Lecture 6, Wednesday Afternoon

$v_1, v_2, \dots, v_n \in H$ $\qquad S = \varepsilon_1 v_1 + \cdots + \varepsilon_n v_n$, $\varepsilon_i = \pm 1$ i.i.d. symmetric

$\exists\, c > 0$ universal

$$P\left(|S|_H \geq \sqrt{|v_1|_H^2 + \cdots + |v_n|_H^2}\right) \geq (c) > 0$$

$$P\left(|S|_H \leq \sqrt{|v_1|_H^2 + \cdots + |v_n|_H^2}\right) \geq c > 0.$$

pf: $Z := |S|_H^2 - \sum_{i=1}^n |v_i|^2 = \left\langle \sum_{i=1}^n \varepsilon_i v_i, \sum_{j=1}^n \varepsilon_j v_j \right\rangle$

$$- \sum |v_i|_H^2$$

$$= 2 \sum_{1 \leq i < j \leq n} \varepsilon_i \varepsilon_j \langle v_i, v_j \rangle. \Rightarrow \text{chaos of degree } 2$$

So $\mathbb{E}[Z^4] \leq 9^d (\mathbb{E}[Z^2]^2)$

and

$$\{|S|_H \geq \sqrt{|v_1|_H^2 + \cdots + |v_n|_H^2}\} = \{Z \geq 0\}$$

$\mathbb{E} Z = 0 \Rightarrow \mathbb{E} Z^4 \leq C (\mathbb{E} Z^2)^2$

$\Rightarrow \mathbb{E} Z^2 \leq K(C)\, \mathbb{E}|Z| \Rightarrow$ $P(Z \geq 0) > c_{K(C)} > 0$, $P(Z \leq 0) \geq c_{K(C)} > 0.$

Alt pf: $\mathbb{E}Z = 0 \Rightarrow \frac{1}{2}\mathbb{E}|Z| = \mathbb{E}Z\mathbb{1}_{Z\ge 0}$

Hölder

$\le (\mathbb{E}Z^4)^{1/4} \cdot (\mathbb{E}\mathbb{1}_{Z\ge 0}^{4/3})^{3/4} = P(Z\ge 0)^{3/4}(\mathbb{E}Z^4)^{1/4}$

* $\mathbb{E}[Z^4]\cdot(\mathbb{E}|Z|)^2 \ge (\mathbb{E}Z^2)^3$ Hölder boosting

chaos

$\left(\frac{\sqrt{\mathbb{E}Z^4}}{9}\right)^3 = \frac{(\mathbb{E}[Z^4])^{3/2}}{9^3}$

$\Rightarrow \sqrt{\mathbb{E}(Z^4)} \le 9^3(\mathbb{E}|Z|)^2$

$\Rightarrow \sqrt[4]{\mathbb{E}(Z^4)} \le 27\cdot(\mathbb{E}|Z|)$

may not be optimal

$\frac{1}{2}\mathbb{E}|Z| \le P(Z\ge 0)^{3/4}\cdot 27\,\mathbb{E}|Z|$

So $P(Z\ge 0) \ge \left(\frac{1}{54}\right)^{4/3}$, $P(Z\le 0) \ge (1/54)^{4/3}$

Conclusion: in a Hilbert space, $\sqrt{\sum |v_i|^2}$ is a good measure of size of Rademachers.

Lemma:
Assume that $Y_1, Y_2, \dots, Y_m$ are ind., sym RV. Then $\mathbb{E}\left(\sum_{i=1}^m Y_i^2\right)^{1/2} \le C_{2,1}\mathbb{E}\left|\sum_{i=1}^m Y_i\right|$, where $C_{2,1}$ is the constant from 2-1 Khintchine ≠ ($C_{2,1} = \sqrt{2}$ by Szarek)

pf: $(Y_1,\dots,Y_n) \sim (\varepsilon_1 Y_1, \dots, \varepsilon_n Y_n)$, where $Y_1,\dots,Y_n,\varepsilon_1,\dots,\varepsilon_n$ are ind., $\varepsilon_i = \pm 1$ w/ prob 1/2.

$\mathbb{E}\left|\sum_{i=1}^m Y_i\right| = \mathbb{E}\left|\sum_{i=1}^m \varepsilon_i Y_i\right| = \mathbb{E}_Y\mathbb{E}_\varepsilon\left|\sum_{i=1}^n \varepsilon_i Y_i\right|$

$\ge \frac{1}{C_{2,1}}\mathbb{E}_Y\sqrt{\sum_{i=1}^m Y_i^2} = \frac{1}{C_{2,1}}\mathbb{E}\sqrt{\sum_{i=1}^m Y_i^2}$ ☐

Thm (Marcinkiewicz, Zygmund) Assume that $Y_1,\dots,Y_m$ are ind., sym, and $\mathbb{E}Y_i^2 \le \rho(\mathbb{E}|Y_i|)^2$ $\forall i \le m$. Let $Z = \sum_{i=1}^m Y_i$. Then

$$\mathbb{E}Z^2 \le C_{2,1}^2\,\rho\,(\mathbb{E}|Z|)^2.$$

+ Hölder boosting

• Sometimes given stronger assumption about controlling 4th moment by 2nd moment squared, easier.

