Norms of structured random matrices

Marta Strzelecka

(based on joint work with R. Adamczak, J. Prochno, and M. Strzelecki)

University of Warsaw

Conference in Honour of Sergey Bobkov

May 29, 2023

Given:

- $1 \leq p, q \leq \infty$;
- a deterministic real $m \times n$ matrix $A = (a_{ij})_{i \leq m, j \leq n}$;
- a random Gaussian $m \times n$ matrix $G = (g_{ij})_{i \leq m, j \leq n}$.

We denote by

$$G_A := A \circ G = (a_{ij} g_{ij})_{i \leq m, j \leq n}$$

the structured Gaussian matrix.

Given:

- $1 \leq p, q \leq \infty$;
- a deterministic real $m \times n$ matrix $A = (a_{ij})_{i \leq m, j \leq n}$;
- a random Gaussian $m \times n$ matrix $G = (g_{ij})_{i \leq m, j \leq n}$.

We denote by

$$G_A := A \circ G = (a_{ij} g_{ij})_{i \leq m, j \leq n}$$

the structured Gaussian matrix.

Aim:

$$\mathbb{E}\| G_A \colon \ell_p^n \to \ell_q^m \| \asymp_{p,q} \quad ?$$

Theorem (Latała-van Handel-Youssef, 2018)

$$\mathbb{E}\|G_A\colon \ell_2^n \to \ell_2^m\| \asymp \mathbb{E}\max_i \|(a_{ij}g_{ij})_j\|_2 + \mathbb{E}\max_j \|(a_{ij}g_{ij})_i\|_2$$
$$\asymp \max_i \|(a_{ij})_j\|_2 + \max_j \|(a_{ij})_i\|_2 + \max_{i,j} |a_{ij}^*|\sqrt{\log(i+1)},$$

where (a_{ij}^*) is obtained by permuting the rows of A so that $\max_j |a_{1j}^*| \geq \ldots \geq \max_j |a_{nj}^*|$.

Theorem (Latała-van Handel-Youssef, 2018)

$$\mathbb{E}\|G_A\colon \ell_2^n \to \ell_2^m\| \asymp \mathbb{E}\max_i \|(a_{ij}g_{ij})_j\|_2 + \mathbb{E}\max_j \|(a_{ij}g_{ij})_i\|_2$$
$$\asymp \max_i \|(a_{ij})_j\|_2 + \max_j \|(a_{ij})_i\|_2 + \max_{i,j} |a_{ij}^*|\sqrt{\log(i+1)}$$

$$\asymp \max_i \|(a_{ij})_j\|_2 + \max_j \|(a_{ij})_i\|_2 + \mathbb{E}\max_{i,j} |a_{ij}g_{ij}|,$$

where (a_{ij}^*) *is obtained by permuting the rows of* A *so that* $\max_j |a_{1j}^*| \geq \ldots \geq \max_j |a_{nj}^*|$.

Theorem (Latała-van Handel-Youssef, 2018)

$$\mathbb{E}\| G_A\colon \ell_2^n \to \ell_2^m\| \asymp \mathbb{E}\max_i \|(a_{ij}g_{ij})_j\|_2 + \mathbb{E}\max_j \|(a_{ij}g_{ij})_i\|_2$$
$$\asymp \max_i \|(a_{ij})_j\|_2 + \max_j \|(a_{ij})_i\|_2 + \max_{i,j} |a^*_{ij}|\sqrt{\log(i+1)}$$
$$\asymp \max_i \|(a_{ij})_j\|_2 + \max_j \|(a_{ij})_i\|_2 + \mathbb{E}\max_{i,j} |a_{ij}g_{ij}|,$$

where (a^*_{ij}) *is obtained by permuting the rows of* A *so that* $\max_j |a^*_{1j}| \geq \ldots \geq \max_j |a^*_{nj}|$.

First guess: is it true that for every $1 \leq p, q \leq \infty$,

$$\mathbb{E}\| G_A\colon \ell_p^n \to \ell_q^m\| \asymp_{p,q} \mathbb{E}\max_i \|(a_{ij}g_{ij})_j\|_{p^*} + \mathbb{E}\max_j \|(a_{ij}g_{ij})_i\|_q,$$

where p^* is Hölder's dual of p, i.e., $\frac{1}{p} + \frac{1}{p^*} = 1$?

Theorem (Latała-van Handel-Youssef, 2018)

$$\mathbb{E}\| G_A \colon \ell_2^n \to \ell_2^m \| \asymp \max_i \|(a_{ij})_j\|_2 + \max_j \|(a_{ij})_i\|_2 + \mathbb{E} \max_{i,j} |a_{ij} g_{ij}|.$$

Theorem (Latała-van Handel-Youssef, 2018)

$$\mathbb{E}\|G_A \colon \ell_2^n \to \ell_2^m\| \asymp \max_i \|(a_{ij})_j\|_2 + \max_j \|(a_{ij})_i\|_2 + \mathbb{E}\max_{i,j} |a_{ij}g_{ij}|.$$

Theorem (Guédon-Hinrichs-Litvak-Prochno, 2017)

Assume that $1 \leq p \leq 2 \leq q \leq \infty$

Theorem (Latała-van Handel-Youssef, 2018)

$$\mathbb{E}\|G_A \colon \ell_2^n \to \ell_2^m\| \asymp \max_i \|(a_{ij})_j\|_2 + \max_j \|(a_{ij})_i\|_2 + \mathbb{E}\max_{i,j} |a_{ij} g_{ij}|.$$

Theorem (Guédon-Hinrichs-Litvak-Prochno, 2017)

Assume that $1 \leq p \leq 2 \leq q \leq \infty$

$$\mathbb{E}\|G_A \colon \ell_p^n \to \ell_q^m\| \lesssim_{p,q} \Big[\big(\log m\big)^{1/q} \max_i \|(a_{ij})_j\|_{p^*} + \max_j \|(a_{ij})_i\|_q + \big(\log m\big)^{1/q} \, \mathbb{E}\max_{i,j} |a_{ij} g_{ij}| \Big].$$

Theorem (Latała-van Handel-Youssef, 2018)

$$\mathbb{E}\|G_A\colon \ell_2^n \to \ell_2^m\| \asymp \max_i \|(a_{ij})_j\|_2 + \max_j \|(a_{ij})_i\|_2 + \mathbb{E}\max_{i,j} |a_{ij} g_{ij}|.$$

Theorem (Guédon-Hinrichs-Litvak-Prochno, 2017)

Assume that $1 \le p \le 2 \le q \le \infty$

$$\mathbb{E}\|G_A\colon \ell_p^n \to \ell_q^m\| \lesssim_{p,q} \Big[(\log m)^{1/q} \max_i \|(a_{ij})_j\|_{p^*} + \max_j \|(a_{ij})_i\|_q + (\log m)^{1/q}\, \mathbb{E}\max_{i,j} |a_{ij} g_{ij}| \Big].$$

Theorem (Latała-van Handel-Youssef, 2018)

$$\mathbb{E}\| G_A \colon \ell_2^n \to \ell_2^m \| \asymp \max_i \|(a_{ij})_j\|_2 + \max_j \|(a_{ij})_i\|_2 + \mathbb{E} \max_{i,j} |a_{ij} g_{ij}|.$$

Theorem (Guédon-Hinrichs-Litvak-Prochno, 2017)

Assume that $1 \le p \le 2 \le q \le \infty$

$$\mathbb{E}\| G_A \colon \ell_p^n \to \ell_q^m \| \lesssim_{p,q} \Big[(\log m)^{1/q} \max_i \|(a_{ij})_j\|_{p^*} + \max_j \|(a_{ij})_i\|_q + (\log m)^{1/q}\, \mathbb{E} \max_{i,j} |a_{ij} g_{ij}| \Big].$$

Theorem (Latała-van Handel-Youssef, 2018)

$$\mathbb{E}\|G_A\colon \ell_2^n \to \ell_2^m\| \asymp \max_i \|(a_{ij})_j\|_2 + \max_j \|(a_{ij})_i\|_2 + \mathbb{E}\max_{i,j}|a_{ij}g_{ij}|.$$

Theorem (Guédon-Hinrichs-Litvak-Prochno, 2017)

Assume that $1 \le p \le 2 \le q \le \infty$

$$\mathbb{E}\|G_A\colon \ell_p^n \to \ell_q^m\| \lesssim_{p,q} \Big[(\log m)^{1/q} \max_i \|(a_{ij})_j\|_{p^*} + \max_j \|(a_{ij})_i\|_q + (\log m)^{1/q}\, \mathbb{E}\max_{i,j}|a_{ij}g_{ij}| \Big].$$

Theorem (Latała-van Handel-Youssef, 2018)

$$\mathbb{E}\| G_A \colon \ell_2^n \to \ell_2^m \| \asymp \max_i \|(a_{ij})_j\|_2 + \max_j \|(a_{ij})_i\|_2 + \mathbb{E} \max_{i,j} |a_{ij} g_{ij}|.$$

Theorem (Guédon-Hinrichs-Litvak-Prochno, 2017)

Assume that $1 \leq p \leq 2 \leq q \leq \infty$

$$\mathbb{E}\| G_A \colon \ell_p^n \to \ell_q^m \| \lesssim_{p,q} \Big[(\log m)^{1/q} \| A \circ A \colon \ell_{p/2}^n \to \ell_{q/2}^m \|^{1/2} + \max_j \|(a_{ij})_i\|_q + (\log m)^{1/q}\, \mathbb{E} \max_{i,j} |a_{ij} g_{ij}| \Big].$$

Theorem (Latała-van Handel-Youssef, 2018)

$$\mathbb{E}\|G_A\colon \ell_2^n \to \ell_2^m\| \asymp \max_i \|(a_{ij})_j\|_2 + \max_j \|(a_{ij})_i\|_2 + \mathbb{E}\max_{i,j}|a_{ij}g_{ij}|.$$

Theorem (Guédon-Hinrichs-Litvak-Prochno, 2017)

Assume that $1 \le p \le 2 \le q \le \infty$

$$\mathbb{E}\|G_A\colon \ell_p^n \to \ell_q^m\| \lesssim_{p,q} \Big[(\log m)^{1/q}\|A\circ A\colon \ell_{p/2}^n \to \ell_{q/2}^m\|^{1/2} + \|(A\circ A)^T\colon \ell_{q^*/2}^m \to \ell_{p^*/2}^n\|^{1/2} + (\log m)^{1/q}\,\mathbb{E}\max_{i,j}|a_{ij}g_{ij}|\Big].$$

