Profunctors and group cohomology

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One of the major contribution of J. Benabou is the notion of profunctor or, more precisely, of the composition of profunctors giving rise to a bicategory Prof. This composition, although technically rather difficult (especially for internal profunctors), is an extraordinary synthesis tool. We shall exemplify this observation by an application to group cohomology.

It is well known that any exact sequence of groups with abelian kernel :

$$1 \rightarrowtail A \rightarrowtail X \twoheadrightarrow Y \twoheadrightarrow 1$$

gives rise to an action ψ of the group Y on the group A, and that the set $Ext_\psi(Y, A)$ of isomorphic classes of extensions between Y and A producing the same group action ψ is endowed, via the Baer sum, with an abelian group structure.

The associated split epimorphism $A \rtimes_\psi Y \rightleftarrows Y$ is actually underlying a abelian group structure in the slice category Gp/Y or, in other words, an internal groupoid structure A_ψ in Gp. The set $Ext_\psi(Y, A)$ is nothing but the "internal Hom" $\underline{\mathsf{Prof}}(A_\psi, A_\psi)$ in the sub-bigroupoid determined by a special class of internal profunctors in Gp, namely the fully faithful internal profunctors. The group structure on $Ext_\psi(Y, A) = \underline{\mathsf{Prof}}(A_\psi, A_\psi)$ is then induced by the profunctor composition. This could seem rather anecdotical. The extension of this kind of description to any exact sequence of groups :

$$1 \rightarrowtail K \rightarrowtail X \twoheadrightarrow Y \twoheadrightarrow 1$$

shows that it is not the case. Eilenberg and Mac Lane observed that, any extension fo this kind produces a group homomorphism $\phi : Y \to AutK/InK$, called the *abstract kernel* of the sequence. Then the set $Ext_\phi(Y, K)$ of isomorphic classes of extensions between Y and K with abstract kernel ϕ is endowed with a simply transitive action of the abelian group $Ext_\psi(Y, ZK)$, where ZK is the center of K and the action ψ is induced by ϕ.

Now, from the abstract kernel ϕ it is possible to construct in a natural way an internal groupoid $D_\phi Y$ in Gp, such that $Ext_\phi(Y, K)$ appears as nothing but the "internal Hom" $\underline{\mathsf{Prof}}(D_\phi Y, ZK_\psi)$ in the same sub-bigroupoid as above. So, the simply transitive action can just be understood as the natural effect of the profunctor composition of the endo-group $\underline{\mathsf{Prof}}(ZK_\psi, ZK_\psi)$ on the "internal Hom" $\underline{\mathsf{Prof}}(D_\phi Y, ZK_\psi)$.

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