↳ Moral: can transfer bounds on each variable to bounds of Rademacher sum w/ small loss in constant.

pf: $C_{2,1}\mathbb{E}|Z| \ge \mathbb{E}\sqrt{\sum_{i=1}^m Y_i^2} = \mathbb{E}\|(|Y_1|,\dots,|Y_d|)\|_{\ell_2^m}$

* $\forall v, w \in \mathbb{R}^m \Rightarrow \langle v, w\rangle \le |v|\cdot|w|$ (Schwartz)

$\ge \mathbb{E}\frac{(\mathbb{E}|Y_1|\cdot Y_1 + \cdots + \mathbb{E}|Y_m|)}{\|(\mathbb{E}|Y_1|, \dots, \mathbb{E}|Y_m|)\|}$

linearity

$= \frac{(\mathbb{E}|Y_1|)^2 + \cdots + (\mathbb{E}|Y_m|)^2}{\sqrt{(\mathbb{E}|Y_1|)^2 + \cdots + (\mathbb{E}|Y_m|)^2}}$

$= \sqrt{(\mathbb{E}|Y_1|)^2 + \cdots + (\mathbb{E}|Y_m|)^2}$

assumption on Y_i: $\ge \frac{1}{\sqrt{\rho}}\sqrt{(\mathbb{E}Y_1)^2 + \cdots + (\mathbb{E}Y_m)^2} = \frac{1}{\sqrt{\rho}}\sqrt{\mathbb{E}Z^2}$ ☐

~ What is currently known?

$p > q > 1$: dominated in a hyper-contractibility sense

Def: Let $(F, \|\cdot\|)$ be a normed vector space; let μ, ν be Borel probability measures on F, then $\mu \prec_{(p,q)} \nu$ if $\forall v \in F$,

$$\left(\int \|v + w\|^p \, d\mu(w)\right)^{1/p} \le \left(\int \|v+w\|^q \, d\nu(w)\right)^{1/q}$$

can induct to get n-convolutions!

Thm: $\mu_1, \mu_2, \nu_1, \nu_2$ probability measures on $(F, \|\cdot\|)$, $\mu_1 \prec_{(p,q)} \mu_2$, $\nu_1 \prec_{(p,q)} \nu_2$. Then

$$\mu_1 * \mu_2 \prec_{(p,q)} \nu_1 * \nu_2$$

pf: $X_1 \sim \mu_1$, $X_2 \sim \mu_2$, $Y_1 \sim \nu_1$, $Y_2 \sim \nu_2$ and

$\boxed{\text{WTS}}$ $\forall v \in F$, $\|v + X_1 + X_2\|_p \le \|v + Y_1 + Y_2\|_q$

We know: $\forall v \in F$, $\|v + X_1\|_p \le \|v + Y_1\|_q$,

$\|v + X_2\|_p \le \|v + Y_2\|_q$

by domination assumption.

pf: $\left(\mathbb{E}\|v+X_1+X_2\|^p\right)^{1/p} = \left(\mathbb{E}_{X_1}\mathbb{E}_{X_2}\|(v+X_1)+X_2\|^p\right)^{1/p}$

$= \left(\mathbb{E}_{X_1}\|(v+X_1)+X_2\|^p_{p,X_2}\right)^{1/p}$ ← inside, $(v+X_1)$ is fixed vector so can use assumption

assumption $\leq \left(\mathbb{E}_{X_1}\|(v+X_1)+Y_2\|^p_{q,Y_2}\right)^{1/p}$

$= \left(\mathbb{E}_{X_1}\|(v+Y_2)+X_1\|^p_{q,Y_2}\right)^{1/p}$

$= \| \|(v+Y_2)+X_1\|_{q,Y_2}\|_{p,X_1}$

greedy-hasty $\leq \| \|(v+Y_2)+X_1\|_{p,X_1}\|_{q,Y_2}$

assumption $\leq \| \|(v+Y_2)+Y_1\|_{q,Y_1}\|_{q,Y_2}$

both q $= \|v+(Y_1+Y_2)\|_q$ $\boxtimes$

Def: let $0<\sigma<1$, $p>q>1$, $(\mathbb{F},\|\cdot\|)$; Then a random F-valued vector X is (p,q,σ)-hypercontractive if

$$\mu_{\sigma X} \prec_{(p,q)} \mu_X.$$

↳ linear contraction of X, but can bound higher moment by lower moment - similar to Khintchine's ≠?

In particular taking $v=0$ in def of hyper-cont. Then $\sigma\|X\|_p \leq \|X\|_q$.
↳ Extra v is important for pf (induction).

* In general a demanding assumption, though for X Gaussian a lot is known.

