A STORY OF ACTION REPRESENTATIVITY

George Janelidze, University of Cape Town

Defining an action of a monoid M on a set X, we can either say that it is a map $h : M \times X \to X$ with $h(1, x) = x$ and $h(m, h(m', x)) = h(mm', x)$ for all $x \in X$ and $m, m' \in M$, or that it is a monoid homomorphism $f : M \to X^X$. In other words, there is a natural in M bijection

$$\mathrm{Act}(M, X) \approx \mathrm{Hom}(M, X^X)$$

and we say that the functor

$$\mathrm{Act}(-, X) : \mathsf{Mon} \to \mathsf{Set}$$

is representable. This *action representability* property has many well-known and less-well-known generalizations and counterparts, some of which are described in [3] and in [2]. The context of the 'widest' generalization described in [2] has:

- a monoidal category $\mathcal{C}$, a category $\mathcal{X}$, and a 'hyper action' ($\otimes : \mathcal{C} \times \mathcal{X} \to \mathcal{X}, \ldots$) of $\mathcal{C}$ on $\mathcal{X}$ (making $\mathcal{X}$ a "$\mathcal{C}$-actegory" in the terminology of Paddy McCrudden [4]) instead of just the category of sets;
- the role of actions $M \times X \to X$ played by morphisms $M \otimes X \to X$ satisfying suitable conditions, in which $M \in \mathsf{Mon}(\mathcal{C})$ is a monoid in $\mathcal{C}$ and X an object in $\mathcal{X}$;
- whether the functor $\mathrm{Act}(-, X) : \mathsf{Mon}(\mathcal{C}) \to \mathsf{Set}$ is representable or not becomes an open question, highly non-trivial in some special cases.

The purpose of this talk is a brief description of joint work (in progress) with Ross Street, where we go 'one dimension up' in order to develop a new context where the above-mentioned "hyper action" itself becomes 'an action'. In particular, it will be explained that in that special case, which goes back Jean Bénabou's Subsection (2.3) of [1], in fact we have a well-known action representability, which uses monoidal functors.

References

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4. P. McCrudden, Categories of Representations of Coalgebroids, Advances in Mathematics 154, 2000, 299-332