Lecture 8, Thursday Afternoon!

S'pose $0 < \tilde{\sigma} \le \sigma$; $p > 1$; $\mathbb{E}X = 0$.

Then:

$$\|v + \tilde{\sigma} \omega X\|_p = \|(1 - \tfrac{\tilde{\sigma}}{\sigma}) v + \tfrac{\tilde{\sigma}}{\sigma}(v + \sigma \omega X)\|_p$$

$$\overset{\text{Min.}}{\underset{\text{for vectors}}{\le}} \|(1 - \tfrac{\tilde{\sigma}}{\sigma}) v\|_p + \|\tfrac{\tilde{\sigma}}{\sigma}(v + \sigma \omega X)\|_p$$

$$\overset{\text{homogeneity}}{=} (1 - \tfrac{\tilde{\sigma}}{\sigma}) \|v\|_p + \tfrac{\tilde{\sigma}}{\sigma} \|v + \sigma \omega X\|_p$$

$$\overset{*}{\le} ((1 - \tfrac{\tilde{\sigma}}{\sigma}) + \tfrac{\tilde{\sigma}}{\sigma}) \|v + \sigma \omega X\|_p = \|v + \sigma \omega X\|_p$$

$*$: $\|v + \sigma \omega X\|_p \ge \|v + \sigma \omega X\|_1 = \mathbb{E}\|v + \sigma \omega X\|$

$$\overset{\text{Jensen}}{\ge} \|v + \sigma \omega \underbrace{\mathbb{E}X}_{=0}\| = \|v\|.$$

$*$ S'pose X $(p, q, \sigma, \mathbb{R})$-hyp, $p > q > 1$. Then

$$\frac{(\mathbb{E}|1 + \sigma \varepsilon X|^p)^{1/p} - 1}{\varepsilon} \le \frac{(\mathbb{E}|1 + \varepsilon X|^q)^{1/q} - 1}{\varepsilon}$$

$\varepsilon \to 0$.

Thm: $\mathbb{P}(r = 1) = 1/2 = \mathbb{P}(r = -1)$, $p > q > 1$, $\sigma = \sqrt{\frac{q-1}{p-1}}$

$\Downarrow$

$r \in \mathcal{H}\mathcal{C}(p, q, \sqrt{\frac{q-1}{p-1}}, F)$ $\quad \forall (F, \|\cdot\|)$ Banach space.

Recall: $\mathbb{E}X^2 \Rightarrow$ this σ is optimal.

pf: Must show: $\forall (F, \|\cdot\|)$ $\forall v, w \in F$,

$$\left(\frac{\|v + \sigma w\|^p + \|v - \sigma w\|^p}{2}\right)^{1/p} \le \left(\frac{\|v + w\|^q + \|v - w\|^q}{2}\right)^{1/q}$$

Claim: True $\Leftrightarrow$

$$\forall x \in [-1, 1] \quad \left(\frac{(1 + \sigma x)^p + (1 - \sigma x)^p}{2}\right)^{1/p} \le \left(\frac{(1 + x)^q + (1 - x)^q}{2}\right)^{1/q}$$

$\Rightarrow$ is clear b/c $(\mathbb{R}, |\cdot|)$ is a Banach space.

$\Updownarrow$ homogeneity

$$\forall A \geq 0,\ |B| \leq A \quad \left(\frac{(A+\sigma B)^p + (A-\sigma B)^p}{2}\right)^{1/p} \leq \left(\frac{(A+B)^q + (A-B)^q}{2}\right)^{1/q}$$

Note That
$$v + \sigma w = \frac{1+\sigma}{2}(v+w) + \frac{1-\sigma}{2}(v-w)$$
$$v - \sigma w = \frac{1-\sigma}{2}(v+w) + \frac{1+\sigma}{2}(v-w)$$

$$\|v+\sigma w\| \leq \frac{1+\sigma}{2}\|v+w\| + \frac{1-\sigma}{2}\|v-w\| = \underbrace{\left(\frac{\|v+w\| + \|v-w\|}{2}\right)}_{A} + \sigma\underbrace{\left(\frac{\|v+w\| - \|v-w\|}{2}\right)}_{B}$$

By same argument, $\|v-\sigma w\| \leq A - \sigma B$.

