Chevet-type inequality for subexponential Weibull processes and norms of iid random matrices

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(based on a joint work in progress with Marta Strzelecka)

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Chevet's inequality

By $g_{i,j}$, g_i, $i,j \geq 1$ we will denote i.i.d. standard Gaussian r.v's.

Theorem (Chevet'1978)

For any bounded nonempty sets $S \subset \mathbb{R}^m$, $T \subset \mathbb{R}^n$ we have

$$\mathbb{E} \sup_{s\in S, t\in T} \sum_{i\leq m, j\leq n} g_{i,j} s_i t_j \sim \sup_{s\in S} \|s\|_2 \mathbb{E} \sup_{t\in T} \sum_{j=1}^{n} g_j t_j + \sup_{t\in T} \|t\|_2 \mathbb{E} \sup_{s\in S} \sum_{i=1}^{m} g_i s_i.$$

$\ell_p^n \to \ell_q^m$ norms of iid Gaussian matrices

Applying Chevet's inequality with $S = B_{q^*}^m$, $T = B_p^n$ and observing that

$$\mathbb{E}\|(g_i)_{i\le k}\|_u \sim \sqrt{u \wedge \operatorname{Log} k}\, k^{1/u}$$

one gets the following two-sided bound
(p^* is the Hölder conjugate of p and $\operatorname{Log} n := \log(n \vee e)$).

Corollary

For any $1 \le p, q \le \infty$,

$$\mathbb{E}\Big\|(g_{i,j})_{i\le m, j\le n}\Big\|_{\ell_p^n \to \ell_q^m}$$
$$\sim \begin{cases} m^{1/q-1/2} n^{1/p^*} + n^{1/p^*-1/2} m^{1/q}, & p^*, q \le 2, \\ \sqrt{p^* \wedge \operatorname{Log} n}\, n^{1/p^*} m^{1/q-1/2} + m^{1/q}, & q \le 2 \le p^*, \\ n^{1/p^*} + \sqrt{q \wedge \operatorname{Log} m}\, m^{1/q} n^{1/p^*-1/2}, & p^* \le 2 \le q, \\ \sqrt{p^* \wedge \operatorname{Log} n}\, n^{1/p^*} + \sqrt{q \wedge \operatorname{Log} m}\, m^{1/q}, & 2 \le q, p^*. \end{cases}$$

$\ell_p^n \to \ell_q^n$ norms of square iid Gaussian matrices

In the case $n = m$ the previous bound has a simpler form.

Corollary

For any $1 \leq p, q \leq \infty$,

$$\mathbb{E}\Big\|(g_{i,j})_{i,j\leq n}\Big\|_{\ell_p^n \to \ell_q^n} \sim \begin{cases} n^{1/p^*+1/q-1/2}, & p^*, q \leq 2, \\ \sqrt{p^* \wedge q \wedge \mathrm{Log}\, n}\; n^{1/(p^*\wedge q)}, & p^* \vee q \geq 2. \end{cases}$$

Questions

- Can one extend Chevet's inequality to more general class of r.v's?
- What are two-sided bounds for $\ell_p^n \to \ell_q^m$ norms of other iid random matrices?

We will first discuss both questions for symmetric Weibull r.v's with shape parameter r, i.e. symmetric r.v's such that $\mathbb{P}(|X_{ij}| \geq t) = e^{-t^r}$, $t \geq 0$.

ALLPT version of Chevet's inequality for exponential r.v's

Theorem (ALLPT'12)

Let $E_{i,j}$, E_i, $i,j \geq 1$ be iid symmetric exponential r.v's with variance 1. For any bounded nonempty sets $S \subset \mathbb{R}^m$, $T \subset \mathbb{R}^n$ we have

$$\mathbb{E} \sup_{s\in S, t\in T} \sum_{i\leq m, j\leq n} E_{i,j} s_i t_j \lesssim \sup_{s\in S} \|s\|_2 \mathbb{E} \sup_{t\in T} \sum_{j=1}^{n} E_j t_j + \sup_{t\in T} \|t\|_2 \mathbb{E} \sup_{s\in S} \sum_{i=1}^{m} E_i s_i .$$

The above bound is not two-sided. If $S = B_\infty^m$ and $T = B_1^n$ then LHS$\sim m + \mathrm{Log}\, n$, whereas RHS$\sim m + \sqrt{m}\,\mathrm{Log}\, n$.

