Multi-Bubble Isoperimetric Problems - Old and New

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61 Probability Encounters
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joint work (in progress) with Joe Neeman (UT Austin)

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The Classical Isoperimetric Inequality

"Among all sets in Euclidean space $\mathbb{R}^n$ having a given volume, Euclidean balls minimize surface area."

$$V(\Omega) = V(\text{Ball}) \;\Rightarrow\; A(\Omega) \geq A(\text{Ball}).$$

$\Omega \in \mathcal{B}(\mathbb{R}^n)$, $V = \text{Leb}^n$, $A =$ Surface Area.

What is Surface Area? Various (non-equivalent) definitions:

- If $\partial\Omega$ smooth, $\int_{\partial\Omega} d\mathrm{Vol}_{\partial\Omega}$.
- Hausdorff measure $\mathcal{H}^{n-1}(\partial\Omega)$.
- Minkowski exterior boundary measure:
  $V^+(\Omega) = \liminf_{\epsilon\to 0+} \frac{V(\Omega_\epsilon \setminus \Omega)}{\epsilon}$, $\Omega_\epsilon := \{y \in \mathbb{R}^n \; ; \; d(y,\Omega) < \epsilon\}$.
- De Giorgi Perimeter $P(\Omega) = \mathcal{H}^{n-1}(\partial^*\Omega)$.
  Stronger than rest, l.s.c., invariant under null-set modifications.

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Isoperimetric Inequalities on Metric-Measure Spaces

Classical isoperimetric inequality is on $\mathbb{R}^n = (\mathbb{R}^n, |\cdot|, \text{Leb}^n)$.
Study in weighted-manifold setting $(M^n, g, \mu = \Psi(x) d\text{Vol}_g)$, $\Psi > 0$.

Weighted Volume and Area:

- $V(\Omega) = \mu(\Omega) = \int_\Omega \Psi(x) d\text{Vol}_g$.
- $A(\Omega) = P_\Psi(\Omega) = \int_{\partial^* \Omega} \Psi(x) d\mathcal{H}^{n-1}(x)$.

Examples:

1. $\mathbb{S}^n = (\mathbb{S}^n, g_{\text{can}}, \lambda_{\mathbb{S}^n} = \frac{\text{Vol}_{\mathbb{S}^n}}{\text{Vol}(\mathbb{S}^n)})$ - P. Lévy, Schmidt 20-30's: geodesic balls are isoperimetric minimizers.
2. $\mathbb{G}^n = (\mathbb{R}^n, |\cdot|, \gamma^n = \frac{1}{(2\pi)^{n/2}} e^{-\frac{|x|^2}{2}} dx)$ - Sudakov–Tsirelson, Borell '75: half-spaces are isoperimetric minimizers.

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Isoperimetric inequalities crucially used in geometry, analysis, PDE, probability, combinatorics, etc...

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Isoperimetric Inequalities for Clusters

Cluster $\Omega = (\Omega_1, \ldots, \Omega_q)$ is a partition $M = \Omega_1 \uplus \ldots \uplus \Omega_q$ (up to null-sets)
Given $V(\Omega) = (V(\Omega_1) \ldots V(\Omega_q))$ minimize $A(\Omega) = \frac{1}{2}\sum_{i=1}^{q} A(\Omega_i) = \sum_{i<j} A_{ij}$.

Previous examples: $q = 2$ $(\Omega_1 = U, \Omega_2 = M \setminus U)$, "Single Bubble".
Would like to study $q \geq 3$, "Multi Bubble" case.
Case $q = 3$ is called "Double Bubble" $(\Omega_1, \Omega_2, M \setminus (\Omega_1 \uplus \Omega_2))$.

0. $\mathbb{R}^n$ - Theorem: for all $V(\Omega) = (v_1, v_2, \infty)$, standard double bubble (3 spherical caps meeting at $120°$ along $(n-2)$-dim sphere) minimizes total surface area:

$\mathbb{R}^2$ - F. Morgan's “SMALL" undergraduate group (Foisy–Alfaro–Brock–Hodges–Zimba) '93.

