spectral estimates on convex domains for statistical purposes: Global Sensitivity Analysis, dimension reduction through active subspaces (Bonnefont-J. '23).
- Stability in BL's inequality (Bonnefont, J., Serres, in progress).

As predicted by Jim Morrison, this is the end...

THANK YOU FOR YOUR ATTENTION

and 61 thanks to Sergey Bobkov for all those wonderful mathematics that inspired several generations !

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And what a soccer player !