Theorem (Guédon-Hinrichs-Litvak-Prochno, 2017)

Assume that $1 \le p \le 2 \le q \le \infty$

$$\mathbb{E}\|G_A\colon \ell_p^n \to \ell_q^m\| \lesssim_{p,q} \Big[\big(\log m\big)^{1/q} \|A \circ A\colon \ell_{p/2}^n \to \ell_{q/2}^m\|^{1/2} + \|(A \circ A)^T\colon \ell_{q^*/2}^m \to \ell_{p^*/2}^n\|^{1/2} + \big(\log m\big)^{1/q}\, \mathbb{E}\max_{i,j} |a_{ij} g_{ij}|\Big].$$

Second guess: Is it true, that for every $1 \le p, q \le \infty$ we have

$$\mathbb{E}\|G_A\colon \ell_p^n \to \ell_q^m\| \asymp_{p,q} \|A \circ A\colon \ell_{p/2}^n \to \ell_{q/2}^m\|^{1/2} + \|(A \circ A)^T\colon \ell_{q^*/2}^m \to \ell_{p^*/2}^n\|^{1/2} + \mathbb{E}\max_{i,j} |a_{ij} g_{ij}|?$$

Theorem (Guédon-Hinrichs-Litvak-Prochno, 2017)

Assume that $1 \le p \le 2 \le q \le \infty$

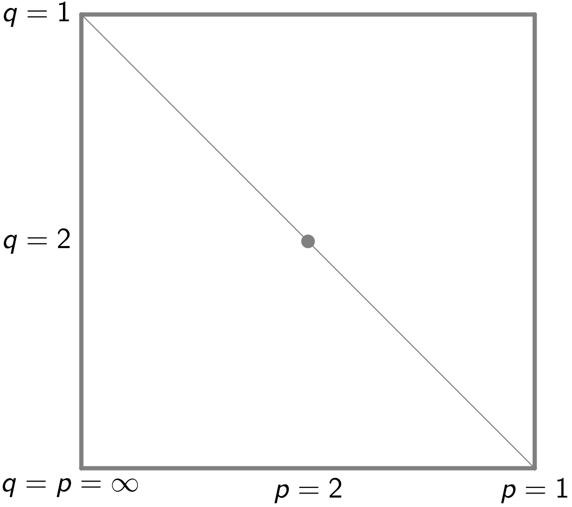
$$\mathbb{E}\|G_A\colon \ell_p^n \to \ell_q^m\| \lesssim_{p,q} \Big[\big(\log m\big)^{1/q} \|A \circ A\colon \ell_{p/2}^n \to \ell_{q/2}^m\|^{1/2} + \|(A \circ A)^T\colon \ell_{q^*/2}^m \to \ell_{p^*/2}^n\|^{1/2} + \big(\log m\big)^{1/q}\, \mathbb{E} \max_{i,j} |a_{ij} g_{ij}| \Big].$$

Second guess: Is it true, that for every $1 \le p, q \le \infty$ we have

$$\mathbb{E}\|G_A\colon \ell_p^n \to \ell_q^m\| \asymp_{p,q} \|A \circ A\colon \ell_{p/2}^n \to \ell_{q/2}^m\|^{1/2} + \|(A \circ A)^T\colon \ell_{q^*/2}^m \to \ell_{p^*/2}^n\|^{1/2} + \mathbb{E} \max_{i,j} |a_{ij} g_{ij}|?$$

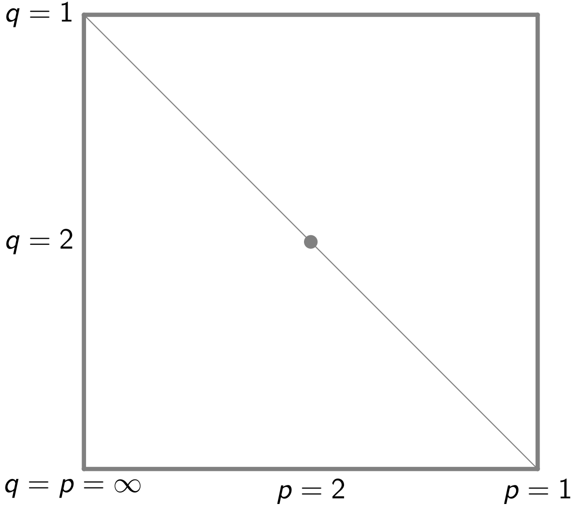
$$D_1 := \|A \circ A\colon \ell_{p/2}^n \to \ell_{q/2}^m\|^{1/2}, \quad D_2 := \|(A \circ A)^T\colon \ell_{q^*/2}^m \to \ell_{p^*/2}^n\|^{1/2}.$$

$$\mathbb{E}\| G_A \colon \ell_p^n \to \ell_q^m \| \asymp_{p,q} D_1 + D_2 + \ldots$$



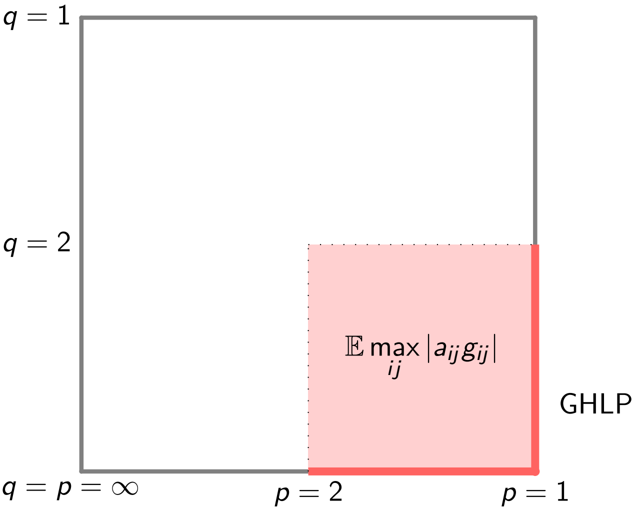
$$\| G_A \colon \ell_p^n \to \ell_q^m \| = \| (G_A)^T \colon \ell_{q^*}^m \to \ell_{p^*}^n \| = \| (G^T)_{A^T} \colon \ell_{q^*}^m \to \ell_{p^*}^n \|$$

$\mathbb{E}\| G_A \colon \ell_p^n \to \ell_q^m \| \asymp_{p,q} D_1 + D_2 + \ldots$

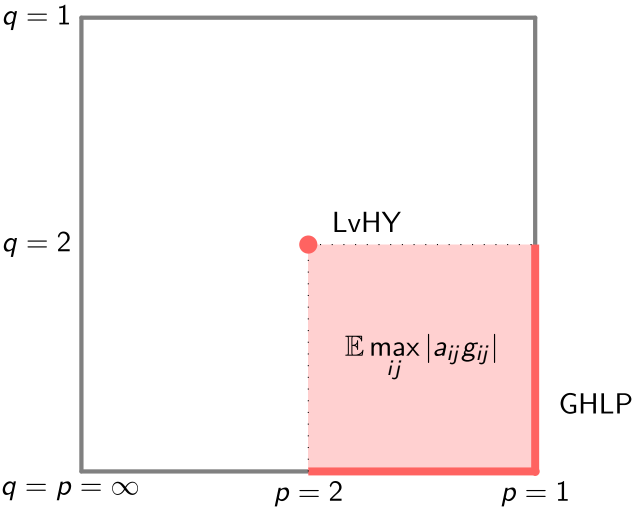


Duality: $(p, q) \longleftrightarrow (q^*, p^*)$.

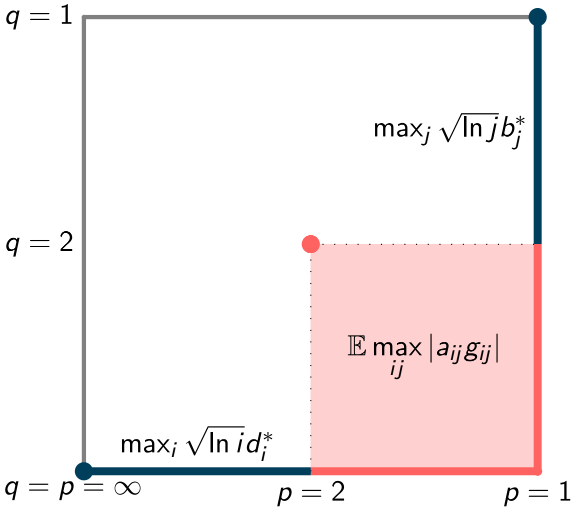
$$\mathbb{E}\| G_A \colon \ell_p^n \to \ell_q^m \| \asymp_{p,q} D_1 + D_2 + \dots$$



$$\mathbb{E}\|G_A\colon \ell_p^n \to \ell_q^m\| \asymp_{p,q} D_1 + D_2 + \ldots$$

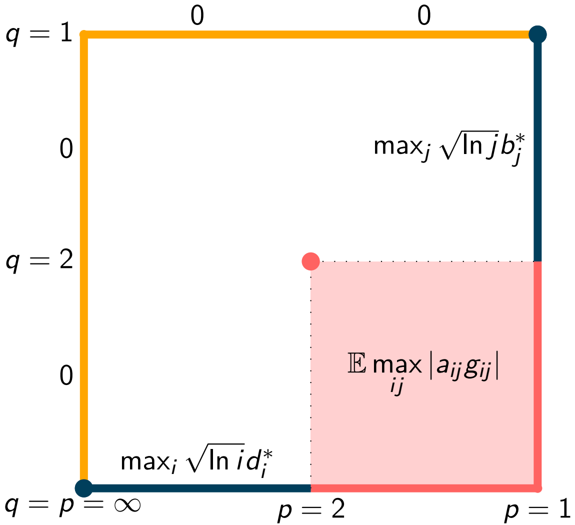


$$\mathbb{E}\|G_A\colon \ell_p^n \to \ell_q^m\| \asymp_{p,q} D_1 + D_2 + \dots$$



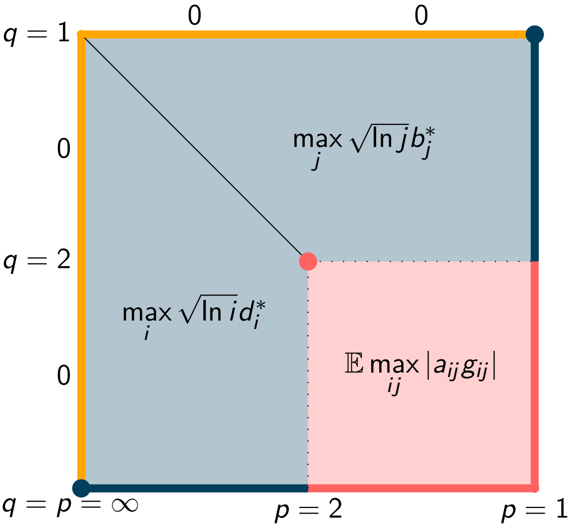
$$b_j := \|(a_{ij})_{i\le m}\|_{2q/(2-q)}, \quad d_i := \|(a_{ij})_{j\le n}\|_{2p/(p-2)}, \quad (b_j^*)_j, (d_i^*)_i \downarrow$$

$$\mathbb{E}\| G_A\colon \ell_p^n \to \ell_q^m \| \asymp_{p,q} D_1 + D_2 + \ldots$$



