Relative Log-concavity

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Sergey's birthday party

Outline

Colaboradores: Sergey Bobkov, Heshan Aravinda, Arturo Jaramillo, Mokshay Madiman, Arnaud Marsiglietti, Gerardo Palafox-Castillo, Cyril Roberto, Tomasz Tkocz.

[redacted]

f $x_n g_n$ without holes, st

$$x_n^2 \qquad x_{n+1}\, x_n \quad 1$$

Log-Concavity

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X log-concave () $x_n = P(X = n)$

convolution

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- Examples: Bernoulli, Binomial, Poisson, Geometric, Hypergeometric, etc

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- Geometry of Polynomials $P[X] = \sum_{i=0}^{n} c_i X^i$ real rooted, $\{c_i\}$ is log-concave

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- Convex Geometry ▸ Alexandrov-Fenchel inequality $\Rightarrow$ \intrinsic volumes" associated to convex bodies are log-concave

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- Probability - Theory of Negative dependence
  Joag-Dev & Proschan '83 Pemantle '00 Borcea, Brändén, and Liggett '09.
- Information Theory - Maximum (Shannon) entropy properties Poisson
  Johnson '06 Johnson, Kontoyannis, & Madiman '11

Relative Log-concavity

Definition

X log-concave wrt Z, $x\ 2\ L(z)$

$$r_n := \frac{x_n}{z_n} := \frac{\mathrm{P}(X = n)}{\mathrm{P}(Z = n)} \qquad \text{Log-concave:}$$

X log-affine wrt Z $(\ )$ r_n log-affine.

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Examples:

- Z geometric: Log-concave, support N,
  $x_n^2 \quad x_{n+1}x_{n\ \ 1}$
- Z Poisson( ): Ultra log-concave. $ULC(\ 1\ )$,
  $x_n^2 \quad 1 + \frac{1}{n} \ x_{n+1}x_{n\ \ 1}$
- Z Binomial$(m; p)$: ultra log-concave of orden m, $ULC(m)$.
  $x_n^2 \quad 1 + \frac{1}{n} \quad 1 + \frac{1}{m\ \ n} \ x_{n+1}x_{n\ \ 1}$
- Z is the intrinsic volume sequence of a Euclidean ball,

Relative Log-concavity

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Study systematically, log-concave variables.

Geometry of Polynomials $P(x) = \sum_{i=0}^{n} a_i x^i$ real roots, $\{a_i\}$ $\iff$ distribution of a Bernoulli sum. ▸ Pitman '97 ▸ Tang y Tang '23 ▸ Madiman, M. y Roberto '23

M matroid m elements, x_n = number of independent sets of size n. ▸ Anari et al '18 ▸ Braden - Huh '20

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Relative Log-concavity

Aim:

Study systematically, log-concave variables.

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$ULC(m)$

"intrinsic volumes" associated to convex bodies
▸ Alexandrov-Fenchel inequality) LC wrt Ball intrinsic volume
$ULC(1)$.

Extreme Points

Theorem: M.- Marsiglietti '23+

$f : [b; c] \to \mathbb{R}$, $L_f^z[b; c] := \{x \in L(z) : \sum_{i=b}^{c} x_i f_i = 0\}$

$$E(L_f[a; b]) = \left\{ a \in \text{log-affine} : \sum_{i=b} a_i f_i \text{ same sign} \right\}$$

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Stolen from Guedon-Fradelizi '04, extreme point interpretation of "localization lemma" of Lovasz and Simonovits '93

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Extreme Points

Theorem: M.- Marsiglietti '23+

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extended by Nayar-Słobodianiuk to "degrees of freedom" '23+

My first paper: "Hyperbolic Measures on infinite dimensional spaces" Bobkov - M. '16

Corollary

$f_1; f_2; f_3; f_4 : Z \; ! \; [0; \mathit{1}\,)$, ;

$$x \;\; (f_1)x \;\; (f_2) \quad x \;\; (f_3)x \;\; (f_4)$$

for all x log-concave if

$$a \;\; (f_1)a \;\; (f_2) \quad a \;\; (f_3)a \;\; (f_4)$$

for all a log-affine.

Corollary

$f_1; f_2; f_3; f_4 : Z \to [0; \infty)$, ;

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Use:

- Sharp small and large deviations from medians for log-concave sequences. (Bobkov-Nazarov '08)
- Walkup $p; q$ log-concave wrt Poisson $\Rightarrow$ $p \quad q$ log-concave wrt Poisson
- Gurvits inequality p log-concave $\Rightarrow$ $t \mapsto \sum \frac{p_n}{n!} t^n$
- Klartag-Lehec $c_k = \sum_{n \quad k} \binom{n}{k} a_n$ log-concave if a_n log-concave.