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Isoperimetric Double-Bubble Conjectures

$q = 3$ regions in dimension $n \geq 2$:

1. $\mathbb{S}^n$ - Double-Bubble Conjecture: for all $V(\Omega) = (v_1, v_2, v_3)$, standard double bubble (3 spherical caps in $\mathbb{S}^n$ meeting at $120°$ along $(n-2)$-dim sphere) minimizes total surface area.

   $\mathbb{S}^2$ - Proved by Masters '96.
   $\mathbb{S}^3$ - Cotton–Freeman '02, Corneli–Hoffman-HLLMS '07, partial.
   $\mathbb{S}^n$ - Corneli–Corwin–Hoffman-HSADLVX '08, if $\left|v_i - \frac{1}{3}\right| \leq 0.04$.

2. $\mathbb{G}^n$ - Double-Bubble Conjecture: for all $V(\Omega) = (v_1, v_2, v_3)$, standard "tripod" / "Y" (3 half-hyperplanes meeting at $120°$ along $(n-2)$-dim plane) minimizes total (Gaussian) surface area.

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   Interaction between $\mathbb{G}$ and $\mathbb{S}$:
   $\mathbb{G}^2 \Rightarrow \mathbb{S}^N \;\; \forall N \gg 1 \Rightarrow \mathbb{S}^n \;\; \forall n \geq 2 \Rightarrow \mathbb{G}^n \;\; \forall n \geq 2$ by projection.

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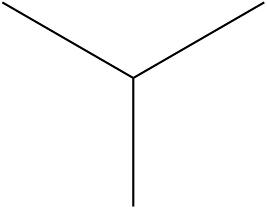
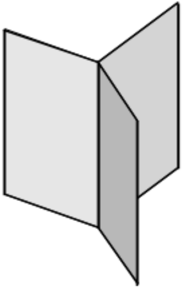
Y cone



In $\mathbb{R}^2$

In $\mathbb{R}^n$ take $\times\mathbb{R}^{n-2}$



In $\mathbb{R}^3$

Isoperimetric Multi-Bubble Conjectures

Higher-order cluster $\Omega = (\Omega_1, \ldots, \Omega_q)$.
There's no reasonable conjecture when $q \gg n$:

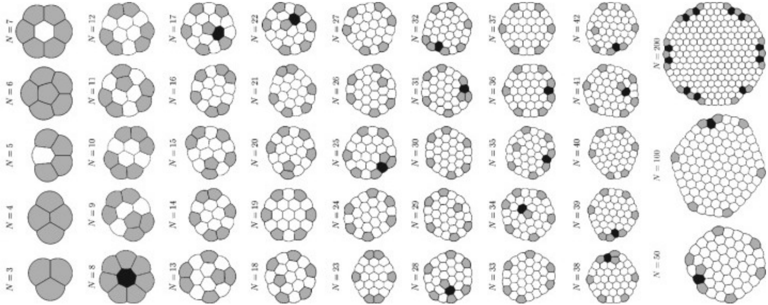


Image from Cox, Graner, et al.

Multi-Bubble Conjecture on $\mathbb{R}^n$ (J. Sullivan '95): If $q - 1 \leq n + 1$, for all $V(\Omega) = (v_1, \ldots, v_{q-1}, \infty)$, the minimizer is a standard $q - 1$ bubble:

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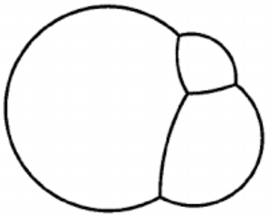

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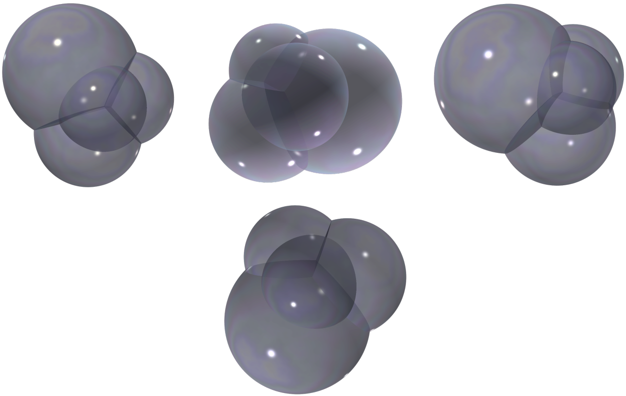

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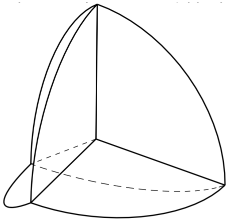

Montesinos Amilibia '01 - standard bubbles exist and are uniquely determined (up to isometries) for all prescribed volumes, for all $q-1 \leq n+1$.