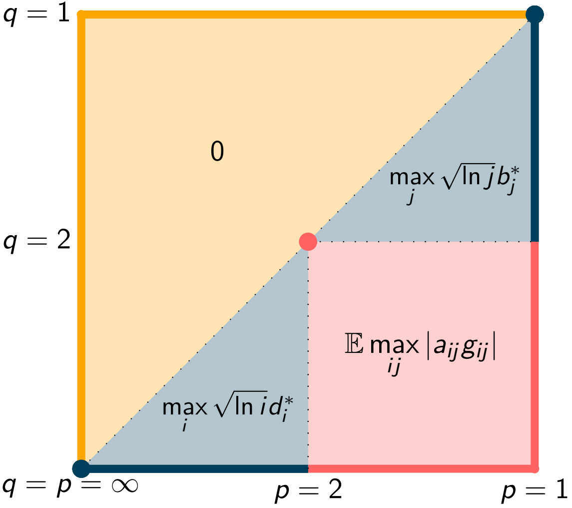
$$b_j := \|(a_{ij})_{i\le m}\|_{2q/(2-q)}, \quad d_i := \|(a_{ij})_{j\le n}\|_{2p/(p-2)}, \quad (b_j^*)_j, (d_i^*)_i \downarrow$$

$$\mathbb{E}\| G_A \colon \ell_p^n \to \ell_q^m \| \asymp_{p,q} D_1 + D_2 + \ldots$$



$$b_j := \|(a_{ij})_{i \le m}\|_{2q/(2-q)}, \quad d_i := \|(a_{ij})_{j \le n}\|_{2p/(p-2)}, \quad (b_j^*)_j, (d_i^*)_i \downarrow$$

$$\mathbb{E}\|G_A\colon \ell_p^n \to \ell_q^m\| \asymp_{p,q} D_1 + D_2 + \ldots$$



$$b_j := \|(a_{ij})_{i\le m}\|_{2q/(2-q)}, \quad d_i := \|(a_{ij})_{j\le n}\|_{2p/(p-2)}, \quad (b_j^*)_j, (d_i^*)_i \downarrow$$

Conjecture (APSS, 2021)

For all $1 \leq p, q \leq \infty$, *we conjecture that*

$$\mathbb{E}\|G_A\colon \ell_p^n \to \ell_q^m\| \asymp_{p,q}$$

$$D_1 + D_2 + \begin{cases} \mathbb{E} \max_{i \leq m, j \leq n} |a_{ij} g_{ij}| & \text{if } p \leq 2 \leq q, \\ \max_{j \leq n} \sqrt{\ln(j+1)} b_j^* & \text{if } p \leq q \leq 2, \\ \max_{i \leq m} \sqrt{\ln(i+1)} d_i^* & \text{if } 2 \leq p \leq q, \\ 0 & \text{if } q < p. \end{cases}$$

$D_1 = \|A \circ A\colon \ell_{p/2}^n \to \ell_{q/2}^m\|^{1/2}$,

$D_2 = \|(A \circ A)^T\colon \ell_{q^*/2}^m \to \ell_{p^*/2}^n\|^{1/2}$,

$b_j := \|(a_{ij})_{i \leq m}\|_{2q/(2-q)}, \quad d_i := \|(a_{ij})_{j \leq n}\|_{2p/(p-2)}$.

and $(x_k^*)_k$ is the non-increasing rearrangement of $(|x_k|)_k$.

First guess: is it true that for every $1 \le p, q \le \infty$,

$$\mathbb{E}\| G_A \colon \ell_p^n \to \ell_q^m \| \asymp_{p,q} \mathbb{E} \max_i \|(a_{ij} g_{ij})_j\|_{p^*} + \mathbb{E} \max_j \|(a_{ij} g_{ij})_i\|_q?$$

Why is this not true?

First guess: is it true that for every $1 \le p, q \le \infty$,

$$\mathbb{E}\|G_A \colon \ell_p^n \to \ell_q^m\| \asymp_{p,q} \mathbb{E}\max_i \|(a_{ij}g_{ij})_j\|_{p^*} + \mathbb{E}\max_j \|(a_{ij}g_{ij})_i\|_q ?$$

Why is this not true?

Example in the case $q < p$

$m = n$, $A = \mathrm{Id}$.

First guess: is it true that for every $1 \le p, q \le \infty$,

$$\mathbb{E}\| G_A \colon \ell_p^n \to \ell_q^m \| \asymp_{p,q} \mathbb{E} \max_i \|(a_{ij} g_{ij})_j\|_{p^*} + \mathbb{E} \max_j \|(a_{ij} g_{ij})_i\|_q ?$$

Why is this not true?

Example in the case $q < p$

$m = n$, $A = \mathrm{Id}$.
Then

$$\mathbb{E} \max_j \|(G_A)_i\|_q + \mathbb{E} \max_i \|(G_A)_j\|_{p^*} = 2\mathbb{E} \max_j |g_{jj}| \approx \sqrt{\ln n}.$$

First guess: is it true that for every $1 \le p, q \le \infty$,

$$\mathbb{E}\|G_A\colon \ell_p^n \to \ell_q^m\| \asymp_{p,q} \mathbb{E}\max_i \|(a_{ij}g_{ij})_j\|_{p^*} + \mathbb{E}\max_j \|(a_{ij}g_{ij})_i\|_q?$$

Why is this not true?

Example in the case $q < p$

$m = n$, $A = \mathrm{Id}$.
Then

$$\mathbb{E}\max_j \|(G_A)_i\|_q + \mathbb{E}\max_i \|(G_A)_j\|_{p^*} = 2\mathbb{E}\max_j |g_{jj}| \approx \sqrt{\ln n}.$$

On the other hand,

$$D_1 = \|\,\mathrm{Id}\colon \ell_{p/2}^n \to \ell_{q/2}^m\|^{1/2} = \sup_{x\in\ell_{p/2}} \Big(\sum_i |x_i|^{q/2}\Big)^{1/q}$$

$$= \Big(\sup_{y\in\ell_{p/q}} \sum_i |y_i|\Big)^{1/q} = \big(n^{1/(p/q)^*}\big)^{1/q} \gg \sqrt{\ln n}.$$

First guess: is it true that for every $1 \le p, q \le \infty$,

$$\mathbb{E}\|G_A\colon \ell_p^n \to \ell_q^m\| \asymp_{p,q} \mathbb{E}\max_i \|(a_{ij}g_{ij})_j\|_{p^*} + \mathbb{E}\max_j \|(a_{ij}g_{ij})_i\|_q \,?$$

Why is this not true??

Example for $2 < p \le q$

First guess: is it true that for every $1 \le p, q \le \infty$,

$$\mathbb{E}\|G_A \colon \ell_p^n \to \ell_q^m\| \asymp_{p,q} \mathbb{E}\max_i \|(a_{ij}g_{ij})_j\|_{p^*} + \mathbb{E}\max_j \|(a_{ij}g_{ij})_i\|_q\,?$$

Why is this not true??

Example for $2 < p \le q$

$p \le q < 2$ is the dual case

First guess: is it true that for every $1 \le p, q \le \infty$,

$$\mathbb{E}\|G_A\colon \ell_p^n \to \ell_q^m\| \asymp_{p,q} \mathbb{E}\max_i \|(a_{ij}g_{ij})_j\|_{p^*} + \mathbb{E}\max_j \|(a_{ij}g_{ij})_i\|_q\,?$$

Why is this not true??

Example for $2 < p \le q$

A matrix of all 1's,

$m, n \to \infty$ satisfy $m^{1/q} \gg n^{1/p^*}$.

First guess: is it true that for every $1 \le p, q \le \infty$,

$$\mathbb{E}\|G_A \colon \ell_p^n \to \ell_q^m\| \asymp_{p,q} \mathbb{E}\max_i \|(a_{ij}g_{ij})_j\|_{p^*} + \mathbb{E}\max_j \|(a_{ij}g_{ij})_i\|_q$$

$$\asymp_{p,q} \max_i \|(a_{ij})_j\|_{p^*} + \max_j \|(a_{ij})_i\|_q + D_3$$

Example for $2 < p \le q$

A matrix of all 1's,
$m, n \to \infty$ satisfy $m^{1/q} \gg n^{1/p^*}$.