Finally, $A+B = \|v+w\|$, $A-B = \|v-w\|$. $\square$

Step 2: WTS: $\forall\, 1 < q < p \leq 2$, $\forall x \in [-1,1]$, $\sigma = \sqrt{\frac{q-1}{p-1}}$,
$$\left(\frac{(1+\sigma x)^p + (1-\sigma x)^p}{2}\right)^{1/p} \overset{?}{\leq} \left(\frac{(1+x)^q + (1-x)^q}{2}\right)^{1/q}$$
$$\Updownarrow$$
$$\frac{(1+\sigma x)^p + (1-\sigma x)^p}{2} \overset{?}{\leq} \left(\left(\frac{(1+x)^q + (1-x)^q}{2}\right)^{p/q}\right)$$

For $|y| < 1$: (and $0 < \sigma < 1$, $|x| < 1$)
$$(1+y)^p = 1 + \binom{p}{1}y + \binom{p}{2}y^2 + \cdots$$
$$(1-y)^p = 1 - \binom{p}{1}y + \binom{p}{2}y^2 + \cdots$$
$$\Rightarrow \frac{(1+y)^p + (1-y)^p}{2} = 1 + \binom{p}{2}y^2 + \binom{p}{4}y^4 + \cdots$$

$*\ \binom{p}{n} := \frac{p(p-1)\cdots(p-n+1)}{n!}$

$$\iff 1 + \binom{p}{2}\sigma^2 x^2 + \binom{p}{4}\sigma^4 x^4 + \cdots \overset{?}{\leq} \left(1 + \binom{q}{2}x^2 + \cdots + \binom{q}{4}x^4 + \cdots\right)^{p/q}$$

Note: $\forall \alpha \geq 1$, $\forall t > 0$, $(1+t)^\alpha \geq 1 + \alpha t$

So $\text{RHS} \geq 1 + \frac{p}{q}\binom{q}{2}x^2 + \frac{p}{q}\binom{q}{4}x^4 + \cdots$

and each term majorizes the LHS terms. $\square$

Step 3: $\forall p > q \geq 2$: $r \in HC\left(p, q, \sqrt{\frac{q-1}{p-1}}, F\right)$ $\forall (F, \|\cdot\|)$ $\Longleftarrow$ $r \in HC\left(q', p', \sqrt{\frac{p'-1}{q'-1}}, F\right)$.

$q' = \frac{q}{q-1}$, $p' = \frac{p}{p-1}$ $(p > q \geq 2 \Rightarrow 2 \geq q' \geq p' > 1)$.

So we know The assumption holds for $p > q = 2$ by Step 2.

$(\{\pm 1\}, \mu)$ $\mu = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1}$ $f : \{\pm 1\} \to \mathbb{R}$

$\longrightarrow Tf : \{-1, 1\} \to \mathbb{R}$, $Tf(+1) = \frac{1+\sigma}{2}f(1) + \frac{1-\sigma}{2}f(-1)$

$Tf(-1) = \frac{1-\sigma}{2}f(1) + \frac{1+\sigma}{2}f(-1)$

$*$ Note That $\sqrt{\frac{q-1}{p-1}} = \sqrt{\frac{p'-1}{q'-1}}$

Eigenfunctions: $T\mathbb{1} = \mathbb{1}$, $Tr = \sigma r$ $\qquad r : \{-1, 1\} \to \mathbb{R}$, $r(\pm 1) = \pm 1$

For: $f, g : \{-1, 1\} \to \mathbb{R}$
$$\langle f, g\rangle = \mathbb{E}_\mu[f, g] = \frac{f(1)g(1) + f(-1)g(-1)}{2}$$

Claim: $\forall f, g : \{-1, 1\} \to \mathbb{R}$, $\langle f, Tg\rangle = \langle Tf, g\rangle$

(i.e. T symmetric).

$\mathbb{1} \perp r$.

$0 = \langle 1, \sigma r\rangle = \langle 1, Tr\rangle = \langle T\mathbb{1}, r\rangle = \langle \mathbb{1}, r\rangle = 0$

$\langle \mathbb{1}, T\mathbb{1}\rangle = \langle T\mathbb{1}, \mathbb{1}\rangle$

$\langle r, Tr\rangle = \langle Tr, r\rangle$ ~~= ⟨1, r⟩~~ $\langle r, T\mathbb{1}\rangle = \langle T\mathbb{1}, r\rangle$.

