Lecture 10, Friday Afternoon

CLT $\left(X_n \xrightarrow{\text{in law}} X \text{ and } \sup_n \mathbb{E}|X_n|^s < \infty\right) \Rightarrow$

$\forall p \in (0,s)$, $\|X_n\| \xrightarrow[n\to\infty]{} \|X\|_p$

$\forall (F, \|\cdot\|)$, $p>q>1$, $r \in \mathcal{HC}\left(p,q,\sqrt{\frac{q-1}{p-1}},F\right)$

$\Rightarrow r_1 + \dots + r_n \in \mathcal{HC}\left(p,q,\sqrt{\frac{q-1}{p-1}},F\right)$, $\forall_{p>q>1} F$ $\forall n \in \mathbb{N}$

$\Rightarrow$

$\forall n$, $\forall v,w \in F$, $\left\|v + \sqrt{\frac{q-1}{p-1}}(r_1+\dots+r_n)w\right\|_p$

$w \rightsquigarrow w/\sqrt{n}$ $\qquad \le \|v + (r_1 + \dots + r_n)w\|_q$

$\Rightarrow \forall n; v,w \in F$,

$$\left\|v + \sqrt{\frac{q-1}{p-1}}\,\frac{r_1+\dots+r_n}{\sqrt{n}}\,w\right\|_p \le \left\|v + \frac{r_1+\dots+r_n}{\sqrt{n}}\right.$$

CLT

$\Rightarrow g \sim N(0,1)$, $\forall v,w \in F$, $\left\|v + \sqrt{\frac{q-1}{p-1}}\,g w\right\|_p \le \|v+g$

$\Rightarrow \forall p > q > 1, \forall (F, \|\cdot\|) \quad g \in \mathcal{HC}(p, q, \sqrt{\frac{q-1}{p-1}}, F)$. $\boxtimes$

↳ So holds for Gaussians also!

Thm: Assume X - real RV, $\mathbb{E}X = 0$, $\mathbb{E}|X|^p < \infty$ for some $p > 2$. Assume $\mathbb{P}(X=0) \neq 1$. Then

$$X \in \mathcal{HC}(p, 2, \eta_p, \mathbb{R}), \text{ with } \underset{\text{not optimal}}{\eta_p} := \frac{\|X\|_2}{2\sqrt{p-1}\|X\|_p}.$$

pf: (Symmetrize) X' - ind. copy of X, $Y := X - X'$. Then

$$\|Y\|_p \overset{\nabla}{\leq} \|X\|_p + \|-X'\|_p = 2\|X\|_p \quad \text{(Minkowski)}.$$

$Y \sim r \cdot Y$ b/c Y symmetric, r ind. Rademacher

$\forall a, b \in \mathbb{R}$:

$$\|a + b\eta_p X\|_p^p = \mathbb{E}|a + b\eta_p X|^p = \mathbb{E}_X|a + b\eta_p X - b\eta_p \underbrace{\mathbb{E}X'}_{=0}|^p$$

$$\overset{\text{Jensen}}{\leq} \mathbb{E}|a + b\eta_p Y|^p = \mathbb{E}_Y \mathbb{E}_r \left|a + \frac{\sqrt{2-1}}{\sqrt{p-1}} \cdot \frac{b\|X\|_2}{2\|X\|_p} rY\right|^p$$

$$= \mathbb{E}_Y \left[\left\|a + \sqrt{\frac{2-1}{p-1}} \frac{b\|X\|_2}{2\|X\|_p} Y \cdot r\right\|_{p,r}^p\right]$$

$$\underset{r \in \mathcal{HC}(p, 2, \frac{1}{\sqrt{p-1}}, \mathbb{R})}{\leq} \mathbb{E}_Y\left[\left\|a + \frac{b\|X\|_2}{2\|X\|_p} Yr\right\|_{2,r}^p\right] = \mathbb{E}\left(a^2 + \frac{b^2\|X\|_2^2}{4\|X\|_p^2}Y^2\right)^{p/2}$$

