

Relative Log-concavity

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Sergey's birthday party

Outline

Colaboradores: **Sergey Bobkov**, Heshan Aravinda, Arturo Jaramillo, Mokshay Madiman, Arnaud Marsiglietti, Gerardo Palafox-Castillo, Cyril Roberto, Tomasz Tkocz.

- 1 Definitions, Background, and Motivation
- 2 “Localization”
- 3 Majorization

Log-Concavity

Definition: Log-Concave

$\{x_n\}_n$ without holes, st

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- Geometry of Polynomials - $P[X] = \sum_{i=0}^n c_i X^i$ real rooted, $\{c_i\}$ is log-concave

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- Convex Geometry - ▶ Alexandrov-Fenchel inequality \Rightarrow “intrinsic volumes” associated to convex bodies are log-concave
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- Information Theory - Maximum (Shannon) entropy properties Poisson
 - ▶ Johnson '06
 - ▶ Johnson, Kontoyannis, & Madiman '11

Relative Log-concavity

Definition

X log-concave wrt Z , $x \in \mathcal{L}(z)$

$$r_n := \frac{x_n}{z_n} := \frac{\mathbb{P}(X = n)}{\mathbb{P}(Z = n)} \quad \text{Log-concave.}$$

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Examples:

- Z geometric: Log-concave, support \mathbb{N} ,
 $x_n^2 \geq x_{n+1}x_{n-1}$
- Z Poisson(λ): Ultra log-concave. $ULC(\infty)$,
 $x_n^2 \geq \left(1 + \frac{1}{n}\right) x_{n+1}x_{n-1}$
- Z Binomial(m, p): ultra log-concave of orden m , $ULC(m)$.
 $x_n^2 \geq \left(1 + \frac{1}{n}\right) \left(1 + \frac{1}{m-n}\right) x_{n+1}x_{n-1}$
- Z is the intrinsic volume sequence of a Euclidean ball,

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Study systematically, log-concave variables.

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↔ distribution of a Bernoulli sum. ► Pitman '97 ► Tang y Tang '23
► Madiman, M. y Roberto '23
- \mathcal{M} matroid m elements, $x_n =$ number of independent sets of size n . ► Anari et al '18 ► Brädén - Huh '20

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 $\subseteq ULC(m)$
- “intrinsic volumes” associated to convex bodies
► Alexandrov-Fenchel inequality $\Rightarrow \subseteq$ LC wrt Ball intrinsic volume
 $\subseteq ULC(\infty)$.

Extreme Points

Theorem: M.- Marsiglietti '23+

$$f : \llbracket b, c \rrbracket \rightarrow \mathbb{R}, \mathcal{L}_f^z \llbracket b, c \rrbracket := \{x \in \mathcal{L}(z) : \sum_{i=b}^c x_i f_i = 0\}$$

$$\mathcal{E}(\mathcal{L}_f \llbracket a, b \rrbracket) = \left\{ a \in \text{log-affine} : \sum_{i=b}^j a_i f_i \text{ same sign} \right\}$$

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- extended by Nayar-Słobodianiuk to “degrees of freedom” '23+
- My first paper: “Hyperbolic Measures on infinite dimensional spaces” Bobkov - M. '16

Corollary

$f_1, f_2, f_3, f_4 : \mathbb{Z} \rightarrow [0, \infty)$, α, β

$$x^\alpha(f_1)x^\beta(f_2) \leq x^\alpha(f_3)x^\beta(f_4)$$

for all x log-concave if

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Use:

- Sharp small and large deviations from medians for log-concave sequences. (Bobkov-Nazarov '08)
- Walkup p, q log-concave wrt Poisson $\Rightarrow p * q$ log-concave wrt Poisson
- **Gurvits inequality** p log-concave $\Rightarrow t \mapsto \sum \frac{p_n}{n!} t^n$
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Theorem: Gurvits

$$p_n\text{-log-concave} \Rightarrow f(t) = \sum_{n=0}^{\infty} \frac{p_n}{n!} t^n \text{ log-concave}$$

Proof: $F = (f')^2 - f(f'') \geq 0$ when $p_n = p^n \mathbb{1}_A$, $A = \mathbb{1}_{[a,b]}$ enough.