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Uses:

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Theorem: Gurvits

p_n-log-concave $\Rightarrow$ $f(t) = \sum_{n=0}^{\infty} \frac{p_n}{n!} t^n$ log-concave

Proof: $F = (f')^2 - f(f'') \geq 0$ when $p_n = p^n \mathbf{1}_A$, $A = [\![a;b]\!]$ enough.

[redacted]

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$$F(t) = \sum_{k=0}^{\infty} p^{k+2} \left[\frac{t^k}{k!} \sum_{n \in \mathbb{Z}} \binom{k}{n} I_A(n;k) \right]$$

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with $I_A(n;k) = \chi_A(n+1)\chi_A(k-n+1) - \chi_A(n)\chi_A(k-n+2)$ coefficients $= 0$ unless $m := (a-1) \vee (l-(b-1)) \leq \frac{l}{2}$, in which case,

$$\sum_{n \in \mathbb{Z}} \binom{l}{n} I_A(n;k) = \binom{l}{m} - \binom{l}{m-1} \geq 0.$$

Majorization

Definition

$X \quad Z$, convex)

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$F_X(t) := \mathrm{P}(X \quad t)$

Equivalent to

$$\mathrm{Z}_{t}^{1} F_X(\ )d \qquad \mathrm{Z}_{t}^{1} F_Z(\ )d \ :$$

Example from earlier work

Z_K Intrinsic volume random variables of convex body K $\quad \mathrm{R}^n$,

$$\mathrm{P}(Z_k = j) \qquad \begin{matrix} n \\ j \end{matrix} \quad \frac{n}{j \quad n \quad j} \mathrm{E}_{\mathbf{Q}}[\mathrm{vol}_j(P_j \, \mathbf{Q} K)]$$

P_j, j-dimensional projection, $\mathbf{Q}$ random rotation, $\quad j = \frac{j=2}{\Gamma(1+j=2)}$

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Theorem: (Lotz, McCoy, Nourdin, Pecatti, Tropp '20)

$$\mathrm{Var}(Z_K) \quad 4n;$$

$$\mathbb{P}(jZ_K \quad \mathbb{E}[Z_K]j \quad t^{\,p}\overline{n}) \quad 2e^{\;3t^2=28}$$

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$$\mathrm{Var}(Z_K) \quad 4n;$$

$$\mathrm{P}(jZ_K \quad \mathrm{E}[Z_K]j \quad t \sqrt[p]{n}) \quad 2e^{\quad 3t^2=28}$$

Inspiration: "cut-off" phenomena in algorithms.

with convexity

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Theorem: (Lotz, McCoy, Nourdin, Pecatti, Tropp '20)

$$\mathrm{Var}(Z_K) \le 4n,$$

$$\mathbb{P}(|Z_K - \mathbb{E}[Z_K]| \ge t\sqrt{n}) \le 2e^{-3t^2/28}$$

Theorem: (Aravinda, Marsiglietti, M. '22)

$Z \in ULC(-1)$ in $\{0, 1, \ldots, n\}$,

$$\mathrm{Var}(Z_K) \le n,$$

$$\mathbb{P}(|Z_K - \mathbb{E}[Z_K]| \ge t\sqrt{n}) \le 2e^{-t^2/2}$$

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Theorem: (Aravinda, Marsiglietti, M. '22)

$Z\ 2\ ULC(\ 1\ )$ in $f0;1;:::;ng$,

$$\mathrm{Var}(Z_K) \quad n;$$

$$\mathrm{P}(jZ_K \quad \mathrm{E}[Z_K]j \quad t^{p}\overline{n}) \quad 2e^{\ t^2=2}$$

Proof idea: $ULC(\ 1\ )$ controlled by Poisson.

Ideas:

Theorem: ▸ Aravinda, Marsiglietti, M. '22

$Z \in ULC(\infty)$ with values in $\{0, 1, \ldots, n\}$,

$$\mathrm{Var}(Z) \leq \mathrm{E}[Z] \leq n;$$

$$\mathrm{P}(|Z - \mathrm{E}[Z]| \geq t\sqrt{n}) \leq 2e^{-t^2/2}$$

- $ULC(\infty)$ controlled by Poisson
- Extreme points of slices of log-concave distributions ▸ Marsiglietti, M. '23
- $\mathrm{E}[X] = \mathrm{E}[Z]$, $X \in ULC(\infty)$ Z Poisson $\Rightarrow$ $\mathrm{E}e^{tX} \leq \mathrm{E}e^{tZ}$
- Taylor expancion y Chernoff bounds (resp).

