Lecture 2, Monday

$(0,\infty) \ni p \longmapsto \mathbb{E} Z^p$ is log convex, (p^{th} moment, $\|Z\|_p$)

i.e. $\varphi : (0,\infty) \to [0,\infty]$, $\varphi(p) := \ln \mathbb{E} Z^p$ is convex.

$\|Z\|_p := (\mathbb{E} Z^p)^{1/p}$

↳ If Z real r.v: $(\mathbb{E}|Z|^p)^{1/p}$

↳ If Z RV w/ $(F, \|\cdot\|)$: $\|\mathbb{E}|Z|^p\|_p$.

Thm: $(0,\infty) \ni p \longmapsto \|Z\|_{1/p}$ is also log-convex.

Convex ⇒ can be supported from below by family of affine fcns

$\exists\, (a_\alpha), (b_\alpha) \in \mathbb{R}$ s.t.
$\alpha \in A$

$$\varphi(p) = \sup_{\alpha \in A} (a_\alpha p + b_\alpha).$$

so $p\varphi\left(\frac{1}{p}\right) = \sup_{\alpha \in A} (b_\alpha p + a_\alpha)$

$\| \ln \|Z\|_{1/p}$

$\ln (\mathbb{E} Z^{1/p})^p$

$= p \ln \mathbb{E} Z^{1/p}$.

$p>q>r>0$, $Z\ge 0$, $\mathbb{E}Z^p<\infty$ } Recall

$(\mathbb{E}Z^q)^{p-r} \le (\mathbb{E}Z^p)^{q-r}\cdot(\mathbb{E}Z^r)^{p-q}$

↳ $q = \frac{q-r}{p-r}\,p + \frac{p-q}{p-r}\,r$

Minkowski's ≠ : X, Y real RVs with finite absolute p^{th} moments. $\forall p\ge 1$, $\quad x^{\langle a\rangle} = \operatorname{sgn} x\cdot|x|^a$

$$\|X+Y\|_p \le \|X\|_p + \|Y\|_p$$

pf: $\|X+Y\|_p^p = \mathbb{E}|X+Y|^p = \mathbb{E}(X+Y)(X+Y)^{\langle p-1\rangle}$

$= \mathbb{E}X(X+Y)^{\langle p-1\rangle} + \mathbb{E}Y(X+Y)^{\langle p-1\rangle}$

$\underset{\text{Hölder }(p,\frac{p}{p-1})}{\le} \|X\|_p\,\|(X+Y)^{\langle p-1\rangle}\|_{\frac{p}{p-1}} + \|Y\|_p\,\|(X+Y)^{\langle p-1\rangle}\|_{\frac{p}{p-1}}$

$= \|X\|_p\,(\mathbb{E}|X+Y|^p)^{\frac{p-1}{p}} + \|Y\|_p\,\|X+Y\|_p^{p-1}$

$= (\|X\|_p+\|Y\|_p)(\|X+Y\|_p^{p-1})$. □

⇓ more generally.

X, Y RVs , $(F,\|\cdot\|)$-valued.

$\|X+Y\|_p = (\mathbb{E}\|X+Y\|^p)^{1/p} \overset{\Delta\neq}{\le} (\mathbb{E}(\|X\|+\|Y\|)^p)^{1/p}$

$= \|\,\|X\|+\|Y\|\,\|_p \underset{\text{Minkowski}}{\le} \|\,\|X\|\,\|_p + \|\,\|Y\|\,\|_p = \|X\|_p+\|Y\|_p$.

Polish space: metric, separable, complete w/ norm induced by metric.

↳ We will use separable Banach spaces

Thm (Talagrand): $\ell^\infty = \{(a_0,a_1,a_2,\dots) : \sup|a_i|<\infty\}$

NOT separable (consider $(\pm1,\pm1,\dots)$) $\quad \|(a)\| := \sup_i |a_i|$

Talagrand: $\exists$ X, Y random vectors ℓ^∞-valued

$\{X\in A\}$ ← random

$\{Y\in A\}$ ← events for all $A\in\mathcal{B}(\ell^\infty)$

but $X+Y$ is not a random variable, i.e. $\exists B\in\mathcal{B}(\ell^\infty)$ s.t. $\{X+Y\in B\}$ is not a random event.

→ This is one of the simplest non-separable space, so justifies why we assume separability!