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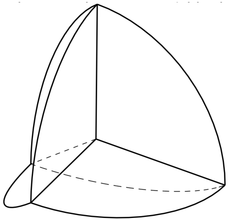

Credit: Quanta.

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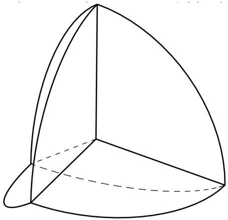

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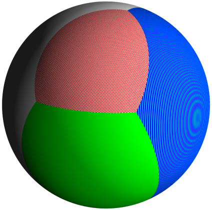

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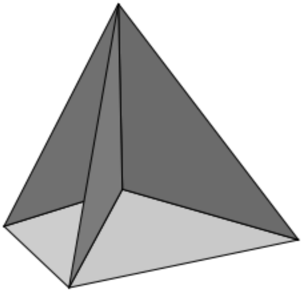

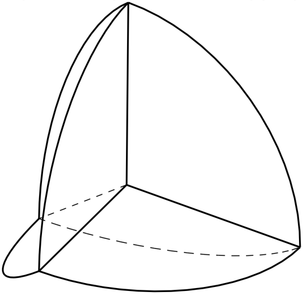

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$q = 2$ corresponds to the classical isoperimetric inqs.
$q = 3$ is the double-bubble theorem ($\mathbb{R}^n$) / conjecture ($\mathbb{S}^n$ / $\mathbb{G}^n$, $n \geq 3$).
$q = 4$ and $n = 2$ in $\mathbb{R}^n$ (planar triple-bubble) proved by Wichiramala '04.
Not aware of any other results when $q \geq 4$ prior to 2018.

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For all $n \geq 2$ and $2 \leq q \leq \min(6, n+1)$, the Multi-Bubble Conjecture on $\mathbb{R}^n$ / $\mathbb{S}^n$ is true: "A standard $q-1$ bubble is an isoperimetric minimizer". In other words, Double-Bubble ($n \geq 2$), Triple-Bubble ($n \geq 3$), Quadruple-Bubble ($n \geq 4$), Quintuple-Bubble ($n \geq 5$).

Additional partial results valid for all $q \leq n+1$ later on.

Multi-Bubble Uniqueness on $\mathbb{R}^n$ / $\mathbb{S}^n$ (M.–Neeman '22)

Uniqueness (up to null-sets) on $\mathbb{S}^n$ for $2 \leq q \leq \min(6, n+1)$.
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Tools in Isoperimetric Problems

Single Bubble ($q = 2$):

0. $\mathbb{R}^n$ - symmetrization, Brunn–Minkowski, L^2, heat-flow, PDE, Localization, Optimal-Transport, Combinatorial, GMT.
1. $\mathbb{S}^n$ - symmetrization, GMT, Localization.
2. $\mathbb{G}^n$ - Projection of $\mathbb{S}^N$, symmetrization (Ehrhard), Brunn-Minkowski (Borell), heat-flow (Bakry–Ledoux), GMT, Bobkov: Tensorizing 2-point inq, Localization, $\ldots$.