First guess: is it true that for every $1 \leq p, q \leq \infty$,

$$\mathbb{E}\|G_A \colon \ell_p^n \to \ell_q^m\| \asymp_{p,q} \mathbb{E}\max_i \|(a_{ij}g_{ij})_j\|_{p^*} + \mathbb{E}\max_j \|(a_{ij}g_{ij})_i\|_q$$
$$\asymp_{p,q} \max_i \|(a_{ij})_j\|_{p^*} + \max_j \|(a_{ij})_i\|_q + D_3$$

Example for $2 < p \leq q$

A matrix of all 1's,
$m, n \to \infty$ satisfy $m^{1/q} \gg n^{1/p^*}$. Then

$$D_1 = \|(1)_{ij} \colon \ell_{p/2}^n \to \ell_{q/2}^m\|^{1/2} = \sup_{x \in \ell_{p/2}} \Big(\sum_i \Big|\sum_j x_j\Big|^{q/2}\Big)^{1/q}$$
$$= m^{1/q} \sup_{x \in \ell_{p/2}} \sqrt{\Big|\sum_j x_j\Big|} = m^{1/q}\sqrt{n^{1/(p/2)^*}} = m^{1/q} n^{\frac{1}{2(p/2)^*}}.$$

First guess: is it true that for every $1 \le p, q \le \infty$,

$$\mathbb{E}\|G_A \colon \ell_p^n \to \ell_q^m\| \asymp_{p,q} \mathbb{E} \max_i \|(a_{ij}g_{ij})_j\|_{p^*} + \mathbb{E} \max_j \|(a_{ij}g_{ij})_i\|_q$$
$$\asymp_{p,q} \max_i \|(a_{ij})_j\|_{p^*} + \max_j \|(a_{ij})_i\|_q + D_3$$

Example for $2 < p \le q$

A matrix of all 1's,
$m, n \to \infty$ satisfy $m^{1/q} \gg n^{1/p^*}$. Then

$$D_1 = \|(1)_{ij} \colon \ell_{p/2}^n \to \ell_{q/2}^m\|^{1/2} = \sup_{x \in \ell_{p/2}} \Big(\sum_i \Big|\sum_j x_j\Big|^{q/2}\Big)^{1/q}$$
$$= m^{1/q} \sup_{x \in \ell_{p/2}} \sqrt{\Big|\sum_j x_j\Big|} = m^{1/q}\sqrt{n^{1/(p/2)^*}} = m^{1/q} n^{\frac{1}{2(p/2)^*}}.$$

$$\max_i \|(a_{ij})_j\|_{p^*} + \max_j \|(a_{ij})_i\|_q = n^{1/p*} + m^{1/q}$$

First guess: is it true that for every $1 \le p, q \le \infty$,

$$\mathbb{E}\| G_A \colon \ell_p^n \to \ell_q^m \| \asymp_{p,q} \mathbb{E} \max_i \|(a_{ij} g_{ij})_j\|_{p^*} + \mathbb{E} \max_j \|(a_{ij} g_{ij})_i\|_q$$
$$\asymp_{p,q} \max_i \|(a_{ij})_j\|_{p^*} + \max_j \|(a_{ij})_i\|_q + D_3$$

Example for $2 < p \le q$

A matrix of all 1's,
$m, n \to \infty$ satisfy $m^{1/q} \gg n^{1/p^*}$. Then

$$D_1 = \|(1)_{ij} \colon \ell_{p/2}^n \to \ell_{q/2}^m\|^{1/2} = \sup_{x \in \ell_{p/2}} \Big(\sum_i \Big| \sum_j x_j \Big|^{q/2} \Big)^{1/q}$$
$$= m^{1/q} \sup_{x \in \ell_{p/2}} \sqrt{\Big| \sum_j x_j \Big|} = m^{1/q} \sqrt{n^{1/(p/2)^*}} = m^{1/q} n^{\frac{1}{2(p/2)^*}}.$$

$$\max_i \|(a_{ij})_j\|_{p^*} + \max_j \|(a_{ij})_i\|_q = n^{1/p*} + m^{1/q} \sim m^{1/q} \ll D_1.$$

First guess: is it true that for every $1 \le p, q \le \infty$,

$$\mathbb{E}\| G_A \colon \ell_p^n \to \ell_q^m \| \asymp_{p,q} \mathbb{E} \max_i \|(a_{ij} g_{ij})_j\|_{p^*} + \mathbb{E} \max_j \|(a_{ij} g_{ij})_i\|_q$$

$$\asymp_{p,q} \max_i \|(a_{ij})_j\|_{p^*} + \max_j \|(a_{ij})_i\|_q + D_3$$

Example for $2 < p \le q$

A matrix of all 1's,

$m, n \to \infty$ satisfy $m^{1/q} \gg n^{1/p^*}$. Then

$$D_1 = \|(1)_{ij} \colon \ell_{p/2}^n \to \ell_{q/2}^m\|^{1/2} = \sup_{x \in \ell_{p/2}} \Big(\sum_i \Big| \sum_j x_j \Big|^{q/2} \Big)^{1/q}$$

$$= m^{1/q} \sup_{x \in \ell_{p/2}} \sqrt{\Big| \sum_j x_j \Big|} = m^{1/q} \sqrt{n^{1/(p/2)^*}} = m^{1/q} n^{\frac{1}{2(p/2)^*}}.$$

$$\max_i \|(a_{ij})_j\|_{p^*} + \max_j \|(a_{ij})_i\|_q = n^{1/p*} + m^{1/q} \sim m^{1/q} \ll D_1.$$

$$D_3 = \max_i \sqrt{\ln i}\, d_i^* = \sqrt{lnm}\, n^{\frac{p-2}{2p}} = \sqrt{lnm}\, n^{\frac{1}{2(p/2)^*}} \ll D_1.$$

Theorem (APSS, 2021)

Assume that $1 \le p, q \le \infty$. *Then,*

$$\mathbb{E}\| G_A \colon \ell_p^n \to \ell_q^m \| \lesssim (\ln n)^{1/p^*} (\ln m)^{1/q} \Big[\sqrt{\ln(mn)} D_1 + \sqrt{\ln n} D_2 \Big].$$

$$D_1 = \|A \circ A \colon \ell_{p/2}^n \to \ell_{q/2}^m \|^{1/2},$$

$$D_2 = \|(A \circ A)^T \colon \ell_{q^*/2}^m \to \ell_{p^*/2}^n \|^{1/2}.$$

Theorem (APSS, 2021)

Assume that $1 \le p, q \le \infty$. *Then,*

$$\mathbb{E}\| G_A \colon \ell_p^n \to \ell_q^m \| \lesssim (\ln n)^{1/p^*} (\ln m)^{1/q} \Big[\sqrt{\ln(mn)} D_1 + \sqrt{\ln n} D_2 \Big].$$

Corollary

Assume that $K, L > 0$, $r \in (0, 2]$, $1 \le p, q \le \infty$, *and* $X = (X_{ij})_{i \le m, j \le n}$ *has independent mean-zero entries satisfying*

$$\mathbb{P}(|X_{ij}| \ge t) \le K e^{-t^r/L} \qquad \text{for all } t \ge 0, i \le m, j \le n.$$

Then

$$\mathbb{E}\| X_A \colon \ell_p^n \to \ell_q^m \| \lesssim_{r,K,L} (\ln n)^{1/p^*} (\ln m)^{1/q} \ln(mn)^{\frac{1}{r} - \frac{1}{2}} \Big[\sqrt{\ln(mn)} D_1 + \sqrt{\ln n} D_2 \Big].$$

Lemma

Assume that $1 \leq p \leq \infty$, $n \in \mathbb{N}$, *and define the convex set*

$$L := \operatorname{conv}\Big\{ \frac{1}{|J|^{1/p}} \big(\varepsilon_j \mathbf{1}_{\{j \in J\}}\big)_{j=1}^n : \\ J \subset \{1, \ldots, n\},\ J \neq \emptyset,\ (\varepsilon_j)_{j=1}^n \in \{-1, 1\}^n \Big\}.$$

Then $B_p^n \subset \ln(en)^{1/p^*} L$.

Lemma

Assume that $1 \le p \le \infty$, $n \in \mathbb{N}$, *and define the convex set*

$$L := \operatorname{conv}\Big\{ \frac{1}{|J|^{1/p}} \big(\varepsilon_j \mathbf{1}_{\{j\in J\}}\big)_{j=1}^n : \\ J \subset \{1,\dots,n\},\ J \neq \emptyset,\ (\varepsilon_j)_{j=1}^n \in \{-1,1\}^n \Big\}.$$

Then $B_p^n \subset \ln(en)^{1/p^*} L$.

Proof of the Theorem.

$$\| G_A \colon \ell_p^n \to \ell_q^m \|$$

Lemma

Assume that $1 \leq p \leq \infty$, $n \in \mathbb{N}$, *and define the convex set*

$$L := \operatorname{conv}\Big\{ \frac{1}{|J|^{1/p}} \big(\varepsilon_j \mathbf{1}_{\{j \in J\}}\big)_{j=1}^n : \\ J \subset \{1, \dots, n\},\ J \neq \emptyset,\ (\varepsilon_j)_{j=1}^n \in \{-1, 1\}^n \Big\}.$$

Then $B_p^n \subset \ln(en)^{1/p^*} L$.

Proof of the Theorem.

$$\| G_A \colon \ell_p^n \to \ell_q^m \| = \sup_{x \in B_p^n} \sup_{y \in B_{q^*}^m} \sum_{i=1}^m \sum_{j=1}^n y_i a_{ij} g_{ij} x_j$$

Lemma

Assume that $1 \leq p \leq \infty$, $n \in \mathbb{N}$, *and define the convex set*

$$L := \operatorname{conv}\Big\{ \frac{1}{|J|^{1/p}} \big(\varepsilon_j \mathbf{1}_{\{j \in J\}}\big)_{j=1}^n : \\ J \subset \{1, \dots, n\},\ J \neq \emptyset,\ (\varepsilon_j)_{j=1}^n \in \{-1, 1\}^n \Big\}.$$

Then $B_p^n \subset \ln(en)^{1/p^*} L$.

Proof of the Theorem.

$$\| G_A \colon \ell_p^n \to \ell_q^m \| = \sup_{x \in B_p^n} \sup_{y \in B_{q^*}^m} \sum_{i=1}^m \sum_{j=1}^n y_i a_{ij} g_{ij} x_j$$

$$\leq \ln(en)^{1/p^*} \ln(em)^{1/q} \sup_{x \in \operatorname{Ext} L} \sup_{y \in \operatorname{Ext} K} \sum_{i=1}^m \sum_{j=1}^n y_i a_{ij} g_{ij} x_j$$

Lemma

Assume that $1 \le p \le \infty$, $n \in \mathbb{N}$, *and define the convex set*

$$L := \operatorname{conv}\Big\{ \frac{1}{|J|^{1/p}} \big(\varepsilon_j \mathbf{1}_{\{j\in J\}}\big)_{j=1}^n : \\ J \subset \{1,\dots,n\},\ J \neq \emptyset,\ (\varepsilon_j)_{j=1}^n \in \{-1,1\}^n \Big\}.$$

Then $B_p^n \subset \ln(en)^{1/p^*} L$.