X random vectors with values in a separable Banach space $(F,\|\cdot\|)$

$\mathbb{E}\|X\|<\infty$ } $\Rightarrow \exists!\, v\in F$ s.t. $\forall \varphi\in F^*$ $\varphi(v)=\mathbb{E}\varphi(X)$

$v=\mathbb{E}X$ $\qquad \varphi(\mathbb{E}X)$

↑ Bochner Integrability

Uniqueness: $v\neq w \Rightarrow v-w\neq 0 \Rightarrow \varphi(v)-\varphi(w)\neq 0$.

↳ Use Hahn-Banach

Simple ≠s: let Z be a RV

(0) · $\forall p\ge 1$, $\|Z\|_p = \|(Z-\mathbb{E}Z)+\mathbb{E}Z\|_p$ ← fixed vector in F

$\le \|Z-\mathbb{E}Z\|_p + \|\mathbb{E}Z\|_p$

$= \|Z-\mathbb{E}Z\|_p + \|\mathbb{E}Z\|_F$ ← well-defined vector by above

(1) • $\forall p\ge 1$, $\|\mathbb{E}Z\|_F \le \|Z\|_p \iff \mathbb{E}\|Z\|_F^p \ge \|\mathbb{E}Z\|_F^p$

$F\ni v\longmapsto \|v\|^p$ convex, Jensen in vector spaces

↓ other explanation

$\mathbb{E}Z\in F \Rightarrow$ (Hahn-Banach) $\exists\, \varphi\in F^*$, $\|\varphi\|_{F^*}=1$ s.t. $\varphi(\mathbb{E}Z)=\|\mathbb{E}Z\|_F$

↳ $\mathbb{E}\|Z\|^p \ge \mathbb{E}|\varphi(Z)|^p \underset{x\mapsto|x|^p}{\overset{\text{Jensen}}{\ge}} |\mathbb{E}\varphi(Z)|^p = |\varphi(\mathbb{E}Z)|^p$

$= \|\mathbb{E}Z\|_F^p$.

Thm: $p > q > 0$, X, Y independent RVs, $\mathbb{E}|h(X,Y)| < \infty$.
h real-valued. $\mathbb{E}_x h(X,y) = \int h(x,Y)\, d\mu_x(x)$
$= \mathbb{E}(h(X,Y) \mid Y)$ b/c X, Y ind. (Note
$\mathbb{E}_x h(X,y)$ is $\sigma(Y)$-measurable.

Similarly: $\mathbb{E}_y h(X,Y) = \mathbb{E}(h(X,Y) \mid X)$
$= \int_F h(X,y)\, d\mu_y(y)$.

↳ By independence: $F \times F$, $d\mu_{(x,y)}(x,y)$
$\mu_{(x,y)} = \mu_{(x)} \otimes \mu_y$ $\quad = d\mu_x(x)\, d\mu_y(y)$

$$\boxed{\mathbb{E}_x \mathbb{E}_y h = \mathbb{E}_y \mathbb{E}_x h = \mathbb{E} h.}$$

General Greedy-Hasty Phenomenon: $p > q > 0$, X, Y ind. r.v. Then for $f: F \times F \to [0, \infty)$, $f \geq 0$,

$$\| \| f(X,Y) \|_{p,x} \|_{q,y} \geq \| \| f(X,Y) \|_{q,y} \|_{p,x}.$$

↳ Can be extended to $-q$ as well.

Moral: to maximize, take higher moment first!

$$\left(\mathbb{E}_y \left(\mathbb{E}_x \left[(f(X,Y))^p \right]\right)^{q/p}\right)^{1/q} \geq$$
$$\left(\mathbb{E}_x \left[\left(\mathbb{E}_y \left[f(X,Y)^q \right]\right)^{p/q} \right]\right)^{1/p}.$$

pf: Change notation s.t. $1 < p/q =: s$, $f^q = g$.