WTS: $\left(\frac{|x+\sigma y|^p + |x-\sigma y|^p}{2}\right)^{1/p} \overset{?}{\leq} \left(\frac{|x+y|^q + |x-y|^q}{2}\right)^{1/q}$

$x+y=a,\ x-y=b \quad \forall a,b \in \mathbb{R}$

$\Rightarrow x+\sigma y = \frac{a+b}{2} + \sigma\frac{a-b}{2} = \frac{1+\sigma}{2}a + \frac{1-\sigma}{2}b.$

$x-\sigma y = \frac{a+b}{2} - \sigma\frac{a-b}{2} = \frac{1-\sigma}{2}a + \frac{1+\sigma}{2}b.$

$\Updownarrow$

$\forall a,b\in\mathbb{R}: \left(\frac{\left|\frac{1+\sigma}{2}a + \frac{1-\sigma}{2}b\right|^p + \left|\frac{1-\sigma}{2}a + \frac{1+\sigma}{2}b\right|^p}{2}\right)^{1/p}$

$\overset{?}{\leq} \left(\frac{|a|^q + |b|^q}{2}\right)^{1/q}$

$\Leftrightarrow T: L^q(\{\pm1\},\mu) \to L^p(\{\pm1\},\mu),\ \|T\|_{l_q\to l_p} \leq 1$

$\|Tf\|_p = \sup_{\|g\|_{p'}=1} \langle Tf, g\rangle = \sup_{\|g\|_{p'}=1} \langle f, Tg\rangle$

We already know That $\|T\|_{L^{p'}(\{\pm1\},\mu) \to L^{q'}(\{\pm1\},\mu)} \leq 1$

$\Leftrightarrow \left(\frac{|x+\sigma y|^{q'} + |x-\sigma y|^{q'}}{2}\right)^{1/q'} \leq \dots$

So $\forall \|g\|_{p'}=1$, Then $\|Tg\|_{q'} \leq 1$

$\leq \sup_{\substack{h:\{\pm1\}\to\mathbb{R} \\ \|h\|_{q'}\leq 1}} \langle f,h\rangle \overset{\text{Hölder}}{\leq} \|f\|_q.$

Thus pf $\Rightarrow$ bd holds for real line. $\Rightarrow$ holds for Banach spaces by Step I.

Step 4: $p\geq 2,\ q\leq 2 \qquad p\geq 2\geq q>1.$

$\forall v,w\in F$, estimate $\left\|v + \sqrt{\frac{q-1}{p-1}}\,wr\right\|_p$

$= \left\|v + \left(\sqrt{\frac{q-1}{p-1}}\,w\right)r\right\| \underset{(p,2)}{\leq}$

$\leq \left\|v + \left(\sqrt{\frac{q-1}{2-1}}\sqrt{\frac{q-1}{p-1}}\right)wr\right\|_2$

$\underset{(2,q)}{\leq} \left\|v + \sqrt{\frac{q-1}{q-1}}\sqrt{\frac{2-1}{2-1}}\sqrt{\frac{q-1}{p-1}}\,wr\right\|_q \cdot \|v+wr\|_q$

* Here we use: $\|v+wr\|_p \underset{w\to w/\sigma}{\leq} \left\|v+\frac{wr}{\sigma}\right\|_q$.

So $\sigma_{p,q} = \sigma_{p,2}\,\sigma_{2,q}$; does not hold in non-symmetric case.

Khintchine - Kahane $\neq$: $v_1,\dots,v_n\in F \quad (F,\|\cdot\|)$
$S = \sum r_i v_i \qquad \|S\|_p \leq \sqrt{\frac{p-1}{q-1}}\,\|S\|_q$

$\hookrightarrow$ Works for $p,q\geq 1$; Hölder bounds gives good constants for This case as well.

Thm:
$p>q>1 \qquad (F,\|\cdot\|)$ - normed linear space

$X_1,\dots,X_n$ - real random variables

$X_1\in HC(p,q,\sigma_1,F),\dots,X_n\in HC(p,q,\sigma_n,F)$.