Double/Multi Bubble ($q \geq 3$):

- Geometric Measure Theory (GMT): existence and (partial) regularity of minimizers.
- Differential-geometric variational arguments: interfaces will have constant mean curvature (CMC) and meet in threes at $120°$.
- Symmetrization: minimizers will have some symmetries. On $\mathbb{R}^n/\mathbb{S}^n$, when $q \leq n+1$ – $\exists$ minimizer with reflection symmetry.
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Proof: Steps 1-2-3 – Minimizer is Spherical Voronoi

Thm (M.-Neeman '22). On $\mathbb{R}^n/\mathbb{S}^n$, $q \leq n+1$:

1. A minimizer's interfaces $\Sigma_{ij} := \partial^* \Omega_i \cap \partial^* \Omega_j$ are locally spherical.
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There exist $\{\mathfrak{c}_i\}_{i=1,\ldots,q} \subset \mathbb{R}^{n+1}/\mathbb{R}^n$ and $\{\kappa_i\}_{i=1,\ldots,q} \subset \mathbb{R}$ so that:

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2. On $\mathbb{S}^n$, the following Voronoi representation holds:

$$\Omega_i = \operatorname{int}\left\{p \in \mathbb{S}^n \,;\, \arg\min_{j=1,\ldots,q} \langle \mathfrak{c}_j, p\rangle + \kappa_j = i\right\} = \bigcap_{j\neq i} \left\{p \in \mathbb{S}^n \,;\, \langle \mathfrak{c}_{ij}, p\rangle + \kappa_{ij} < 0\right\}.$$

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3. Furthermore, each Ω_i is connected.

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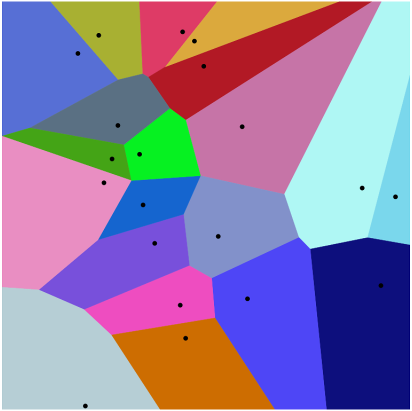

Euclidean Voronoi Cells:

$$\Omega_i = \{x : \arg\min_j |x - x_j|^2 = i\}$$

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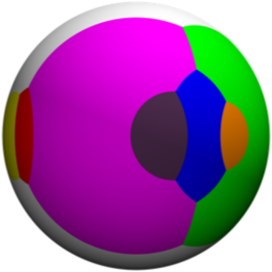

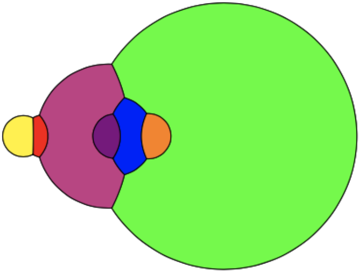

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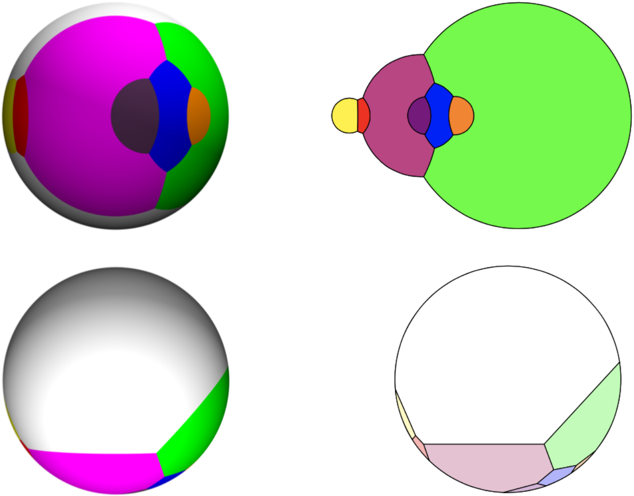

Proof: Step 4 – Need Global Information

At this point, we know that our cluster is spherical Voronoi.
We are almost done! **Fact:** class of Voronoi clusters with $\Sigma_{ij} \neq \emptyset \; \forall i < j$ coincides with the class of conjectured minimizers.

We now need to incorporate a global argument, as local arguments (e.g. stability) will never be enough to exclude configurations like:

Typical GMT argument: if cluster non-rigid, move bubbles until they touch, forming an illegal singularity for an isoperimetric cluster.
$\Rightarrow$ Resolve $q \leq 5$ (double, triple, quadruple-bubble on $\mathbb{R}^n$, $\mathbb{S}^n$).