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coefficients = 0 unless $m := (a-1) \vee (l-(b-1)) \leq \frac{l}{2}$, in which case,

$$\sum_{n \in \mathbb{Z}} \binom{l}{n} I_A(n, k) = \binom{l}{m} - \binom{l}{m-1} \geq 0.$$

Majorization

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- Sufficient to check $\phi_t(x) := [x - t]_+$.
- $F_X(t) := \mathbb{P}(X \geq t)$
- Equivalent to

$$\int_t^\infty F_X(\lambda) d\lambda \leq \int_t^\infty F_Z(\lambda) d\lambda.$$

Example from earlier work

Z_K Intrinsic volume random variables of convex body $K \subseteq \mathbb{R}^n$,

$$\mathbb{P}(Z_k = j) \asymp \binom{n}{j} \frac{\kappa_n}{\kappa_j \kappa_{n-j}} \mathbb{E}_{\mathbf{Q}}[\text{vol}_j(P_j \mathbf{Q} K)]$$

P_j , j -dimensional projection, \mathbf{Q} random rotation, $\kappa_j = \frac{\pi^{j/2}}{\Gamma(1+j/2)}$

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Theorem: (Lotz, McCoy, Nourdin, Pecatti, Tropp '20)

$$\text{Var}(Z_K) \leq 4n,$$

$$\mathbb{P}(|Z_K - \mathbb{E}[Z_K]| \geq t\sqrt{n}) \leq 2e^{-3t^2/28}$$

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- Inspiration: “cut-off” phenomena in algorithms.

with convexity

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- Proof idea: $ULC(\infty)$ controlled by Poisson.

Ideas:

Theorem: ▶ Aravinda, Marsiglietti, M. '22

$Z \in ULC(\infty)$ with values in $\{0, 1, \dots, n\}$,

$$\text{Var}(Z) \leq \mathbb{E}[Z] \leq n,$$

$$\mathbb{P}(|Z - \mathbb{E}[Z]| \geq t\sqrt{n}) \leq 2e^{-t^2/2}$$

- $ULC(\infty)$ controlled by Poisson
- Extreme points of slices of log-concave distributions
▶ Marsiglietti, M. '23
- $\mathbb{E}[X] = \mathbb{E}[Z]$, $X \in ULC(\infty)$ Z Poisson $\Rightarrow \mathbb{E}e^{tX} \leq \mathbb{E}e^{tZ}$
- Taylor expansion y Chernoff bounds (resp).

Majorization and convexity

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► Whitt '85

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- ► Karlin-Novikoff cut criteria criteria, densities cross twice \Rightarrow convex majorization
- log-concavity \Rightarrow cannot cross more than twice
- Probability distributions same mean \Rightarrow cross at least twice.

Transference of Chernoff Bounds

Lemma: ▶ Marsiglietti, M. '23+

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$$\mathbb{P}(X \geq t) \leq \inf_{\lambda > 0} \mathbb{E}[e^{\lambda(Z-t)}].$$

- Recovers ▶ Aravinda, Marsiglietti, M. '22 .

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Demostración:

$$\begin{aligned}\mathbb{P}(X \geq t) &= \mathbb{P}(e^{\lambda X} \geq e^{\lambda t}) \\ &\leq \mathbb{E} e^{\lambda(X-t)} \\ &\leq \mathbb{E} e^{\lambda(Z-t)}.\end{aligned}$$

Corollaries of transfer

Theorem: Hoeffding's inequality for $ULC(n)$

$X \in ULC(n)$, $\mu := \mathbb{E}[X]/n$,

$$\mathbb{P}(X \geq (\mu + t)n) \leq e^{-nD(\mu+t||\mu)},$$

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 $ULC(m) + ULC(n) \subseteq ULC(m+n)$ (Liggett '97)

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 $ULC(m) + ULC(n) \subseteq ULC(m+n)$ (Liggett '97)
- Matroid RVs have binomial concentration.

Recent and Future directions

- Moment inequalities
- Maximum entropy distributions (Intrinsic volume random variables)
- Quantitative limit theorems (w/ Arturo Jaramillo)
- Feige Conjecture (w/ Alqasem, Aravinda, & Marsiglietti).
- Renyi entropy comparisons (w/ Tkocz), for monotone sequences (w/ Palafox-Castillo)

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ULC(∞)

Theorem: (Marsiglietti, M. '23+)

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$$p \mapsto \frac{\mathbb{E}[(X)_p]}{(n)_p}$$

is log-concave or

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- See ▶ Bobkov y Madiman '11 for analogous results in the continuous setting. Rényi entropy ▶ M.y Palafax-Castillo '23+

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- Idea: Two-crossing densities preserved under monotone maps.

Moment Inequalities

Theorem: (Marsiglietti, M. '23+)

$X \in ULC(n)$,

$$p \mapsto \frac{\mathbb{E}[(X)_p]}{(n)_p}.$$

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Theorem: ▶ Bobkov y Madiman '11

For $X > 0$ log-concave,

$$p \mapsto \mathbb{E}[X^p]/\Gamma(p + 1)$$

is log-concave.

Bibliography

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The end



Thank you Sergey!