Def: A real random variable X is (p,q,σ)-hypercontractive if $\forall v,w \in F$,

$$\|v+\sigma Xw\|_p \leq \|v+Xw\|_q.$$

Thm 1 (Kwapień, Szulga): TFAE; for $p>1$:
1) $X \in \mathcal{HC}(p,q,\sigma,\mathbb{R})$ for some $\sigma\in(0,1)$ for all $q\in(1,p)$.
(1a) ‖‖ some, b/c higher power implied by lower.
2) For all $q\in(0,p)$, $\exists C>0$ s.t. $\forall$ sequence in $\mathbb{R}$,
(2a) ‖‖ some
$a_1,a_2,\dots,a_n$,

$$\left(\mathbb{E}\left[\left|\sum_{i=1}^n a_iX_i\right|^p\right]\right)^{1/p} \leq C\left(\mathbb{E}\left|\sum_{i=1}^n a_iX_i\right|^q\right)^{1/q}$$

where $X_1,\dots,X_n$ i.i.d. copies of X.

3) $\forall s>0\ \exists K>0$ s.t. $\forall\ 0\leq t\leq s$,

min

$$\mathbb{E}\left(|tX|^p-1\right)_+ \leq K\,\mathbb{E}\left(|tX|^2\wedge 1\right)$$

↳ Gives equivalence btwn universal moment comparison and hyper-contractibility.

all ⟺ some by Hölder boosting

Thm 2: TFAE: for $p>1$
1) $X\in\mathcal{HC}(p,q,\sigma,F)$ for some $\sigma\in(0,1)$, some (all) $q\in(1,p)$ and for every normed linear space F. σ may depend on q.
↳ σ works for any F.
⟺?
2) For all (some) $q\in(0,p)$, there is a $C>0$ s.t. $\forall(F,\|\cdot\|)$, $n\in\mathbb{N}$ $\forall v_1,\dots,v_n\in F$,

$$\left(\mathbb{E}\left\|\sum X_iv_i\right\|^p\right)^{1/p} \leq C_q\left(\mathbb{E}\left\|\sum_{i=1}^n X_iv_i\right\|^q\right)^{1/q}.$$

3) $\forall s>0\ \exists k>0\ \forall 0\leq t\leq s$, $\mathbb{E}(|tX|^p-1)_+ \leq k\mathbb{P}(|tX|>1)$

* Condition 3 in Thms 1 & 2 are easy to verify given an explicit RV.

For $p>2$, X real RV;

$$X\in\mathcal{HC}(p,q,\mathbb{R}) \iff \mathbb{E}|X|^p<\infty \text{ and } \mathbb{E}X=0.$$

for some σ
↓ not necessarily optimal

"Hopeless" open problem: given explicit Banach spaces, what RVs are hyper-contractible?
↳ Can be done in nice cases

Known That you cannot deduce hypercontractibility from having hypercontractibility on all 2d-subspaces.
↳ Possibly with stronger assumptions in space, may work.

X (p,q)-hypercontractive σ in F $\Rightarrow$ $X - X'$ (p,q)-hyp in F.

$\overset{?}{\Leftarrow}$ Known for $F = \mathbb{R}$, or if it happens for any F.
But in fixed Banach space, not known if symmetrization of RV hypercontractible, Then is The original RV?

A few simple observations: isometric embedding, norm restriction

- X (p,q,σ,F_1)-hyp, $F_2 \subseteq F_1$, Then

X is (p,q,σ,F_2)-hyp. Thus,

$\exists X \in (F, \|\cdot\|)$ (p,q,σ,F)-hyp $\Rightarrow$ X is $(p,q,\sigma,\mathbb{R})$-hyp.
$F \neq \{0\}$

↳ Any non-trivial Banach space has isometric copy of $\mathbb{R}$ embedded (scale any vector).

(?)
- Holds for σ $\Rightarrow$ holds for $\tilde{\sigma} \in (0,\sigma)$. $\sigma \mapsto \mathbb{E}\|v+\sigma X w\|^p$ convexity argument.

- X is $(p,q,\sigma,\mathbb{R})$-hyp $\overset{X \neq \text{const}}{\Rightarrow}$ $\mathbb{E}X = 0$.
↳ if X constant, then $(p,q,1,F)$-hyp.

- X is $(p,q,\sigma,\mathbb{R})$-hyp and $\mathbb{E}X^2 < \infty$, Then

$$\sigma \leq \sqrt{\frac{q-1}{p-1}}$$

← simple exercise: Taylor exp, use def of hyp on real line.

• Rademacher and $N(0,1)$ are $\left(p, q, \sqrt{\frac{q-1}{p-1}}, F\right)$-hyp in every Banach space.

↳ This is the best case possible for reasonable RV! ($\mathbb{E}X^2 < \infty$)

↳ Implies $K_{p,q} \leq \sqrt{\frac{p-1}{q-1}}$ for Kahane $\neq$, for all vector sums in any VS!

Lecture 7 Thursday ...