Then: $\forall\, v_0,\ v_1,\dots,v_n,\ v_{1,2},\dots,v_{n-1,n},\ v_{1,2,3},\dots,v_{n-2,n-1,n}$

$\|v_0 + \sigma_1 X_1 v_1 + \sigma_2 X_2 v_2 + \dots + \sigma_n X_n v_n + \sigma_1\sigma_2 X_1 X_2 \cdot v_{1,2} + \dots + \sigma_{n-1}\sigma_n X_{n-1}X_n v_{n-1,n} + \sigma_1\sigma_2\sigma_3 X_1X_2X_3 v_{1,2,3} + \dots\|_p$

$\vdots$

$\leq \|v_0 + X_1v_1 + X_2v_2 + \dots + X_nv_n + X_1X_2v_{1,2} + \dots + X_{n-1}X_nv_{n-1,n} + \dots\|_q$

If strategy: induction, take out all terms w/ X_1, etc.

* This removing J idea w/ Stein ...

$\mu = (\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{1})^{\otimes n}$ $\forall B \subseteq \{\pm 1\}^n$, $\mu(B) = \frac{|B|}{2^n}$

$f : \{\pm 1\}^n \to \mathbb{R}$

ex: $w_{\{1,2\}}(x) = x_1 x_2$

$A \subseteq [n]$ $w_A(x) = \prod_{i \in A} x_i$ (Walsh fcns)

$\hookrightarrow r_i(x) = w_{\{i\}}(x) = x_i$ (rademacher fcns)

$W_A = \prod_{i \in A} r_i$, $w_\emptyset = 1$

Note that $r_1, \dots, r_n$ are independent (product of prob. spaces)

$H = L^2(\{-1,1\}, \mu)$,

$\langle f, g \rangle = \int_{\{-1,1\}^n} fg \, d\mu = \sum_{x \in \{\pm 1\}^n} f(x) g(x) / 2^n$

$L^p(\{\pm 1\}, \mu)$, $\|f\|_p = \left(\int_{\{-1,1\}^n} |f|^p d\mu\right)^{1/p} = \left(\frac{\sum |f(x)|^p}{2^n}\right)^{1/p}$

$\hookrightarrow \|a_1 r_1 + \dots + a_n r_n\|_p = \|r_1 v_1 + \dots + r_n v_n\|$

- $\forall A, B \subseteq [n]$, $w_A \cdot w_B = \prod_{i \in A} r_i \cdot \prod_{j \in B} r_j$

$= \prod_{i \in A \cap B} r_i^2 \cdot \prod_{i \in A \Delta B} r_i = \prod_{i \in A \Delta B} = w_{A \Delta B}$.

So: $\langle w_A, w_B \rangle = \begin{cases} 0 \iff A \neq B \\ 1 \iff A = B \end{cases}$ orthonormal basis for H, linear space of dim 2^n

$= \mathbb{E}[w_{A \Delta B}]$

$= \mathbb{E}[\prod_{i \in A \Delta B} r_i]$

- $\forall f : \{\pm 1\}^n \to \mathbb{R}$ $\exists! (a_A)_{A \subseteq [n]}$ s.t. $f = \sum_{A \subseteq [n]} a_A w_A$ → Fourier Walsh expansion

$a_A := \hat{f}(A)$

$\|f\|_2^2 = \langle f, f \rangle = \langle \sum_A \hat{f}(A) w_A, \sum_B \hat{f}(B) w_B \rangle$

$= \sum_{A \subseteq [n]} \sum_{B \subseteq [n]} \hat{f}(A) \hat{f}(B) \langle w_A, w_B \rangle = \sum_{A \subseteq [n]} (\hat{f}(A))^2$

- $\mathbb{E} f = \langle f, 1 \rangle = \langle f, w_\emptyset \rangle = \hat{f}(\emptyset)$.