Proof of the Theorem.

$$\|G_A\colon \ell_p^n \to \ell_q^m\| = \sup_{x\in B_p^n} \sup_{y \in B_{q^*}^m} \sum_{i=1}^m \sum_{j=1}^n y_i a_{ij} g_{ij} x_j$$

$$\le \ln(en)^{1/p^*} \ln(em)^{1/q} \sup_{x\in \operatorname{Ext} L} \sup_{y\in \operatorname{Ext} K} \sum_{i=1}^m \sum_{j=1}^n y_i a_{ij} g_{ij} x_j$$

$$= \text{logs} \max_{\substack{k\le m \\ l \le n}} \frac{1}{k^{1/q^*} l^{1/p}} \max_{\substack{I\subset[m] \\ |I|=k}} \max_{\substack{J\subset[n] \\ |J|=l}} \sup_{x\in B_\infty^n} \sup_{y\in B_\infty^m} \sum_{i\in I} \sum_{j\in J} y_i a_{ij} g_{ij} x_j$$

Lemma

Assume that $1 \le p \le \infty$, $n \in \mathbb{N}$, *and define the convex set*

$$L := \operatorname{conv}\Big\{ \frac{1}{|J|^{1/p}} \big(\varepsilon_j \mathbf{1}_{\{j\in J\}}\big)_{j=1}^n : \\ J \subset \{1,\dots,n\},\ J \neq \emptyset,\ (\varepsilon_j)_{j=1}^n \in \{-1,1\}^n \Big\}.$$

Then $B_p^n \subset \ln(en)^{1/p^*} L$.

Proof of the Theorem.

$$\|G_A\colon \ell_p^n \to \ell_q^m\| = \sup_{x\in B_p^n} \sup_{y\in B_{q^*}^m} \sum_{i=1}^m \sum_{j=1}^n y_i a_{ij} g_{ij} x_j$$

$$\le \ln(en)^{1/p^*} \ln(em)^{1/q} \sup_{x\in \operatorname{Ext} L} \sup_{y\in \operatorname{Ext} K} \sum_{i=1}^m \sum_{j=1}^n y_i a_{ij} g_{ij} x_j$$

$$= \text{logs} \max_{\substack{k\le m \\ l\le n}} \frac{1}{k^{1/q^*} l^{1/p}} \max_{\substack{I\subset[m] \\ |I|=k}} \max_{\substack{J\subset[n] \\ |J|=l}} \sup_{x\in B_\infty^n} \sup_{y\in B_\infty^m} \sum_{i\in I} \sum_{j\in J} y_i a_{ij} g_{ij} x_j$$

Let $(\widetilde{g}_{ij})_{i\le m, j\le n}$ be an independent copy of $(g_{ij})_{i\le m, j\le n}$.

Let $(\widetilde{g}_{ij})_{i\le m, j\le n}$ be an independent copy of $(g_{ij})_{i\le m, j\le n}$.

$$\mathbb{E} \sup_{I,J} \sup_{x\in B_\infty^n} \sup_{y\in B_\infty^m} \sum_{i\in I, j\in J} y_i a_{ij} g_{ij} x_j$$

Let $(\widetilde{g}_{ij})_{i\le m, j\le n}$ be an independent copy of $(g_{ij})_{i\le m, j\le n}$.

$$\mathbb{E} \sup_{I,J} \sup_{x\in B_\infty^n} \sup_{y\in B_\infty^m} \sum_{i\in I, j\in J} y_i a_{ij} g_{ij} x_j = \mathbb{E} \sup_{I,J} \sup_{x\in B_\infty^n} \sum_{i\in I} \Big| \sum_{j\in J} a_{ij} g_{ij} x_j \Big|$$

Let $(\widetilde{g}_{ij})_{i\le m, j\le n}$ be an independent copy of $(g_{ij})_{i\le m, j\le n}$.

$$\begin{aligned}
\mathbb{E}\sup_{I,J}\sup_{x\in B_\infty^n}\sup_{y\in B_\infty^m}\sum_{i\in I, j\in J} y_i a_{ij} g_{ij} x_j &= \mathbb{E}\sup_{I,J}\sup_{x\in B_\infty^n}\sum_{i\in I}\Big|\sum_{j\in J} a_{ij} g_{ij} x_j\Big| \\
&\le \mathbb{E}\sup_{I,J}\sup_{x\in B_\infty^n}\sum_{i\in I}\Big(\Big|\sum_{j\in J} a_{ij} g_{ij} x_j\Big| - \mathbb{E}\Big|\sum_{j\in J} a_{ij}\widetilde{g}_{ij} x_j\Big|\Big) \\
&\quad + \sup_{I,J}\sup_{x\in B_\infty^n}\sum_{i\in I}\mathbb{E}\Big|\sum_{j\in J} a_{ij}\widetilde{g}_{ij} x_j\Big|
\end{aligned}$$

Let $(\widetilde{g}_{ij})_{i\le m, j\le n}$ be an independent copy of $(g_{ij})_{i\le m, j\le n}$.

$$\mathbb{E} \sup_{I,J} \sup_{x\in B_\infty^n} \sup_{y\in B_\infty^m} \sum_{i\in I, j\in J} y_i a_{ij} g_{ij} x_j = \mathbb{E} \sup_{I,J} \sup_{x\in B_\infty^n} \sum_{i\in I} \Big| \sum_{j\in J} a_{ij} g_{ij} x_j \Big|$$

$$\le \mathbb{E} \sup_{I,J} \sup_{x\in B_\infty^n} \sum_{i\in I} \Big(\Big| \sum_{j\in J} a_{ij} g_{ij} x_j \Big| - \mathbb{E} \Big| \sum_{j\in J} a_{ij} \widetilde{g}_{ij} x_j \Big| \Big)$$

$$+ \sup_{I,J} \sup_{x\in B_\infty^n} \sum_{i\in I} \mathbb{E} \Big| \sum_{j\in J} a_{ij} \widetilde{g}_{ij} x_j \Big|$$

Let $(\widetilde{g}_{ij})_{i\le m, j\le n}$ be an independent copy of $(g_{ij})_{i\le m, j\le n}$.

$$\begin{aligned}\mathbb{E} \sup_{I,J} \sup_{x\in B_\infty^n} \sup_{y\in B_\infty^m} \sum_{i\in I, j\in J} y_i a_{ij} g_{ij} x_j &= \mathbb{E} \sup_{I,J} \sup_{x\in B_\infty^n} \sum_{i\in I} \Big| \sum_{j\in J} a_{ij} g_{ij} x_j \Big| \\ &\le \mathbb{E} \sup_{I,J} \sup_{x\in B_\infty^n} \sum_{i\in I} \Big(\Big| \sum_{j\in J} a_{ij} g_{ij} x_j \Big| - \Big| \sum_{j\in J} a_{ij} \widetilde{g}_{ij} x_j \Big| \Big) \\ &\quad + \sup_{I,J} \sup_{x\in B_\infty^n} \sum_{i\in I} \mathbb{E} \Big| \sum_{j\in J} a_{ij} \widetilde{g}_{ij} x_j \Big|\end{aligned}$$

Let $(\widetilde{g}_{ij})_{i\le m, j\le n}$ be an independent copy of $(g_{ij})_{i\le m, j\le n}$.

$$
\begin{aligned}
\mathbb{E} \sup_{I,J} \sup_{x\in B_\infty^n} \sup_{y\in B_\infty^m} \sum_{i\in I, j\in J} y_i a_{ij} g_{ij} x_j &= \mathbb{E} \sup_{I,J} \sup_{x\in B_\infty^n} \sum_{i\in I} \Big| \sum_{j\in J} a_{ij} g_{ij} x_j \Big| \\
&\le \mathbb{E} \sup_{I,J} \sup_{x\in B_\infty^n} \sum_{i\in I} \Big(\Big| \sum_{j\in J} a_{ij} g_{ij} x_j \Big| - \Big| \sum_{j\in J} a_{ij} \widetilde{g}_{ij} x_j \Big| \Big) \\
&\quad + \sup_{I,J} \sup_{x\in B_\infty^n} \sum_{i\in I} \mathbb{E} \Big| \sum_{j\in J} a_{ij} \widetilde{g}_{ij} x_j \Big|
\end{aligned}
$$

Let $(\widetilde{g}_{ij})_{i\le m, j\le n}$ be an independent copy of $(g_{ij})_{i\le m, j\le n}$.

$$\begin{aligned}
\mathbb{E} \sup_{I,J} \sup_{x\in B_\infty^n} \sup_{y\in B_\infty^m} \sum_{i\in I, j\in J} y_i a_{ij} g_{ij} x_j &= \mathbb{E} \sup_{I,J} \sup_{x\in B_\infty^n} \sum_{i\in I} \Big| \sum_{j\in J} a_{ij} g_{ij} x_j \Big| \\
&\le \mathbb{E} \sup_{I,J} \sup_{x\in B_\infty^n} \sum_{i\in I} \Big(\Big| \sum_{j\in J} a_{ij} g_{ij} x_j \Big| - \Big| \sum_{j\in J} a_{ij} \widetilde{g}_{ij} x_j \Big| \Big) \\
&\quad + \sup_{I,J} \sup_{x\in B_\infty^n} \sum_{i\in I} \mathbb{E} \Big| \sum_{j\in J} a_{ij} \widetilde{g}_{ij} x_j \Big| \\
&= \mathbb{E} \sup_{I,J} \sup_{x\in B_\infty^n} \sum_{i\in I} \Big(\Big| \sum_{j\in J} a_{ij} g_{ij} x_j \Big| - \Big| \sum_{j\in J} a_{ij} \widetilde{g}_{ij} x_j \Big| \Big) \\
&\quad + \sup_{I,J} \sup_{x\in B_\infty^n} \sum_{i\in I} \sqrt{\sum_{j\in J} a_{ij}^2 x_j^2}\, \mathbb{E}|g|
\end{aligned}$$

Let $(\widetilde{g}_{ij})_{i\le m, j\le n}$ be an independent copy of $(g_{ij})_{i\le m, j\le n}$.