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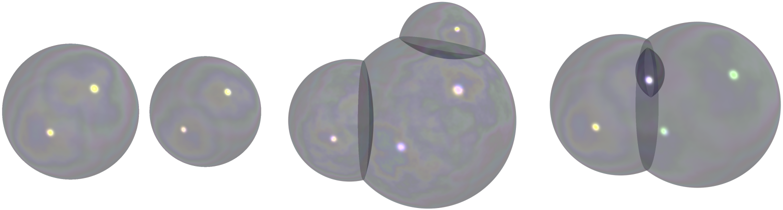

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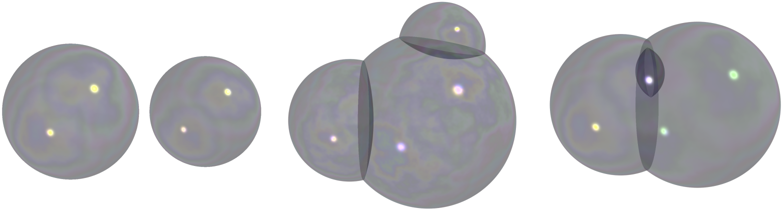

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Bobkov's Functional Gaussian Isoperimetric Inq

Gaussian Isoperimetric Profile: $I : [0,1] \to \mathbb{R}_+$,

$$I(v) := \min\{A_{\gamma^n}(\Omega); \Omega \subset \mathbb{R}^n, V_{\gamma^n}(\Omega) = v\} = \varphi \circ \Phi^{-1}(v).$$

$$\left(\varphi(s) = \frac{1}{\sqrt{2\pi}} \exp(-s^2/2)\ ,\ \Phi(t) = \int_{-\infty}^{t} \varphi(s) ds\right)$$

Thm (Sergey Bobkov 96'-97')

$$\forall f : \mathbb{R}^n \to [0,1] \quad \int_{\mathbb{R}^n} \sqrt{I(f)^2 + |\nabla f|^2} d\gamma^n \geq I\left(\int_{\mathbb{R}^n} f \, d\gamma^n\right).$$

The above is an equivalent functional version of the (single-bubble) Gaussian isoperimetric inequality $A_{\gamma^n}(\Omega) \geq I(V_{\gamma^n}(\Omega))$.

Thm (M.–Neeman '18, unpublished)

For all $f : \mathbb{R}^n \to \Delta^{(2)} = \{(v_1, v_2, v_3); v_i \geq 0, \sum_{i=1}^3 v_i = 1\}$:

$$\int_{\mathbb{R}^n} \sum_{1 \leq i < j \leq 3} \sqrt{I_{ij}^{(2)}(f)^2 + |w_j(f)\nabla f_i - w_i(f)\nabla f_j|^2} d\gamma^n \geq I^{(2)}\left(\int_{\mathbb{R}^n} f \, d\gamma^n\right).$$

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Multi-bubble Isoperimetric Profile: $I^{(q-1)} : \Delta^{(q-1)} \to \mathbb{R}_+$, $n \geq q-1$:

$$I^{(q-1)}(v) := \min \left\{ A_{\gamma^n}(\Omega) ; q\text{-cluster } \Omega = (\Omega_1, \ldots, \Omega_q) \text{ in } \mathbb{R}^n, V_{\gamma^n}(\Omega) = v \right\}.$$

What is the functional version of $A_{\gamma^n}(\Omega) \geq I^{(q-1)}(V_{\gamma^n}(\Omega))$?

Thm (M.–Neeman '18, unpublished)

For all $f : \mathbb{R}^n \to \Delta^{(2)} = \{(V_1, V_2, V_3) : V \geq 0, \sum^3 V_i = 1\}$:

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Here $I_{ij}^{(2)}$ is the Gaussian area of Σ_{ij} in an optimal tripod cluster, $I^{(2)} = I_{12}^{(2)} + I_{13}^{(2)} + I_{23}^{(2)}$, and $w_i = \frac{I_{ij}^{(2)} I_{ik}^{(2)}}{I_{ij}^{(2)} I_{ik}^{(2)} + I_{ji}^{(2)} I_{jk}^{(2)} + I_{ki}^{(2)} I_{kj}^{(2)}}$.