Chevet-type inequality for Weibull processes

Theorem

Let $X_{i,j}$, X_i be iid symmetric Weibull r.v's with parameter $r \in [1,2]$. For every nonempty bounded $S \subset \mathbb{R}^m$ and $T \subset \mathbb{R}^n$,

$$\mathbb{E} \sup_{s\in S, t\in T} \sum_{i\le m, j\le n} X_{i,j} s_i t_j$$

$$\sim \sup_{s\in S} \|s\|_{r^*} \mathbb{E} \sup_{t\in T} \sum_{j=1}^{n} X_j t_j + \sup_{t\in T} \|t\|_{r^*} \mathbb{E} \sup_{s\in S} \sum_{i=1}^{m} X_i s_i$$

$$+ \mathbb{E} \sup_{s\in S, t\in T} \sum_{i\le m, j\le n} g_{i,j} s_i t_j$$

$$\sim \sup_{s\in S} \|s\|_{r^*} \mathbb{E} \sup_{t\in T} \sum_{j=1}^{n} X_j t_j + \sup_{s\in S} \|s\|_2 \mathbb{E} \sup_{t\in T} \sum_{j=1}^{n} g_j t_j$$

$$+ \sup_{t\in T} \|t\|_{r^*} \mathbb{E} \sup_{s\in S} \sum_{i=1}^{m} X_i s_i + \sup_{t\in T} \|t\|_2 \mathbb{E} \sup_{s\in S} \sum_{i=1}^{m} g_i s_i.$$

Sketch of the proof

For a metric space (U,ρ), let

$$\gamma_r(U,\rho) := \inf_{(U_k)_{k=0}^{\infty}} \sup_{u\in U} \sum_{k=0}^{\infty} 2^{k/r}\rho(u,U_k),$$

where the infimum is taken over all admissible sequences of sets, i.e., all sequences $(U_k)_{k=0}^{\infty}$ of subsets of U, such that $|U_0|=1$, and $|U_k|\le 2^{2^k}$ for $k\ge 1$.

Talagrand'1994 result state that for nonempty bounded $U\subset\mathbb{R}^d$

$$\mathbb{E}\sup_{u\in U}\sum_{i=1}^{d} u_i X_i \sim \gamma_r(U,\rho_{r^*}) + \gamma_2(U,\rho_2).$$

To show the upper bound in the theorem it suffices to prove

$$\gamma_r(S\otimes T,\rho_{r^*}) \lesssim \sup_{t\in T}\|t\|_{r^*}\gamma_r(S,\rho_{r^*}) + \sup_{s\in S}\|s\|_{r^*}\gamma_r(T,\rho_{r^*}), \quad (1)$$

where $S\otimes T=\{s\otimes t\colon s\in S, t\in T\}$ and $s\otimes t := (s_i t_j)_{i\le m, j\le n}$.

Sketch of the proof ctd

Let $S_k \subset S$ and $T_k \subset T$ be admissible sequences of sets. Put $T_{-1} := T_0$, $S_{-1} := S_0$ and $U_k := T_{k-1} \otimes S_{k-1}$. Then $(U_k)_{k\geq 0}$ is an admissible sequence of subsets of $S \otimes T$.
For all $s', s'' \in S$, and $t', t'' \in T$ we have

$$\begin{aligned}\|s' \otimes t' - s'' \otimes t''\|_{r^*} &\leq \|s' \otimes (t' - t'')\|_{r^*} + \|(s' - s'') \otimes t''\|_{r^*} \\ &= \|s'\|_{r^*}\|t' - t''\|_{r^*} + \|t''\|_{r^*}\|s' - s''\|_{r^*} \\ &\leq \sup_{s\in S}\|s\|_{r^*}\|t' - t''\|_{r^*} + \sup_{t\in T}\|t\|_{r^*}\|s' - s''\|_{r^*}.\end{aligned}$$

Therefore

$$\gamma_r(S \otimes T, \rho_{r^*}) \leq \sup_{s\in S, t\in T} \sum_{k=0}^{\infty} 2^{\frac{k}{r}} \rho_{r^*}(s \otimes t, U_k)$$

$$\leq \sup_{s\in S}\|s\|_{r^*} \sup_{t\in T} \sum_{k=0}^{\infty} 2^{\frac{k}{r}} \rho_{r^*}(t, T_{k-1}) + \sup_{t\in T}\|t\|_{r^*} \sup_{s\in S} \sum_{k=0}^{\infty} 2^{\frac{k}{r}} \rho_{r^*}(s, S_{k-1}).$$

Taking the infimum over all admissible sequences $(S_k)_{k\geq 0}$ and $(T_k)_{k\geq 0}$ we get the upper bound (1).