$$= \left\|a^2 + \frac{b^2\|X\|_2^2}{4\|X\|_p^2} Y^2\right\|_{p/2}^{p/2} \overset{\text{Minkowski}}{\leq} \left(a^2 + \frac{b^2\|X\|_2^2}{4\|X\|_p^2}\|Y^2\|_{p/2}\right)^{p/2}$$

$$= \left(a^2 + b^2\mathbb{E}X^2 \cdot \left(\frac{\|Y\|_p}{2\|X\|_p}\right)^2\right)^{p/2} \overset{\nabla}{\leq} (a^2 + b^2\mathbb{E}X^2)^{p/2}$$

$$= (\mathbb{E}(a + bX)^2)^{p/2} = \|a + bX\|_2^p. \quad \boxtimes$$

For $g : \{-1, 1\}^n \to \mathbb{R}$, let $(Lg)(x) := \sum_{\substack{y \in \{\pm 1\}^n \\ y \sim x}} \frac{g(x) - g(y)}{2}$

$y \sim x$ (differ in exactly one coordinate)

$$\forall f, g : \{\pm 1\}^n \to \mathbb{R}, \quad \underset{\langle f, Lg \rangle}{\mathbb{E}[f \cdot Lg]} = \frac{1}{2^n}\sum_{x \in \{\pm 1\}^n} f(x)(Lg)(x)$$

$$= \frac{1}{2^{n+1}} \sum_{x \in \{\pm 1\}^n} \sum_{\substack{y \in \{\pm 1\}^n \\ y \sim x}} f(x)(g(x) - g(y))$$

$$= \frac{1}{2^{n+1}} \sum_{\substack{x,y\in\{\pm1\}^n \\ x\sim y}} \left(f(x)g(x) - f(x)g(y) \right)$$

$(x \leftrightarrow y)$

$$= \frac{1}{2^{n+1}} \sum_{\substack{x,y\in\{\pm1\}^n \\ x\sim y}} \left(f(y)g(y) - f(y)g(x) \right)$$

$a = b \Downarrow \frac{a+b}{2} = a$

$$\frac{1}{2^{n+1}} \sum_{\substack{x,y\in\{\pm1\}^n \\ x\sim y}} \frac{f(x)g(x) - f(x)g(y) + f(y)g(y) - f(y)g(x)}{2}$$

$$= \frac{1}{4\cdot 2^n} \sum_{x,y\in\{\pm1\}^n} \left(f(x) - f(y)\right)\left(g(x) - g(y)\right).$$

$\underline{\text{Moral}}$: sym. for f, g!

The Stroock-Varopoulos $\neq$: $\forall f : \{\pm1\}^n \to [0, \mathbb{R})$ ($\mathbb{R}$) also

$$\mathbb{E}\left[f^2 \cdot L(f^2)\right] \leq \frac{4}{3}\mathbb{E}\left[f^3 \cdot Lf\right].$$

$(f(x) - f(y))^4 \geq 0$

pf: Suffices to show: $\forall x,y \in \{\pm1\}^n$, $x \sim y$

$\Updownarrow$

$$(f^2(x) - f^2(y))^2 \overset{?}{\leq} \frac{4}{3}\left(f^3(x) - f^3(y)\right)\left(f(x) - f(y)\right)$$

$\forall a,b \in \mathbb{R}$, $3(a^2 - b^2)^2 \leq 4(a^3 - b^3)(a-b)$

$\Updownarrow$

$$(a-b)^2 \cdot 3(a+b)^2 \leq (a-b)^2 \cdot 4(a^2 + ab + a^2)$$

$\Uparrow$

$$3a^2 + 6ab + 3b^2 \leq 4a^2 + 8ab + 4b^2 \Leftarrow (a-b)^2 \geq 0$$

$\square$

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We already know That $\forall f : \{\pm1\}^n \to \mathbb{R}$ even,

$$\mathbb{E}[f^2] - (\mathbb{E}[f])^2 \leq \frac{1}{2}\mathbb{E}[f \cdot Lf].$$

$\hookrightarrow h(x) = \left\|\sum_{i=1}^n x_i v_i\right\|$ is even on $\{-1,1\}^n$, $\therefore$