Sergey,
thank you for your
contributions,
inspiration,
and Happy Birthday!

The Isoperimetric Profile for Multi-Bubbles

$(M^n, g, \mu) \in \{\mathbb{G}^n, \mathbb{S}^n\}$. Need finite volume, so cannot work on $\mathbb{R}^n$.
$V(\Omega) = (V(\Omega_1), \ldots, V(\Omega_q)) \in \Delta^{(q-1)} := \left\{ v \in \mathbb{R}^q \; ; v_i \geq 0 \; , \; \sum_{i=1}^q v_i = 1 \right\}$.

Isoperimetric Profile: $I^{(q-1)} : \Delta^{(q-1)} \to \mathbb{R}_+$,

$$I^{(q-1)}(v) := \inf \{ A(\Omega); V(\Omega) = v \} .$$

Model Isoperimetric Profile: $I_m^{(q-1)} : \text{int}\, \Delta^{(q-1)} \to \mathbb{R}_+$,
(denoting by Ω^m the conjectured model standard q-cluster),

$$I_m^{(q-1)}(v) = A(\Omega^m) \text{ s.t. } V(\Omega^m) = v \in \text{int}\, \Delta^{(q-1)};$$

extend continuously to $\partial \Delta^{(q-1)}$.

Obviously $I^{(q-1)} \leq I_m^{(q-1)}$; want to show: $I^{(q-1)} \geq I_m^{(q-1)}$ on $\Delta^{(q-1)}$.
Inducting on q, can assume $I^{(q-1)} = I_m^{(q-1)}$ on the boundary $\partial \Delta^{(q-1)}$.

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Partial Differential Inequality for Profile

On $\mathbb{G}^n$, one can show that a fully non-linear elliptic PDE holds:

$$\mathrm{tr}((-\nabla^2 \mathcal{I}_m)^{-1}) = 2\mathcal{I}_m \text{ on } \Delta^{(q-1)}.$$

Similar (but more complicated) PDE holds on $\mathbb{S}^n$.

If we could show that the following PDI holds (in the viscosity sense):

$$\nabla^2 \mathcal{I} < 0 \ , \ \mathrm{tr}((-\nabla^2 \mathcal{I})^{-1}) \leq 2\mathcal{I} \text{ on } \operatorname{int} \Delta^{(q-1)},$$

since $\mathcal{I} = \mathcal{I}_m$ on $\partial \Delta^{(q-1)}$ by induction, $\mathcal{I} \geq \mathcal{I}_m$ by maximum-principle.

This is our global information!! PDI takes into account entire $\Delta^{(q-1)}$.

Hence, need upper bounds on $\nabla^2 \mathcal{I}(v)$ for a given $v \in \operatorname{int} \Delta^{(q-1)}$.
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The index-form $Q(X)$

Recall $\frac{d}{dt}F_t = X \circ F_t$ diffeo, $\Omega_t = F_t(\Omega)$, $\mathcal{I}(V(\Omega_t)) \leq A(\Omega_t)$. Hence:

$$\left\langle \nabla \mathcal{I}, \delta^1_X V \right\rangle = \delta^1_X A = \left\langle \lambda, \delta^1_X V \right\rangle \Rightarrow \nabla \mathcal{I} = \lambda.$$

$$(\delta^1_X V)^T \, \nabla^2 \mathcal{I} \, \delta^1_X V \leq \delta^2_X A - \left\langle \nabla \mathcal{I}, \delta^2_X V \right\rangle = \delta^2_X A - \left\langle \lambda, \delta^2_X V \right\rangle =: Q(X).$$

This generalizes stability: $\delta^1_X V = 0 \;\Rightarrow\; 0 \leq Q(X)$.

The goal: choose X well to get a sharp PDI for $\mathcal{I}$.