$$
\begin{aligned}
\mathbb{E}\sup_{I,J}\sup_{x\in B_\infty^n}\sup_{y\in B_\infty^m}\sum_{i\in I, j\in J} y_i a_{ij} g_{ij} x_j &= \mathbb{E}\sup_{I,J}\sup_{x\in B_\infty^n}\sum_{i\in I}\Big|\sum_{j\in J} a_{ij} g_{ij} x_j\Big| \\
&\le \mathbb{E}\sup_{I,J}\sup_{x\in B_\infty^n}\sum_{i\in I}\Big(\Big|\sum_{j\in J} a_{ij} g_{ij} x_j\Big| - \Big|\sum_{j\in J} a_{ij}\widetilde{g}_{ij} x_j\Big|\Big) \\
&\quad + \sup_{I,J}\sup_{x\in B_\infty^n}\sum_{i\in I}\mathbb{E}\Big|\sum_{j\in J} a_{ij}\widetilde{g}_{ij} x_j\Big| \\
&= \mathbb{E}\sup_{I,J}\sup_{x\in B_\infty^n}\sum_{i\in I}\Big(\Big|\sum_{j\in J} a_{ij} g_{ij} x_j\Big| - \Big|\sum_{j\in J} a_{ij}\widetilde{g}_{ij} x_j\Big|\Big) \\
&\quad + \sup_{I,J}\sup_{x\in B_\infty^n}\sum_{i\in I}\sqrt{\sum_{j\in J} a_{ij}^2 x_j^2}\,\mathbb{E}|g| \\
&= \mathbb{E}\sup_{I,J}\sup_{x\in B_\infty^n}\sum_{i\in I}\Big(\Big|\sum_{j\in J} a_{ij} g_{ij} x_j\Big| - \Big|\sum_{j\in J} a_{ij}\widetilde{g}_{ij} x_j\Big|\Big) + \sqrt{\frac{2}{\pi}}\sup_{I,J}\sum_{i\in I}\sqrt{\sum_{j\in J} a_{ij}^2}.
\end{aligned}
$$

Let $(\widetilde{g}_{ij})_{i\le m, j\le n}$ be an independent copy of $(g_{ij})_{i\le m, j\le n}$.

$$\begin{aligned}
\mathbb{E}\sup_{I,J}\sup_{x\in B_\infty^n}\sup_{y\in B_\infty^m}\sum_{i\in I, j\in J} y_i a_{ij} g_{ij} x_j &= \mathbb{E}\sup_{I,J}\sup_{x\in B_\infty^n}\sum_{i\in I}\Big|\sum_{j\in J} a_{ij} g_{ij} x_j\Big| \\
&\le \mathbb{E}\sup_{I,J}\sup_{x\in B_\infty^n}\sum_{i\in I}\Big(\Big|\sum_{j\in J} a_{ij} g_{ij} x_j\Big| - \Big|\sum_{j\in J} a_{ij}\widetilde{g}_{ij} x_j\Big|\Big) \\
&\quad + \sup_{I,J}\sup_{x\in B_\infty^n}\sum_{i\in I}\mathbb{E}\Big|\sum_{j\in J} a_{ij}\widetilde{g}_{ij} x_j\Big| \\
&= \mathbb{E}\sup_{I,J}\sup_{x\in B_\infty^n}\sum_{i\in I}\Big(\Big|\sum_{j\in J} a_{ij} g_{ij} x_j\Big| - \Big|\sum_{j\in J} a_{ij}\widetilde{g}_{ij} x_j\Big|\Big) \\
&\quad + \sup_{I,J}\sup_{x\in B_\infty^n}\sum_{i\in I}\sqrt{\sum_{j\in J} a_{ij}^2 x_j^2}\,\mathbb{E}|g| \\
&= \mathbb{E}\sup_{I,J}\sup_{x\in B_\infty^n}\sum_{i\in I}\Big(\Big|\sum_{j\in J} a_{ij} g_{ij} x_j\Big| - \Big|\sum_{j\in J} a_{ij}\widetilde{g}_{ij} x_j\Big|\Big) + \sqrt{\frac{2}{\pi}}\sup_{I,J}\sum_{i\in I}\sqrt{\sum_{j\in J} a_{ij}^2}.
\end{aligned}$$

$\left(\left|\sum_{j\in J} a_{ij}g_{ij}x_j\right| - \left|\sum_{j\in J} a_{ij}\widetilde{g}_{ij}x_j\right|\right)$ are indep. and symmetric, so

$$\mathbb{E}\sup_{I,J}\sup_{x\in B_\infty^n}\sum_{i\in I}\left(\left|\sum_{j\in J} a_{ij}g_{ij}x_j\right| - \left|\sum_{j\in J} a_{ij}\widetilde{g}_{ij}x_j\right|\right)$$

$$\leq 2\mathbb{E}\sup_{I,J}\sup_{x\in B_\infty^n}\sum_{i=1}^{m}\varepsilon_i\left|\sum_{j\in J} a_{ij}g_{ij}x_j\mathbf{1}_{\{i\in I\}}\right|$$

$\left(\left|\sum_{j\in J} a_{ij}g_{ij}x_j\right| - \left|\sum_{j\in J} a_{ij}\widetilde{g}_{ij}x_j\right|\right)$ are indep. and symmetric, so

$$\mathbb{E}\sup_{I,J}\sup_{x\in B_\infty^n}\sum_{i\in I}\left(\left|\sum_{j\in J} a_{ij}g_{ij}x_j\right| - \left|\sum_{j\in J} a_{ij}\widetilde{g}_{ij}x_j\right|\right)$$
$$\leq 2\mathbb{E}\sup_{I,J}\sup_{x\in B_\infty^n}\sum_{i=1}^{m}\varepsilon_i\left|\sum_{j\in J} a_{ij}g_{ij}x_j\mathbf{1}_{\{i\in I\}}\right|$$

Lemma (Contraction principle)

Let $T\subset\mathbb{R}^m$, $\varphi_i:\mathbb{R}\to\mathbb{R}$ *are* 1*-Lipschitz and* $\varphi_i(0)=0$*. Then*

$$\mathbb{E}\sup_{t\in T}\sum_{i=1}^{m}\varepsilon_i\varphi_i(t_i)\leq\mathbb{E}\sup_{t\in T}\sum_{i=1}^{m}\varepsilon_i t_i.$$

$\left(\left|\sum_{j\in J} a_{ij}g_{ij}x_j\right| - \left|\sum_{j\in J} a_{ij}\widetilde{g}_{ij}x_j\right|\right)$ are indep. and symmetric, so

$$\mathbb{E}\sup_{I,J}\sup_{x\in B_\infty^n}\sum_{i\in I}\left(\left|\sum_{j\in J} a_{ij}g_{ij}x_j\right| - \left|\sum_{j\in J} a_{ij}\widetilde{g}_{ij}x_j\right|\right)$$

$$\le 2\mathbb{E}\sup_{I,J}\sup_{x\in B_\infty^n}\sum_{i=1}^m \varepsilon_i\left|\sum_{j\in J} a_{ij}g_{ij}x_j\mathbf{1}_{\{i\in I\}}\right|$$

Lemma (Contraction principle)

Let $T\subset\mathbb{R}^m$, $\varphi_i:\mathbb{R}\to\mathbb{R}$ *are* 1*-Lipschitz and* $\varphi_i(0)=0$. *Then*

$$\mathbb{E}\sup_{t\in T}\sum_{i=1}^m \varepsilon_i\varphi_i(t_i) \le \mathbb{E}\sup_{t\in T}\sum_{i=1}^m \varepsilon_i t_i.$$

We use it with $\varphi_i(t) = |t|$.

$\left(\left|\sum_{j\in J} a_{ij}g_{ij}x_j\right| - \left|\sum_{j\in J} a_{ij}\widetilde{g}_{ij}x_j\right|\right)$ are indep. and symmetric, so

$$\mathbb{E}\sup_{I,J}\sup_{x\in B_\infty^n}\sum_{i\in I}\left(\left|\sum_{j\in J} a_{ij}g_{ij}x_j\right| - \left|\sum_{j\in J} a_{ij}\widetilde{g}_{ij}x_j\right|\right)$$

$$\leq 2\mathbb{E}\sup_{I,J}\sup_{x\in B_\infty^n}\sum_{i=1}^{m}\varepsilon_i\left|\sum_{j\in J} a_{ij}g_{ij}x_j\mathbf{1}_{\{i\in I\}}\right|$$

$$\leq 2\mathbb{E}\sup_{I,J}\sup_{x\in B_\infty^n}\sum_{i=1}^{m}\varepsilon_i\sum_{j\in J} a_{ij}g_{ij}x_j\mathbf{1}_{\{i\in I\}}$$

Lemma (Contraction principle)

Let $T \subset \mathbb{R}^m$, $\varphi_i : \mathbb{R} \to \mathbb{R}$ are 1-Lipschitz and $\varphi_i(0) = 0$. Then

$$\mathbb{E}\sup_{t\in T}\sum_{i=1}^{m}\varepsilon_i\varphi_i(t_i) \leq \mathbb{E}\sup_{t\in T}\sum_{i=1}^{m}\varepsilon_i t_i.$$

We use it with $\varphi_i(t) = |t|$.

$\left(\left|\sum_{j\in J} a_{ij}g_{ij}x_j\right| - \left|\sum_{j\in J} a_{ij}\widetilde{g}_{ij}x_j\right|\right)$ are indep. and symmetric, so

$$\mathbb{E}\sup_{I,J}\sup_{x\in B_\infty^n}\sum_{i\in I}\left(\left|\sum_{j\in J} a_{ij}g_{ij}x_j\right| - \left|\sum_{j\in J} a_{ij}\widetilde{g}_{ij}x_j\right|\right)$$

$$\leq 2\mathbb{E}\sup_{I,J}\sup_{x\in B_\infty^n}\sum_{i=1}^{m}\varepsilon_i\left|\sum_{j\in J} a_{ij}g_{ij}x_j\mathbf{1}_{\{i\in I\}}\right|$$

$$\leq 2\mathbb{E}\sup_{I,J}\sup_{x\in B_\infty^n}\sum_{i=1}^{m}\varepsilon_i\sum_{j\in J} a_{ij}g_{ij}x_j\mathbf{1}_{\{i\in I\}} = 2\mathbb{E}\sup_{I,J}\sup_{x\in B_\infty^n}\sum_{j\in J}\sum_{i\in I} a_{ij}g_{ij}x_j.$$

Lemma (Contraction principle)

Let $T\subset\mathbb{R}^m$, $\varphi_i:\mathbb{R}\to\mathbb{R}$ *are* 1*-Lipschitz and* $\varphi_i(0)=0$*. Then*

$$\mathbb{E}\sup_{t\in T}\sum_{i=1}^{m}\varepsilon_i\varphi_i(t_i) \leq \mathbb{E}\sup_{t\in T}\sum_{i=1}^{m}\varepsilon_i t_i.$$

We use it with $\varphi_i(t) = |t|$.