$f = h^2$ also even on $\{\pm1\}^n$. Thus:

$$\mathbb{E}[h^4] - (\mathbb{E}h^2)^2 \leq \frac{1}{2}\mathbb{E}\left[h^2 \cdot L(h^2)\right]$$

$$\overset{\text{Stroock Varopoulos}}{\leq} \frac{1}{2}\cdot\frac{4}{3}\mathbb{E}\left[\underset{0}{\underset{\vee|}{h^3}} \cdot \underset{h}{\overset{\wedge|}{Lh}}\right] \leq \frac{2}{3}\mathbb{E}\left[h^3 \cdot h\right] = \frac{2}{3}\mathbb{E}[h^4]$$

$\Delta\neq$

So $\mathbb{E}[h^4] \le 3(\mathbb{E}[h^2])^2$ i.e.

$\forall (F, \|\cdot\|), n \in \mathbb{N}, v_1, \dots, v_n \in F,$

$$\Big\| \sum_{i=1}^n r_i v_i \Big\|_4 \le \sqrt[4]{3} \cdot \Big\| \sum_{i=1}^n r_i v_i \Big\|_2 .$$

So we have proved Kahane-Khintchine $\neq$ w/ optimal constant $K_{4,2} = \sqrt[4]{3}$. The optimality follows from $F = \mathbb{R}$, $a_1 = a_2 = \dots = a_n = \frac{1}{\sqrt{n}}$ as $n \to \infty$.

General Stroock-Varopoulos $\neq$:

$\forall p > 1$, $\forall f \in \{\pm 1\}^n \to [0, \infty)$,

$$\mathbb{E}\left[f^{p/2} \cdot L(f^{p/2})\right] \le \frac{p^2}{4(p-1)} \mathbb{E}\left[f^{p-1} \cdot Lf\right]$$

$$\Uparrow$$

$$\forall a,b \ge 0 : (p-2)^2(a^p + b^p) - p^2(a^{p-1}b + b^{p-1}a) + 8(p-1)a^{p/2}b^{p/2} \ge 0$$

$$\Uparrow$$

$$\forall t \ge 1 : (p-2)^2(t^p + 1) - p^2(t^{p-1} + t) + 8(p-1)t^{p/2} \ge 0.$$

$a_1 \ge a_2 \ge a_3 \ge \dots \ge a_n \ge 0$

Hitczenko's Formula: $\exists$ universal $C_1, C_2 > 0$ s.t. $\forall p > 2$,

$$C_1 \cdot \Big(a_1 + \dots + a_{[p]} + \sqrt{p}\sqrt{\sum_{k > [p]} a_k^2}\Big)$$

$$\wedge\!|$$

$$\|a_1 r_1 + \dots + a_n r_n\|_p$$

$$\wedge\!|$$

$$C_2 \cdot \Big(a_1 + a_2 + \dots + a_{[p]} + \sqrt{p}\sqrt{\sum_{k > [p]} a_k^2}\Big).$$

↳ Constants C_1, C_2 do not depend on p for $p > 2$.

↳ Can modify cutoff formula, $[p]$ not only option

* Consider all possible (b_i), (c_i) s.t. $(a_1, \dots, a_n) = (b_1 + c_1, \dots, b_n + c_n)$

Using $\inf(\|b\|_1 + \sqrt{p}\,\|c\|_2)$, same up to universal constant.

