How to be concrete when you don't have a choice

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The classical homotopy category is famously not a concrete category, but of course, it is the quotient of the concrete category of topological spaces by a congruence on its hom-sets. While not every locally small category is concretisable, it was proved by Kučera that every locally small category C is likewise a quotient of a concrete category—under the assumption of the axiom of Global Choice. Indeed, the proof starts by assuming that the objects of C are literally ordinals, because the class of objects can itself be well-ordered.

Dissatistfaction with this strategy leads us to reprove Kučera's theorem in the setting of Algebraic Set Theory, with the only extra assumption that the category of classes is boolean. Weaker analogues are possible when removing assumptions like smallness of powersets.

The general strategy can also be applied to Freyd's theorem characterising concrete categories as those satisfying Isbell's criterion, which was also proved using Global Choice. In this instance, we have a proof that works in both ZF and choiceless NBG, or more generally in material set theories with a cumulative hierarchy with well-ordered.