$Q(X)$ index-form, depends only on $f_{ij} = \left\langle X, \mathfrak{n}_{ij} \right\rangle$ on $\Sigma^1 = \sqcup_{i<j} \Sigma_{ij}$.

$$Q(f) = -\left\langle L_{Jac} f, f \right\rangle_{\Sigma^1} + \int_{\partial^* \Sigma^1} \mathrm{bdry}(f, \mathrm{II}).$$

L_{Jac} is the Jacobi operator:

$$-\delta^1_{f\mathfrak{n}} H_{\Sigma,\mu} = L_{Jac} f = \Delta_{\Sigma,\mu} f + (\mathrm{Ric}_{g,\mu}(\mathfrak{n},\mathfrak{n}) + \|\mathrm{II}\|^2) f.$$

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Obtaining the Partial Differential Inequality

$$(\delta_X^1 V)^T \, \nabla^2 \mathcal{I} \, \delta_X^1 V \leq -\langle L_{Jac} X^{\mathfrak{n}}, X^{\mathfrak{n}} \rangle_{\Sigma^1} + \int_{\partial^* \Sigma^1} \mathrm{bdry}(X^{\mathfrak{n}}, \mathrm{II}),$$

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Here $\mathrm{Ric}_{g,\mu}(\mathfrak{n},\mathfrak{n}) = 1$ on $\mathbb{G}^n$ and $\mathrm{Ric}_{g,\mu}(\mathfrak{n},\mathfrak{n}) = (n-1)$ on $\mathbb{S}^n$.
And we already know that $\mathrm{II} = 0$ on $\mathbb{G}^n$ and $\mathrm{II} = \kappa_{ij} \mathrm{Id}$ on $\mathbb{S}^n$.

On $\mathbb{G}^n$: $L_{Jac} 1 = 1$, so if $X^{\mathfrak{n}_{ij}} = a_i - a_j$ then $L_{Jac} X^{\mathfrak{n}_{ij}} = a_i - a_j$.
As $\mathfrak{n}_{ij} + \mathfrak{n}_{jk} + \mathfrak{n}_{ki} = 0$ on $\partial^* \Sigma^1$, possible to (approximately) construct X.
This yields sharp PDI, and we conclude the proof that $\mathcal{I} \geq \mathcal{I}_m$.

On $\mathbb{S}^n$: fields yielding sharp PDI exist (non-trivial). But we don't have explicit formula, unless cluster is (pseudo)-conformally-flat ($\{\mathfrak{c}_i, \kappa_i\}$).
E.g.: • when cluster is full-dimenional, i.e. $\text{affine-rank}\{\mathfrak{c}_i\}_{i=1}^q = q - 1$;
• if all bubbles have a mutual common point.
In those cases, we obtain the sharp PDI for $\mathcal{I}$.

But what if the cluster is not pseudo-conformally-flat???
While this should never happen, we cannot a-priori exclude this.
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And we already know that $\mathrm{II} = 0$ on $\mathbb{G}^n$ and $\mathrm{II} = \kappa_{ij}\mathrm{Id}$ on $\mathbb{S}^n$.

On $\mathbb{G}^n$: $L_{Jac} 1 = 1$, so if $X^{\mathfrak{n}_{ij}} = a_i - a_j$ then $L_{Jac} X^{\mathfrak{n}_{ij}} = a_i - a_j$.
As $\mathfrak{n}_{ij} + \mathfrak{n}_{jk} + \mathfrak{n}_{ki} = 0$ on $\partial^* \Sigma^1$, possible to (approximately) construct X.
This yields sharp PDI, and we conclude the proof that $\mathcal{I} \geq \mathcal{I}_m$.

On $\mathbb{S}^n$: fields yielding sharp PDI exist (non-trivial). But we don't have explicit formula, unless cluster is (pseudo)-conformally-flat ($\{\mathfrak{c}_i, \kappa_i\}$).
E.g.:
- when cluster is full-dimenional, i.e. $\text{affine-rank}\{\mathfrak{c}_i\}_{i=1}^q = q - 1$;
- if all bubbles have a mutual common point.

In those cases, we obtain the sharp PDI for $\mathcal{I}$.

But what if the cluster is not pseudo-conformally-flat???
While this should never happen, we cannot a-priori exclude this.
Using Step 6 (= some tricks), we can go up to $q \leq 6$ on $\mathbb{S}^n$.