We have just proven that

$$\mathbb{E} \sup_{I,J} \sup_{x \in B_\infty^n} \sup_{y \in B_\infty^m} \sum_{i \in I, j \in J} y_i a_{ij} g_{ij} x_j$$

$$\leq \sqrt{\frac{2}{\pi}} \sup_{I,J} \sum_{i \in I} \sqrt{\sum_{j \in J} a_{ij}^2} + 2\mathbb{E} \sup_{I,J} \sup_{x \in B_\infty^n} \sum_{j \in J} \sum_{i \in I} a_{ij} g_{ij} x_j .$$

We have just proven that

$$\mathbb{E} \sup_{I,J} \sup_{x\in B_\infty^n} \sup_{y\in B_\infty^m} \sum_{i\in I, j\in J} y_i a_{ij} g_{ij} x_j$$

$$\leq \sqrt{\frac{2}{\pi}} \sup_{I,J} \sum_{i\in I} \sqrt{\sum_{j\in J} a_{ij}^2} + 2\mathbb{E} \sup_{I,J} \sup_{x\in B_\infty^n} \sum_{j\in J} \sum_{i\in I} a_{ij} g_{ij} x_j .$$

Proceeding similarly we may prove that

$$\mathbb{E} \sup_{I,J} \sup_{x\in B_\infty^n} \sum_{j\in J} \sum_{i\in I} a_{ij} g_{ij} x_j$$

$$\leq \sqrt{\frac{2}{\pi}} \sup_{I,J} \sum_{j\in J} \sqrt{\sum_{i\in I} a_{ij}^2} + 2\mathbb{E} \sup_{I,J} \sum_{j\in J} \sum_{i\in I} a_{ij} g_{ij} .$$

We have just proven that

$$\mathbb{E} \sup_{I,J} \sup_{x \in B_\infty^n} \sup_{y \in B_\infty^m} \sum_{i \in I, j \in J} y_i a_{ij} g_{ij} x_j \leq \sqrt{\frac{2}{\pi}} \sup_{I,J} \sum_{i \in I} \sqrt{\sum_{j \in J} a_{ij}^2} + 2\mathbb{E} \sup_{I,J} \sup_{x \in B_\infty^n} \sum_{j \in J} \sum_{i \in I} a_{ij} g_{ij} x_j .$$

Proceeding similarly we may prove that

$$\mathbb{E} \sup_{I,J} \sup_{x \in B_\infty^n} \sum_{j \in J} \sum_{i \in I} a_{ij} g_{ij} x_j \leq \sqrt{\frac{2}{\pi}} \sup_{I,J} \sum_{j \in J} \sqrt{\sum_{i \in I} a_{ij}^2} + 2\mathbb{E} \sup_{I,J} \sum_{j \in J} \sum_{i \in I} a_{ij} g_{ij} .$$

We have just proven that

$$\mathbb{E} \sup_{I,J} \sup_{x \in B_\infty^n} \sup_{y \in B_\infty^m} \sum_{i \in I, j \in J} y_i a_{ij} g_{ij} x_j \leq \sqrt{\frac{2}{\pi}} \sup_{I,J} \sum_{i \in I} \sqrt{\sum_{j \in J} a_{ij}^2} + 2\mathbb{E} \sup_{I,J} \sup_{x \in B_\infty^n} \sum_{j \in J} \sum_{i \in I} a_{ij} g_{ij} x_j .$$

Proceeding similarly we may prove that

$$\mathbb{E} \sup_{I,J} \sup_{x \in B_\infty^n} \sum_{j \in J} \sum_{i \in I} a_{ij} g_{ij} x_j \leq \sqrt{\frac{2}{\pi}} \sup_{I,J} \sum_{j \in J} \sqrt{\sum_{i \in I} a_{ij}^2} + 2\mathbb{E} \sup_{I,J} \underbrace{\sum_{j \in J} \sum_{i \in I} a_{ij} g_{ij}}_{X_{I,J}} .$$

Lemma (Slepian's lemma)

Let $(X_t)_{t\in T}$ and $(Y_t)_{t\in T}$ be two Gaussian random vectors satisfying $\mathbb{E}[X_t] = \mathbb{E}[Y_t]$ for all $t \in T$. Assume that, for all $s, t \in T$, we have $\mathbb{E}[(X_s - X_t)^2] \le \mathbb{E}[(Y_s - Y_t)^2]$. Then

$$\mathbb{E} \sup_{t\in T} X_t \le \mathbb{E} \sup_{t\in T} Y_t.$$

Lemma (Slepian's lemma)

Let $(X_t)_{t\in T}$ and $(Y_t)_{t\in T}$ be two Gaussian random vectors satisfying $\mathbb{E}[X_t] = \mathbb{E}[Y_t]$ for all $t \in T$. Assume that, for all $s, t \in T$, we have $\mathbb{E}[(X_s - X_t)^2] \leq \mathbb{E}[(Y_s - Y_t)^2]$. Then

$$\mathbb{E} \sup_{t\in T} X_t \leq \mathbb{E} \sup_{t\in T} Y_t.$$

$$Y_{I,J} = \sum_{i\in I} g_i \sqrt{\sum_{j\in J} a_{ij}^2} + \sum_{j\in J} \widetilde{g}_j \sqrt{\sum_{i\in I} a_{ij}^2}$$

$$(g_i)_i \sim \mathcal{N}(0, \mathrm{Id}_m), \qquad (\widetilde{g}_j)_j \sim \mathcal{N}(0, \mathrm{Id}_n), \qquad g \perp \widetilde{g}.$$

Lemma (Slepian's lemma)

Let $(X_t)_{t\in T}$ and $(Y_t)_{t\in T}$ be two Gaussian random vectors satisfying $\mathbb{E}[X_t] = \mathbb{E}[Y_t]$ for all $t \in T$. Assume that, for all $s, t \in T$, we have $\mathbb{E}[(X_s - X_t)^2] \le \mathbb{E}[(Y_s - Y_t)^2]$. Then

$$\mathbb{E} \sup_{t\in T} X_t \le \mathbb{E} \sup_{t\in T} Y_t.$$

$$Y_{I,J} = \sum_{i\in I} g_i \sqrt{\sum_{j\in J} a_{ij}^2} + \sum_{j\in J} \widetilde{g}_j \sqrt{\sum_{i\in I} a_{ij}^2}$$

$$(g_i)_i \sim \mathcal{N}(0, \mathrm{Id}_m), \qquad (\widetilde{g}_j)_j \sim \mathcal{N}(0, \mathrm{Id}_n), \qquad g \perp \widetilde{g}.$$

$$\mathbb{E} \sup_{I,J} Y_{I,J} \le \mathbb{E} \max_{i\le m} |g_i| \sup_{I,J} \sum_{i\in I} \sqrt{\sum_{j\in J} a_{ij}^2} + \mathbb{E} \max_{j\le n} |\widetilde{g}_j| \sup_{I,J} \sum_{j\in J} \sqrt{\sum_{i\in I} a_{ij}^2}$$

Lemma (Slepian's lemma)

Let $(X_t)_{t\in T}$ and $(Y_t)_{t\in T}$ be two Gaussian random vectors satisfying $\mathbb{E}[X_t] = \mathbb{E}[Y_t]$ for all $t \in T$. Assume that, for all $s, t \in T$, we have $\mathbb{E}[(X_s - X_t)^2] \le \mathbb{E}[(Y_s - Y_t)^2]$. Then

$$\mathbb{E} \sup_{t\in T} X_t \le \mathbb{E} \sup_{t\in T} Y_t.$$

$$Y_{I,J} = \sum_{i\in I} g_i \sqrt{\sum_{j\in J} a_{ij}^2} + \sum_{j\in J} \widetilde{g}_j \sqrt{\sum_{i\in I} a_{ij}^2}$$

$$\begin{aligned}\mathbb{E} \sup_{I,J} Y_{I,J} &\le \mathbb{E} \max_{i\le m} |g_i| \sup_{I,J} \sum_{i\in I} \sqrt{\sum_{j\in J} a_{ij}^2} + \mathbb{E} \max_{j\le n} |\widetilde{g}_j| \sup_{I,J} \sum_{j\in J} \sqrt{\sum_{i\in I} a_{ij}^2} \\ &\lesssim \sqrt{\ln m} \sup_{I,J} \sum_{i\in I} \sqrt{\sum_{j\in J} a_{ij}^2} + \sqrt{\ln n} \sup_{I,J} \sum_{j\in J} \sqrt{\sum_{i\in I} a_{ij}^2}.\end{aligned}$$

Lemma (Slepian's lemma)

Let $(X_t)_{t\in T}$ and $(Y_t)_{t\in T}$ be two Gaussian random vectors satisfying $\mathbb{E}[X_t] = \mathbb{E}[Y_t]$ for all $t \in T$. Assume that, for all $s, t \in T$, we have $\mathbb{E}[(X_s - X_t)^2] \le \mathbb{E}[(Y_s - Y_t)^2]$. Then

$$\mathbb{E}\sup_{t\in T} X_t \le \mathbb{E}\sup_{t\in T} Y_t.$$

$$Y_{I,J} = \sum_{i\in I} g_i \sqrt{\sum_{j\in J} a_{ij}^2} + \sum_{j\in J} \widetilde{g}_j \sqrt{\sum_{i\in I} a_{ij}^2}$$

$$\begin{aligned}\mathbb{E}\sup_{I,J} Y_{I,J} &\le \mathbb{E}\max_{i\le m}|g_i| \sup_{I,J}\sum_{i\in I}\sqrt{\sum_{j\in J} a_{ij}^2} + \mathbb{E}\max_{j\le n}|\widetilde{g}_j| \sup_{I,J}\sum_{j\in J}\sqrt{\sum_{i\in I} a_{ij}^2} \\ &\lesssim \sqrt{\ln m}\sup_{I,J}\sum_{i\in I}\sqrt{\sum_{j\in J} a_{ij}^2} + \sqrt{\ln n}\sup_{I,J}\sum_{j\in J}\sqrt{\sum_{i\in I} a_{ij}^2}.\end{aligned}$$

Lemma (Slepian's lemma)