$\ell_p^n \to \ell_q^m$ norms of iid Weibull matrices

Applying Chevet-type inequality with $S = B_{q^*}^m$, $T = B_p^n$ and observing that

$$\mathbb{E}\|(X_i)_{i\leq k}\|_u \sim (u \wedge \operatorname{Log} k)^{1/r}\, k^{1/u}$$

one gets the following two-sided bound

Theorem

Let $(X_{ij})_{i\leq m, j\leq n}$ be iid symmetric Weibull r.v's with parameter $r \in [1,2]$. Then for $1 \leq p, q \leq \infty$,

$$\mathbb{E}\big\|(X_{i,j})_{i\leq m, j\leq n}\big\|_{\ell_p^n \to \ell_q^m}$$

$$\sim \begin{cases} m^{1/q-1/2} n^{1/p^*} + n^{1/p^*-1/2} m^{1/q}, & p^*, q \leq 2, \\ (p^* \wedge \operatorname{Log} n)^{1/r} n^{1/p^*} m^{(1/r^*-1/q^*)\vee 0} & \\ +\sqrt{p^* \wedge \operatorname{Log} n}\, n^{1/p^*} m^{1/q-1/2} + m^{1/q}, & q \leq 2 \leq p^*, \\ n^{1/p^*} + (q \wedge \operatorname{Log} m)^{1/r} m^{1/q} n^{(1/r^*-1/p)\vee 0} & \\ +\sqrt{q \wedge \operatorname{Log} m}\, m^{1/q} n^{1/p^*-1/2}, & p^* \leq 2 \leq q, \\ (p^* \wedge \operatorname{Log} n)^{1/r} n^{1/p^*} + (q \wedge \operatorname{Log} m)^{1/r} m^{1/q}, & 2 \leq p^*, q. \end{cases}$$

$\ell_p^n \to \ell_q^n$ norms of square iid Weibull matrices

Corollary

Let $(X_{ij})_{i,j\le n}$ *be iid symmetric Weibull r.v's with parameter* $r \in [1,2]$. *Then for* $1 \le p, q \le \infty$,

$$\mathbb{E}\|(X_{i,j})_{i,j\le n}\|_{\ell_p^n \to \ell_q^n} \sim \begin{cases} n^{1/p^*+1/q-1/2}, & p^*, q \le 2, \\ (p^* \wedge q \wedge \operatorname{Log} n)^{1/r} n^{1/(p^* \wedge q)}, & p^* \vee q \ge 2. \end{cases}$$

$\ell_p^n \to \ell_q^n$ norms of square iid Rademacher matrices

Let (ε_{ij}) be iid symmetric ± 1 r.v's (Rademacher sequence)

It is well known (at least since Bennett-Goodman-Newman'1975) that

$$\mathbb{E}\left\|(\varepsilon_{i,j})_{i\le m, j\le n}\right\|_{\ell_p^n\to\ell_q^m} \sim_{p,q} \begin{cases} m^{1/q-1/2}n^{1/p^*} + n^{1/p^*-1/2}m^{1/q}, & p^*, q \le 2, \\ m^{1/q-1/2}n^{1/p^*} + m^{1/q}, & q \le 2 \le p^*, \\ n^{1/p^*} + n^{1/p^*-1/2}m^{1/q}, & p^* \le 2 \le q, \\ n^{1/p^*} + m^{1/q}, & 2 \le p^*, q. \end{cases}$$

It turns out that in the square case the constant could be chosen independently on p and q.

Proposition

For $1 \le p, q \le \infty$,

$$\mathbb{E}\left\|(\varepsilon_{i,j})_{i,j\le n}\right\|_{\ell_p^n\to\ell_q^n} \sim \begin{cases} n^{1/p^*+1/q-1/2}, & p^*, q \le 2, \\ n^{1/(p^*\wedge q)}, & p^* \vee q \ge 2. \end{cases}$$

$\ell_p^n \to \ell_q^m$ norms of rectangular iid Rademacher matrices

In the case of rectangular matrices one needs to modify the previous formula to get two-sided bound.