Thm (Latala, 1997) $X_1, \dots, X_n$ symmetric, real RV, $p > 2$; $t > 0$ s.t.

$$\sum_{k=1}^{n} \ln \|1 + \frac{X_k}{t}\|_p = 1$$

$\Rightarrow$ $Bt \le \|X_1 + \dots + X_n\|_p \le At$, A, B universal

↳ Exchanges probability for purely analytic condition.

Key Tool: for ε, δ small, $1 + \varepsilon + \delta \approx (1+\varepsilon)(1+\delta)$

Open Question: can a version of this result be stated for complex RV, maybe rotation invariant?

Latala $\Rightarrow$ Hitczenko, many other $\neq$

Including: for $p > 2$, $\|X_1 + \dots + X_n\|_p \le C_p(\|X_1 + \dots + X_n\|_2 + (\sum_{i=1}^{n} \mathbb{E}|X_i|^p)^{1/p})$

→ On RHS can also take max, take weighted sums, etc – same up to universal constant.

Kwapien's Thm: $(F, \|\cdot\|)$ – Banach space. Assume $\exists A, B > 0$ s.t. $\forall n$, $\forall v_1, \dots, v_n \in F$,

$$A(\|v_1\|^2 + \dots + \|v_n\|^2) \le \mathbb{E}\|r_1 v_1 + \dots + r_n v_n\|^2 \le B(\|v_1\|^2 + \dots + \|v_n\|^2).$$

Then F is isomorphic to a Hilbert space.

In general:

$\forall n$, $A_p(\sum \|v_i\|^p) \le \mathbb{E}\|r_1 v_1 + \dots + r_n v_n\|^p \le B_p(\sum \|v_i\|^p)$

Def: $(F, \|\cdot\|)$ - Banach space is isomorphic to a Hilbert space if $\exists \langle \cdot, \cdot \rangle$ on F s.t. $\exists\, C_1, C_2 > 0$ $\forall v \in F$, $C_1 \|v\|_F^2 \leq \langle v, v \rangle \leq C_2 \|v\|_F^2$.

→ This Thm shows That only those fcns with type and cotype 2 are isomorphic to a Hilbert space; converse, That type = cotype = 2 ⇒ Hilbert space.

↳ Rademacher type/cotype related to Gaussian type/cotype. ($g_i \sim N(0,1)$ instead of r_i)

Type, cotype help show when an isomorphic embedding cannot happen.

Cotype ⇐⇒ exists such a lower bound
Type ⇐⇒ exists such an upper bound.

Kwapien's Conjecture: $\forall p > q > 1$, $K_{p,q} = C_{p,q}$

↳ Why might This be The case? We know it holds for Hilbert spaces (but This only shows There is no easy counter-ex; Hilbert spaces behave very similarly to real line in This setting).

→ $g_1, \ldots, g_n \sim N(0,1)$ ind. $\|\sum_{i=1}^{n} g_i v_i\|_p \leq \frac{\gamma_p}{\gamma_q} \|\sum g_i v_i\|_q$

where $\gamma_p = (\mathbb{E}|g|^p)^{1/p}$, $g \sim N(0,1)$.

↳ Get equality for $v_i \in \mathbb{R}$. $\Uparrow$ Szarek

This ≠ is ~~[illegible]~~ ≠ S ≠ : $K \subseteq \mathbb{R}^n$ convex, symmetric, $N(0, I_n) \sim \gamma_n$. Then

$$\forall t \geq 1, \quad \gamma_n(tK) \geq \gamma_n(tS),$$ (Isoperimetric type statement)

where S is a symmetric strip:

$$S = \{x \in \mathbb{R}^n : |\langle v, x \rangle| \leq 1\}, \quad \gamma_n(S) = \gamma_n(K).$$

↳ Strip has Gaussian measure growing most slowly under dilation

$\| \sum_{i=1}^n v_i g_i \|_q = \| a g \|_q$ (choose such an a)

i.e. $\mathbb{E} \| \sum v_i g_i \|^q = \mathbb{E} | a g |^q = \mathbb{E} \| (a g_1, \dots, a g_n) \|_S$