Let $(X_t)_{t\in T}$ and $(Y_t)_{t\in T}$ be two Gaussian random vectors satisfying $\mathbb{E}[X_t]=\mathbb{E}[Y_t]$ for all $t\in T$. Assume that, for all $s,t\in T$, we have $\mathbb{E}[(X_s-X_t)^2]\le\mathbb{E}[(Y_s-Y_t)^2]$. Then

$$\mathbb{E}\sup_{t\in T} X_t \le \mathbb{E}\sup_{t\in T} Y_t.$$

$$Y_{I,J}=\sum_{i\in I} g_i\sqrt{\sum_{j\in J} a_{ij}^2}+\sum_{j\in J}\widetilde{g}_j\sqrt{\sum_{i\in I} a_{ij}^2}$$

$$\begin{aligned}\mathbb{E}\sup_{I,J} Y_{I,J} &\le \mathbb{E}\max_{i\le m}|g_i|\sup_{I,J}\sum_{i\in I}\sqrt{\sum_{j\in J} a_{ij}^2}+\mathbb{E}\max_{j\le n}|\widetilde{g}_j|\sup_{I,J}\sum_{j\in J}\sqrt{\sum_{i\in I} a_{ij}^2}\\ &\lesssim \sqrt{\ln m}\sup_{I,J}\sum_{i\in I}\sqrt{\sum_{j\in J} a_{ij}^2}+\sqrt{\ln n}\sup_{I,J}\sum_{j\in J}\sqrt{\sum_{i\in I} a_{ij}^2}.\end{aligned}$$

$X_{I,J} = \sum_{i \in I} \sum_{j \in J} a_{ij} g_{ij}$.

$$\mathbb{E}(X_{I,J} - X_{\widetilde{I},\widetilde{J}})^2 = \mathbb{E}X_{I,J}^2 + \mathbb{E}X_{\widetilde{I},\widetilde{J}}^2 - 2\mathbb{E}X_{I,J}X_{\widetilde{I},\widetilde{J}}$$
$$= \sum_{I,J} a_{ij}^2 + \sum_{\widetilde{I},\widetilde{J}} a_{ij}^2 - 2 \sum_{I \cap \widetilde{I}, J \cap \widetilde{J}} a_{ij}^2$$

$X_{I,J} = \sum_{i\in I} \sum_{j\in J} a_{ij} g_{ij}.$

$$\begin{aligned}\mathbb{E}(X_{I,J} - X_{\widetilde{I},\widetilde{J}})^2 &= \mathbb{E}X_{I,J}^2 + \mathbb{E}X_{\widetilde{I},\widetilde{J}}^2 - 2\mathbb{E}X_{I,J}X_{\widetilde{I},\widetilde{J}} \\ &= \sum_{I,J} a_{ij}^2 + \sum_{\widetilde{I},\widetilde{J}} a_{ij}^2 - 2\sum_{I\cap\widetilde{I},J\cap\widetilde{J}} a_{ij}^2\end{aligned}$$

$$= \sum_{I,J} a_{ij}^2 + \sum_{\widetilde{I},\widetilde{J}} a_{ij}^2 - \sum_{I\cap\widetilde{I},J} a_{ij}^2 - \sum_{I\cap\widetilde{I},\widetilde{J}} a_{ij}^2 + \sum_{I\cap\widetilde{I},J\setminus\widetilde{J}} a_{ij}^2 + \sum_{I\cap\widetilde{I},\widetilde{J}\setminus J} a_{ij}^2.$$

$$Y_{I,J} = \sum_{i\in I} g_i \sqrt{\sum_{j\in J} a_{ij}^2} + \sum_{j\in J} \widetilde{g}_j \sqrt{\sum_{i\in I} a_{ij}^2}.$$

$$\mathbb{E}(Y_{I,J} - Y_{\widetilde{I},\widetilde{J}})^2 = \mathbb{E}Y_{I,J}^2 + \mathbb{E}Y_{\widetilde{I},\widetilde{J}}^2 - 2\mathbb{E}Y_{I,J}Y_{\widetilde{I},\widetilde{J}}$$

$$Y_{I,J} = \sum_{i \in I} g_i \sqrt{\sum_{j \in J} a_{ij}^2} + \sum_{j \in J} \widetilde{g}_j \sqrt{\sum_{i \in I} a_{ij}^2}.$$

$$\begin{aligned}
\mathbb{E}(Y_{I,J} - Y_{\widetilde{I},\widetilde{J}})^2 &= \mathbb{E} Y_{I,J}^2 + \mathbb{E} Y_{\widetilde{I},\widetilde{J}}^2 - 2\mathbb{E} Y_{I,J} Y_{\widetilde{I},\widetilde{J}} \\
&= 2 \sum_{I,J} a_{ij}^2 + 2 \sum_{\widetilde{I},\widetilde{J}} a_{ij}^2 \\
&\quad - 2 \sum_{I \cap \widetilde{I}} \sqrt{\sum_{J} a_{ij}^2} \sqrt{\sum_{\widetilde{J}} a_{ij}^2} - 2 \sum_{J \cap \widetilde{J}} \sqrt{\sum_{I} a_{ij}^2} \sqrt{\sum_{\widetilde{I}} a_{ij}^2}
\end{aligned}$$

$$Y_{I,J} = \sum_{i\in I} g_i \sqrt{\sum_{j\in J} a_{ij}^2} + \sum_{j\in J} \widetilde{g}_j \sqrt{\sum_{i\in I} a_{ij}^2}.$$

$$\begin{aligned}
\mathbb{E}(Y_{I,J} - Y_{\widetilde{I},\widetilde{J}})^2 &= \mathbb{E}Y_{I,J}^2 + \mathbb{E}Y_{\widetilde{I},\widetilde{J}}^2 - 2\mathbb{E}Y_{I,J}Y_{\widetilde{I},\widetilde{J}} \\
&= 2\sum_{I,J} a_{ij}^2 + 2\sum_{\widetilde{I},\widetilde{J}} a_{ij}^2 \\
&\quad - 2\sum_{I\cap\widetilde{I}} \sqrt{\sum_{J} a_{ij}^2}\sqrt{\sum_{\widetilde{J}} a_{ij}^2} - 2\sum_{J\cap\widetilde{J}} \sqrt{\sum_{I} a_{ij}^2}\sqrt{\sum_{\widetilde{I}} a_{ij}^2} \\
&\overset{2\sqrt{ab}\le a+b}{\ge} 2\sum_{I,J} a_{ij}^2 + 2\sum_{\widetilde{I},\widetilde{J}} a_{ij}^2 - \sum_{I\cap\widetilde{I},J} a_{ij}^2 - \sum_{I\cap\widetilde{I},\widetilde{J}} a_{ij}^2 - \sum_{I,J\cap\widetilde{J}} a_{ij}^2 - \sum_{\widetilde{I},J\cap\widetilde{J}} a_{ij}^2
\end{aligned}$$

$$Y_{I,J} = \sum_{i\in I} g_i \sqrt{\sum_{j\in J} a_{ij}^2} + \sum_{j\in J} \widetilde{g}_j \sqrt{\sum_{i\in I} a_{ij}^2}.$$

$$\begin{aligned}
\mathbb{E}(Y_{I,J} - Y_{\widetilde{I},\widetilde{J}})^2 &= \mathbb{E}Y_{I,J}^2 + \mathbb{E}Y_{\widetilde{I},\widetilde{J}}^2 - 2\mathbb{E}Y_{I,J}Y_{\widetilde{I},\widetilde{J}} \\
&= 2\sum_{I,J} a_{ij}^2 + 2\sum_{\widetilde{I},\widetilde{J}} a_{ij}^2 \\
&\quad - 2\sum_{I\cap\widetilde{I}} \sqrt{\sum_{J} a_{ij}^2}\sqrt{\sum_{\widetilde{J}} a_{ij}^2} - 2\sum_{J\cap\widetilde{J}} \sqrt{\sum_{I} a_{ij}^2}\sqrt{\sum_{\widetilde{I}} a_{ij}^2} \\
&\overset{2\sqrt{ab}\le a+b}{\geq} 2\sum_{I,J} a_{ij}^2 + 2\sum_{\widetilde{I},\widetilde{J}} a_{ij}^2 - \sum_{I\cap\widetilde{I},J} a_{ij}^2 - \sum_{I\cap\widetilde{I},\widetilde{J}} a_{ij}^2 - \sum_{I,J\cap\widetilde{J}} a_{ij}^2 - \sum_{\widetilde{I},J\cap\widetilde{J}} a_{ij}^2 \\
&= \sum_{I,J} a_{ij}^2 + \sum_{\widetilde{I},\widetilde{J}} a_{ij}^2 - \sum_{I\cap\widetilde{I},J} a_{ij}^2 - \sum_{I\cap\widetilde{I},\widetilde{J}} a_{ij}^2 + \sum_{I,J\setminus\widetilde{J}} a_{ij}^2 + \sum_{\widetilde{I},\widetilde{J}\setminus J} a_{ij}^2.
\end{aligned}$$

$$\mathbb{E}(Y_{I,J} - Y_{\widetilde{I},\widetilde{J}})^2 = \mathbb{E}Y_{I,J}^2 + \mathbb{E}Y_{\widetilde{I},\widetilde{J}}^2 - 2\mathbb{E}Y_{I,J}Y_{\widetilde{I},\widetilde{J}}$$

$$\geq \sum_{I,J} a_{ij}^2 + \sum_{\widetilde{I},\widetilde{J}} a_{ij}^2 - \sum_{I\cap\widetilde{I},J} a_{ij}^2 - \sum_{I\cap\widetilde{I},\widetilde{J}} a_{ij}^2 + \sum_{I,J\setminus\widetilde{J}} a_{ij}^2 + \sum_{\widetilde{I},\widetilde{J}\setminus J} a_{ij}^2.$$

$$\mathbb{E}(X_{I,J} - X_{\widetilde{I},\widetilde{J}})^2 =$$

$$\sum_{I,J} a_{ij}^2 + \sum_{\widetilde{I},\widetilde{J}} a_{ij}^2 - \sum_{I\cap\widetilde{I},J} a_{ij}^2 - \sum_{I\cap\widetilde{I},\widetilde{J}} a_{ij}^2 + \sum_{I\cap\widetilde{I},J\setminus\widetilde{J}} a_{ij}^2 + \sum_{I\cap\widetilde{I},\widetilde{J}\setminus J} a_{ij}^2.$$