- $f : \{\pm 1\}^n \to \mathbb{R} \rightsquigarrow Lf : \{\pm 1\}^n \to \mathbb{R}$

$(Lf)(x) := \sum_{\substack{y \in \{\pm 1\}^n \\ y \text{ differs from } x \text{ by one coordinate (exactly)}}} \frac{f(x) - f(y)}{2}$ linear

$L w_A = |A| w_A$

$(L w_A)(x) = \frac{n w_A(x)}{2} - \frac{1}{2} \sum_{y : y \sim x}$

$= \frac{n}{2} w_A(x) - \frac{1}{2} \sum (-w_A(x)) - \frac{1}{2} \sum_{i \in [n] \setminus A} w_A(x)$

$= \frac{n}{2} w_A(x) + \frac{|A|}{2} w_A(x) - \frac{(n - |A|)}{2} w_A(x) = |A| w_A(x)$.

Connection to Khintchine ≠:

$v_1, \dots, v_n \in F$, $(F, \|\cdot\|)$

$R : \{\pm 1\}^n \to \mathbb{R}$ $-(x_1, \dots, x_n) = (-x_1, \dots, -x_n)$

$h(x) = \|x_1 v_1 + \dots + x_n v_n\|$ $h(-x) = h(x)$

$A \subseteq [n]$, A odd $\Rightarrow \hat{h}(A) = 0$

$\hookrightarrow \hat{h}(A) = \langle h, w_A \rangle = \sum_{x \in \{\pm 1\}^n} \frac{h(x) w_A(x)}{2^n} = 0$ (even · odd; odd on $\{-1,1\}^n$)

Poincaré $\neq$: $\mathbb{E}[f^2] - (\mathbb{E}f)^2 \leq \mathbb{E}[f \cdot Lf]$

$$\mathbb{E}[f^2] - (\mathbb{E}f)^2 = \sum_{A \subseteq [n]} (\hat{f}(A))^2 - (\hat{f}(\emptyset))^2 = \sum_{\substack{A \subseteq [n] \\ A \neq \emptyset}} (\hat{f}(A))^2$$

$$\mathbb{E}[f \cdot Lf] = \Big\langle \sum_{A \subseteq [n]} \hat{f}(A) \omega_A, \sum_{B \subseteq [n]} |B| \hat{f}(B) \omega_B \Big\rangle = \sum_{A, B \subseteq [n]} |A||B| \langle \omega_A, \omega_B \rangle = \sum_{A \subseteq [n]} \underbrace{|A|}_{\geq 1} (\hat{f}(A))^2 \quad ✓$$

We show: For even fcns,

$$\mathbb{E}[h^2] - (\mathbb{E}h)^2 \leq \tfrac{1}{2} \mathbb{E}[h \cdot Lh]$$

b/c h even $\Rightarrow$ $|A| \geq 2$ as 1 is odd so $\hat{h}(A) = 0$.

Last step: $Lh \overset{?}{\leq} h$:

$$\sum_{y : y \sim x} \frac{h(x) - h(y)}{2} \leq h(x)$$

$$(n-2) h(x) \overset{?}{\leq} \sum_{y : y \sim x} h(y) = \sum_{i=1}^{n} \| x_1 v_1 + \cdots + x_n v_n - 2 x_i v_i \|$$

$$\overset{\Delta \neq}{\geq} \Big\| \sum_i (x_1 v_1 + \cdots + x_n v_n - 2 x_i v_i) \Big\| = \| n(x_1 v_1 + \cdots + x_n v_n) - 2(x_1 v_1 + \cdots + x_n v_n) \| = (n-2) \| x_1 v_1 + \cdots + x_n v_n \|$$

So: $\mathbb{E}h^2 - (\mathbb{E}h)^2 \leq \frac{1}{2} \mathbb{E}[\underset{\overset{\vee\!\mid}{0}}{h} \cdot \underset{\overset{\vee\!\mid}{h}}{Lh}] \leq \frac{1}{2} \mathbb{E}h^2$

So $\mathbb{E}h^2 \leq 2 (\mathbb{E}h)^2 \Rightarrow$

$$\| r_1 v_1 + \cdots + r_n v_n \|_2 \leq \sqrt{2} \, \| r_1 v_1 + \cdots + r_n v_n \|_1 .$$

Moral: Szarek's $\neq$ for 1st, 2nd moment holds for vector sums as well!

Exercise*: Modification of pf gives

$$\underset{\substack{\uparrow \\ \text{Kahane}}}{K_{4,2}} = \sqrt[4]{3} = C_{4,2}$$