Majorization and convexity

[redacted]

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Karlin-Noviko cut criteria criteria, densities cross twice) convex majorization

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log-concavity) cannot cross more than twice

Majorization and convexity

Lemma: Whitt '85

$X \quad _{LC} Z$ y $\mathrm{E}[X] = \mathrm{E}[Z]$) $X \quad _{cx} Z$

Karlin-Novikoff cut criteria criteria, densities cross twice) convex majorization

log-concavity) cannot cross more than twice

Probability distributions same mean) cross at least twice.

Transference of Chernoff Bounds

Lemma: Marsiglietti, M. '23+

$X \quad {}_{LC}\, Z$,

$$P(X \quad t) \quad \inf_{>0} E[e^{(Z \quad t)}].$$

Recovers Aravinda, Marsiglietti, M. '22.

Transference of Chernoff Bounds

Lemma: Marsiglietti, M. '23+

$X \quad {}_{LC}\, Z$,

$$\mathrm{P}(X \quad t) \quad \inf_{>0} \mathrm{E}[e^{\ (Z \quad t)}]:$$

Recovers Aravinda, Marsiglietti, M. '22.

Demostración:

$$\begin{aligned} \mathrm{P}(X \quad t) &= \mathrm{P}(e^{\ X} \quad e^{\ t}) \\ & \quad \mathrm{E}e^{\ (X \quad t)} \\ & \quad \mathrm{E}e^{\ (Z \quad t)}: \end{aligned}$$

Corollaries of transfer

[redacted]

X 2 ULC(n), $\quad := E[X]/n$,

$$P(X \quad (\quad + t)n) \quad e^{nD(\quad + t || \quad)};$$

$$P(X \quad (\quad + t)n) \quad e^{nD(\quad t || \quad)}.$$

Corollaries of transfer

[redacted]

X 2 ULC(n), := E[X]=n,

$$P(X \quad (\; + t)n) \quad e^{\;nD(\; + t || \;)};$$
$$P(X \quad (\; \; t)n) \quad e^{\;nD(\; \; t || \;)}:$$

$D(p||q) := p \log \frac{p}{q} + (1 \quad p) \log \frac{1 \; p}{1 \; q}$.

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Recovers Hoe ding for Bernoulli sums by

ULC(m) + ULC(n) ULC(m + n) (Liggett '97)

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Recovers Hoe ding for Bernoulli sums by

ULC(m) + ULC(n) ULC(m + n) (Liggett '97)

Matroid RVs have binomial concentration.

Recent and Future directions

Moment inequalities

Maximum entropy distributions (Intrinsic volume random variables)

Quantiative limit theorems (w/ Arturo Jaramillo)

Feige Conjecture (w/ Alqasem, Aravinda, & Marsiglietti).

Renyi entropy comparisons (w/ Tkocz), for monotone sequences (w/ Palafox-Castillo)

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$ULC(\,1\,)$

Theorem: (Marsiglietti, M. '23+)

$X\ 2\ ULC(n), (x)_n := \frac{x!}{(x\ \ n)!}$

$$p\ 7\!\!\!!\ \ \frac{\mathrm{E}[(X)_p]}{(n)_p}$$

is log-concave or

$$\mathrm{E}^2[(X)_p] \qquad 1 + \frac{1}{n \quad p}\ \ \mathrm{E}[(X)_{p+1}]\mathrm{E}[(X)_{p\ \ 1}]$$

$ULC(1)$

Theorem: (Marsiglietti, M. '23+)

$X \in ULC(n)$, $(x)_n := \frac{x!}{(x-n)!}$

$$p \mapsto \frac{\mathbb{E}[(X)_p]}{(n)_p}$$

is log-concave or

$$\mathbb{E}^2[(X)_p] \geq \left(1 + \frac{1}{n-p}\right) \mathbb{E}[(X)_{p+1}]\mathbb{E}[(X)_{p-1}]$$

See Bobkov y Madiman '11 for analogous results in the continuous setting. Rényi entropy M.y Palafox-Castillo '23+

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Idea: Two-crossing densities preserved under monotone maps.

Moment Inequalities

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Moment Inequalities

Theorem: (Marsiglietti, M. '23+)

$X \in ULC(n)$,

$$p \mapsto \frac{E[(X)_p]}{(n)_p}.$$

Theorem: Bobkov y Madiman '11

For $X > 0$ log-concave,

$$p \mapsto E[X^p]/\Gamma(p+1)$$

is log-concave.

Bibliography

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