(for $S = \{ |x_1| \le 1 ; x_2, \dots, x_n \in \mathbb{R} \}$)

$= q \int_0^\infty x^{q-1} \mathbb{P}(\| \sum v_i g_i \| > x) \, dx \overset{(*)}{=} q \int_0^\infty x^{q-1} \mathbb{P}(|ag| > \dots$

$* \; \| a g \|_q = \frac{\gamma_q}{\gamma_p} \| a g \|_p$ b/c on real line.

$\frac{\gamma_q}{\gamma_p} \cancel{\|} \| \sum v_i g_i \| \overset{?}{\le} \| \sum_{i=1}^n v_i g_i \|_q$

Does $(*) \Rightarrow$

$p \int_0^\infty x^{p-1} \mathbb{P}(\| \sum v_i g_i \| > x) \, dx \overset{?}{\le} p \int_0^\infty x^{p-1} \mathbb{P}(|ag| > x) \, dx$

WTS: $\int_0^\infty x^{p-1} (\mathbb{P}(|ag| > x) - \mathbb{P}(\| \sum v_i g_i \|_K > x)) \overset{?}{\ge} 0$

(b/c $? \; 0$ w/ q by $(*)$)

$\| $

$\int_0^\infty \left(\left(\frac{x}{x_0}\right)^{p-1} - \left(\frac{x}{x_0}\right)^{q-1} \right) \left(\mathbb{P}(|ag| > x) - \mathbb{P}(\| \sum v_i g_i | > x) \right) dx$

changes sign - only once - by S $\ne$, at x_0, get $\ne$. $\square$

Note: Kwapien's conjecture $\Rightarrow$ This result easily by CLT.

$p \ge q \ge 1$

~~What~~ Let $C_{p,q}$ optimal in Khintchine $\ne$, $K_{p,q}$ optimal in Kahane $\ne$.

$$\text{CLT} \to \frac{\gamma_p}{\gamma_q} \le C_{p,q} \le K_{p,q} \underset{p > q > 1}{\le} \sqrt{\frac{p-1}{q-1}} \quad \text{hypercontractive}$$

$\frac{K_{p,q}}{C_{p,q}}$

As $p \to \infty$, $\frac{\gamma_p}{\sqrt{p-1}} \to 1/\sqrt{e}$.

$\sup_{q \in [1, \frac{5}{2}]}$

$\approx$

So as $p, q \to \infty$, Kwapien follows.

$\Rightarrow$ Interesting case: q small; $\sup_{q \in [1,p]} \frac{K_{p,q}}{C_{p,q}}$

$\mathcal{O}.$: $\sup_{q \in [1,p]} \frac{K_{p,q}}{C_{p,q}} \longrightarrow 1$ as $p \to \infty$.

Key Tool:

$\forall \delta \in (0,1), \forall p \geq 1$, S - Rademacher vector sum with values in $(E, \|\cdot\|)$

$$\left\| \left(\|S\| - \frac{(1+\delta)^2}{\delta} \mathbb{E}\|S\| \right)_+ \right\|_p \leq (1+\delta)\, \sigma_p(S)$$

(*)

weak p^{th} moment of S

$$\sigma_p(S) = \sup_{\substack{\varphi \in E^* \\ \|\varphi\|_{E^*} \leq 1}} \|\varphi(S)\|_p .$$

Then: optimize over $\delta \in [0,1]$; use that for $\varphi(S)$, have Khintchine:
real Rademacher!

$$\|\varphi(S)\|_p \leq C_{p,q} \|\varphi(S)\|_q .$$

$\Rightarrow$ Can bound Kahane constant from above by Khintchine constant + lower order terms. ☒

* Take S, $S - \delta S'$ perturbation (S', S i.i.d.), look at small rotation (Pisier), to show bd (*).