Theorem

Let (ε_{ij}) *be iid symmetric* ± 1 *r.v's. Then for* $1 \le p, q \le \infty$*,*

$$\mathbb{E}\left\|(\varepsilon_{i,j})_{i\le m, j\le n}\right\|_{\ell_p^n\to\ell_q^m} \sim \begin{cases} m^{1/q-1/2}n^{1/p^*} + n^{1/p^*-1/2}m^{1/q}, & p^*, q \le 2, \\ \sqrt{p^*\wedge m}\, m^{1/q-1/2}n^{1/p^*} + m^{1/q}, & q \le 2 \le p^*, \\ n^{1/p^*} + \sqrt{q\wedge n}\, n^{1/p^*-1/2}m^{1/q}, & p^* \le 2 \le q, \\ (m\wedge p^*)^{1/q}n^{1/p^*} + (n\wedge q)^{1/p^*} m^{1/q}, & 2 \le p^*, q. \end{cases}$$

General conjecture

All of previously stated two-sided bounds for $\ell_p^n \to \ell_q^m$ norms have the form

$$m^{1/q} \sup_{t \in B_p^n} \Big\| \sum_{j=1}^n t_j X_{1,j} \Big\|_{q \wedge \mathrm{Log}\, m} + n^{1/p^*} \sup_{s \in B_{q^*}^m} \Big\| \sum_{i=1}^m s_i X_{i,1} \Big\|_{p^* \wedge \mathrm{Log}\, n}.$$

Conjecture

Let $(X_{ij})_{i \le m, j \le n}$ be i.i.d. centered random variables satisfying some regularity conditions. Then for $1 \le p, q \le \infty$,

$$\mathbb{E}\Big\| (X_{i,j})_{i \le m, j \le n} \Big\|_{\ell_p^n \to \ell_q^m}$$

$$\sim m^{1/q} \sup_{t \in B_p^n} \Big\| \sum_{j=1}^n t_j X_{1,j} \Big\|_{q \wedge \mathrm{Log}\, m} + n^{1/p^*} \sup_{s \in B_{q^*}^m} \Big\| \sum_{i=1}^m s_i X_{i,1} \Big\|_{p^* \wedge \mathrm{Log}\, n}.$$

Conjecture in the square case

In the case $n = m$ the conjecture has a simpler form.

Conjecture

Let $(X_{ij})_{i,j\le n}$ are centered r.v's satisfying some regularity conditions. Then for $1 \le p, q \le \infty$,

$$\mathbb{E}\Big\|(X_{i,j})_{i,j=1}^n\Big\|_{\ell_p^n \to \ell_q^n} \sim \begin{cases} n^{1/q+1/p^*-1/2}\|X_{1,1}\|_2, & q, p^* \le 2, \\ n^{1/(p^*\wedge q)}\|X_{1,1}\|_{p^*\wedge q\wedge \mathrm{Log}\, n}, & q \vee p^* \ge 2. \end{cases}$$

Lower bounds under comparison of L^{2u} and L^u-norms

Proposition

Let $(X_{i,j})$ be centered iid r.v's satisfying regularity condition

$$\|X_{i,j}\|_{2u} \le \alpha \|X_{i,j}\|_u \quad \textit{for } u \ge 1.$$

Then for $1 \le p, q \le \infty$,

$$\mathbb{E}\Big\|(X_{i,j})_{i\le m, j\le n}\Big\|_{\ell_p^n \to \ell_q^m}$$

$$\gtrsim_\alpha m^{1/q} \sup_{t\in B_p^n} \Big\|\sum_{j=1}^n t_j X_{1,j}\Big\|_{q\wedge \mathrm{Log}\, m} + n^{1/p^*} \sup_{s\in B_{q^*}^m} \Big\|\sum_{i=1}^m s_i X_{i,1}\Big\|_{p^*\wedge \mathrm{Log}\, n}.$$

Key lemma for lower bounds

Lemma

Let $r \geq 1$ and $Y_1, Y_2, \ldots, Y_k$ be iid random variables such that $\|Y_i\|_{2r} \leq \alpha \|Y_i\|_r$. Then for $k \geq 4\alpha^{2r}$,

$$\mathbb{E}\Big(\sum_{i=1}^{k} |Y_i|^r\Big)^{1/r} \geq \frac{1}{128\alpha^2} k^{1/r} \|Y_1\|_r.$$

Proof. Define $A_i := \{Y_i^r \geq \frac{1}{2}\mathbb{E}Y_i^r\}$ and $Z := \sum_{i=1}^k 1_{A_i}$. Then

$$\mathbb{P}(A_i) \geq \frac{1}{4}\frac{(\mathbb{E}Y_i^r)^2}{\mathbb{E}Y_i^{2r}} \geq \frac{1}{4}\alpha^{-2r}.$$

Thus $\mathbb{E}Z \geq \frac{k}{4}\alpha^{-2r} \geq 1$ and $\mathbb{E}Z^2 \leq (\mathbb{E}Z)^2 + \mathbb{E}Z \leq 2(\mathbb{E}Z)^2$. Hence

$$\mathbb{P}\Big(Z \geq \frac{1}{2}\mathbb{E}Z\Big) \geq \frac{1}{4}\frac{(\mathbb{E}Z)^2}{\mathbb{E}Z^2} \geq \frac{1}{8}.$$

and

$$\mathbb{E}\Big(\sum_{i=1}^{k} Y_i^r\Big)^{1/r} \geq \mathbb{P}\Big(Z \geq \frac{1}{2}\mathbb{E}Z\Big)\Big(\frac{1}{2}\mathbb{E}Z\frac{1}{2}\mathbb{E}Y_i^r\Big)^{1/r} \geq \frac{1}{128\alpha^2} k^{1/r}\|Y_i\|_r.$$

$\ell_p^n \to \ell_q^m$ norms in the case $p^*, q \le 2$

Theorem

Let $(X_{i,j})_{i\le n, j\le n}$ be iid centered r.v's such that $\|X_{i,j}\|_4 \le \alpha \|X_{i,j}\|_2$. Then for $p^*, q \le 2$ we have

$$\mathbb{E}\Big\|(X_{i,j})_{i\le m, j\le n}\Big\|_{\ell_p^n \to \ell_q^m} \sim_\alpha (m^{1/q-1/2} n^{1/p^*} + n^{1/p^*-1/2} m^{1/q}) \|X_{1,1}\|_2.$$

Idea of the proof. The lower bound is easy. For the upper we use

$$\begin{aligned}\Big\|(X_{i,j})\Big\|_{\ell_p^n\to\ell_q^m} &\le \|\mathrm{Id}\|_{\ell_p^n\to\ell_2^n}\Big\|(X_{i,j})\Big\|_{\ell_2^n\to\ell_2^m}\|\mathrm{Id}\|_{\ell_2^m\to\ell_q^m}\\ &= n^{1/p^*-1/2} m^{1/q-1/2}\Big\|(X_{i,j})\Big\|_{\ell_2^n\to\ell_2^m}\end{aligned}$$

and we estimate $\ell_2 \to \ell_2$ norm via the result of L'2005

$$\begin{aligned}\mathbb{E}\Big\|(X_{i,j})\Big\|_{\ell_2^n\to\ell_2^m} &\lesssim \max_i\Big(\sum_j \mathbb{E}X_{i,j}^2\Big)^{1/2} + \max_j\Big(\sum_i \mathbb{E}X_{i,j}^2\Big)^{1/2} + \Big(\sum_{i,j}\mathbb{E}X_{i,j}^4\Big)^{1/4}\\ &\lesssim_\alpha (\sqrt{n}+\sqrt{m})\|X_{1,1}\|_2.\end{aligned}$$

$\ell_p^n \to \ell_q^m$ norms in the case $p^* \geq \mathrm{Log}\, n$ or $q \geq \mathrm{Log}\, m$

Theorem

Let $(X_{i,j})_{i\leq m, j\leq n}$ be iid centered r.v's such that

$$\|X_{i,j}\|_{2u} \leq \alpha \|X_{i,j}\|_u \quad \textit{for } u \geq 1.$$

Then for $q \geq \mathrm{Log}\, m$,

$$\mathbb{E}\left\|(X_{i,j})_{i\leq m, j\leq n}\right\|_{\ell_p^n \to \ell_q^m} \sim_\alpha \sup_{t\in B_p^n} \Big\|\sum_{j\leq n} t_j X_{1,j}\Big\|_{\mathrm{Log}\, m} + n^{\frac{1}{p^*}} \|X_{1,1}\|_{p^* \wedge \mathrm{Log}\, n}.$$

Analogously, for $p^* \geq \mathrm{Log}\, n$,

$$\mathbb{E}\left\|(X_{i,j})_{i\leq m, j\leq n}\right\|_{\ell_p^n \to \ell_q^m} \sim_\alpha \sup_{s\in B_{q^*}^m} \Big\|\sum_{i\leq m} s_i X_{i,1}\Big\|_{\mathrm{Log}\, n} + m^{\frac{1}{q}} \|X_{1,1}\|_{q \wedge \mathrm{Log}\, m}.$$

Idea of the proof. We have $\|(x_i)_{i\leq m}\|_q \sim \|(x_i)_{i\leq m}\|_\infty$ for $q \geq \mathrm{Log}\, m$ and we use comparison of weak and strong moments.

Comparison of weak and strong moments

Theorem (L.-Strzelecka'2018)

Let $(X_i)_{i\leq n}$ *be independent centered r.v's such that*

$$\|X_i\|_{2u}\leq \alpha\|X_i\|_u \quad \textit{for } u\geq 1.$$

Then for any bounded nonempty $T\subset\mathbb{R}^n$ *and any* $p\geq 1$*,*

$$\Big(\mathbb{E}\sup_{t\in T}\Big|\sum_{i=1}^n t_iX_i\Big|^p\Big)^{1/p}\sim_\alpha \mathbb{E}\sup_{t\in T}\Big|\sum_{i=1}^n t_iX_i\Big|+\sup_{t\in T}\Big(\mathbb{E}\Big|\sum_{i=1}^n t_iX_i\Big|^p\Big)^{1/p}.$$

$\ell_p^n \to \ell_q^m$ norms in the case $p^* \wedge q \geq \mathrm{LogLog}(n+m)$

Theorem

Let $(X_{i,j})_{i\leq m, j\leq n}$ be iid centered r.v's such that

$$\|X_{i,j}\|_{2u} \leq \alpha \|X_{i,j}\|_u \quad \text{for } u \geq 1.$$

Then for $\mathrm{LogLog}(n+m) \leq p^*, q \leq \infty$,

$$\mathbb{E}\left\|(X_{i,j})_{i\leq m, j\leq n}\right\|_{\ell_p^n \to \ell_q^m}$$

$$\sim_\alpha m^{1/q} \sup_{t\in B_p^n} \left\|\sum_{j=1}^n t_j X_{1,j}\right\|_{q\wedge \mathrm{Log}\, m} + n^{1/p^*} \sup_{s\in B_{q^*}^m} \left\|\sum_{i=1}^m s_i X_{i,1}\right\|_{p^*\wedge \mathrm{Log}\, n}.$$

Bounds in the iid square case up to (powers of) $\mathrm{LogLog}\, n$

Using the previous result and the Riesz-Thorin interpolation we get

Corollary

Let $(X_{i,j})_{i,j\le n}$ be iid centered r.v's such that

$$\|X_{i,j}\|_{2u} \le \alpha \|X_{i,j}\|_u \quad \text{for } u \ge 1.$$

Then for $1 \le p, q \le \infty$,

$$\mathbb{E}\Big\|(X_{i,j})_{i,j\le n}\Big\|_{\ell_p^n\to\ell_q^n} \lesssim_\alpha \begin{cases} n^{1/q+1/p^*-1/2}\|X_{1,1}\|_2, & q, p^* \le 2,\\ (\mathrm{LogLog}\, n)^\beta n^{1/(p^*\wedge q)}\|X_{1,1}\|_{p^*\wedge q\wedge \mathrm{Log}\, n}, & q\vee p^* \ge 2, \end{cases}$$

where $\beta = \log_2 \alpha \vee \frac{1}{2}$.

$\ell_p^n \to \ell_q^n$ norms of iid square subexponential matrices

Theorem

Let $(X_{i,j})_{i,j\le n}$ *be iid centered r.v's such that*

$$\|X_{i,j}\|_{2u} \le \alpha \|X_{i,j}\|_u \quad \text{for } u \ge 1.$$

and

$$\|X_{i,j}\|_{\Psi_1} \le \beta \|X_{i,j}\|_2.$$

Then for $1 \le p, q \le \infty$,

$$\mathbb{E}\Big\|(X_{i,j})_{i,j=1}^n\Big\|_{\ell_p^n \to \ell_q^n} \sim_{\alpha,\beta} \begin{cases} n^{1/q+1/p^*-1/2}\|X_{1,1}\|_2, & q, p^* \le 2, \\ n^{1/(p^*\wedge q)}\|X_{1,1}\|_{p^*\wedge q \wedge \mathrm{Log}\, n}, & q \vee p^* \ge 2. \end{cases}$$

Thank you for your attention!

Many happpy returns Sergey!