WTS:

$$\mathbb{E}_x \left[\left(\mathbb{E}_y [g^s]\right)^{1/s} \right] \geq \left(\mathbb{E}_y \left[(\mathbb{E}_x g)^s \right]\right)^{1/s}$$
$$\Updownarrow$$
$$\mathbb{E}_x \|g\|_{s,y} \geq \|\mathbb{E}_x g\|_{s,y}$$

← use Jensen's $\geq$ on space

Alt pf: $\|\mathbb{E}_x g\|_{s,y}^s = \mathbb{E}_y[(\mathbb{E}_x g)^s]$

$= \mathbb{E}_y[\mathbb{E}_x g \cdot \underbrace{(\mathbb{E}_x g)^{\langle s-1\rangle}}_{\sigma(y)\text{ measurable}}]$

$= \mathbb{E}_y[\mathbb{E}_x[g \cdot (\mathbb{E}_x g)^{\langle s-1\rangle}]]$

$= \mathbb{E}_x[\mathbb{E}_y[g \cdot (\mathbb{E}_x g)^{\langle s-1\rangle}]]$ $(s, \frac{s}{s-1})$

$\overset{\text{Hölder}}{\le} \mathbb{E}_x(\|g\|_{s,y} \|(\mathbb{E}_x g)^{\langle s-1\rangle}\|_{\frac{s}{s-1}}$

$= \mathbb{E}_x[\|g\|_{s,y} \cdot \|\mathbb{E}_x g\|_s^{s-1}]$

$= \|\mathbb{E}_x g\|_{s,y}^{s-1} \cdot \mathbb{E}_x\|g\|_{s,y}$. $\square$

Recall: Khintchine: $\forall p > q > 0$, $\exists C_{p,q} > 0$ s.t.

$$\|S\|_p \le C_{p,q}\|S\|_q,$$

where $S = \sum_{i=1}^n a_i r_i$, $r_1, \dots, r_n$ ± 1 symmetric, ind. $a_1, \dots, a_n \in \mathbb{R}$.

Thm: $\|S\|_p \le C_{p,q}\|S\|_q$, where $S = \sum_{i=1}^n r_i v_i$

$v_1, \dots, v_n \in \underset{\text{Hilbert space}}{H}$ (Kahane-Khintchine)

$$\|S\|_p = \left(\mathbb{E}\left|\sum_{i=1}^n r_i v_i\right|_H^p\right)^{1/p}$$

↳ Bound holds for any Banach space (normed space), but in general not known if constant is the same. But for Hilbert spaces it is!

$A = (\langle v_i, v_j\rangle_H)_{i,j}^{n\times n}$ PSD $A = UDU^T = \underbrace{(U\sqrt{D})(U\sqrt{D})^T}_{BB^T = (\langle w_i, w_j\rangle_{\mathbb{R}^n})}$

$U \in O(n)$, $D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n\end{pmatrix}$, $\sqrt{D} = \begin{pmatrix} \sqrt{d_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{d_n}\end{pmatrix}$

↳ i.e. finite dim subspace in Hilbert space looks like $\mathbb{R}^n$.

$G = (g_1, \dots, g_n)$ $g_1, \dots, g_n$ i.i.d. $N(0,1)$

$\forall a, b \in \mathbb{R}^n$ $\langle a, G\rangle_{\mathbb{R}^n} = a_1 g_1 + \dots + a_n g_n$

$\langle b, G\rangle_{\mathbb{R}^n} = b_1 g_1 + \dots + b_n g_n$

$\mathbb{E}[\langle a, G\rangle_{\mathbb{R}^n} \langle b, G\rangle_{\mathbb{R}^n}] = a_1 b_1 + \dots + a_n b_n = \langle a, b\rangle_{\mathbb{R}^n}$.

Let $G_i = \langle w_i, G\rangle_{\mathbb{R}^n}$, Then $\mathbb{E}[G_i G_j] = \langle w_i, w_j\rangle_{\mathbb{R}^n} = \langle v_i, v_j\rangle_H$.

$G = (G_1, \dots, G_n)$ Gaussian ↳ not necessarily independent!

$\forall \alpha_1, \dots, \alpha_n \in \mathbb{R}$, $\alpha_1 G_1 + \dots + \alpha_n G_n$ is a centered Gaussian r.v.

$\forall p > 0$: $\|\alpha_1 G_1 + \dots + \alpha_n G_n\|_p = \gamma_p \|\alpha_1 G_1 + \dots + \alpha_n G_n\|_2$

where $\gamma_p = \|g\|_p$, $g \sim N(0,1)$. (b/c $\|g\|_2 = 1$).

$\|\alpha_1 G_1 + \alpha_2 G_2 + \dots + \alpha_n G_n\|_2^2$

$= \mathbb{E}(\alpha_1 G_1 + \dots + \alpha_n G_n)^2 = \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \mathbb{E}[G_i G_j]$

$= \sum_{i=1}^n \sum_{j=1}^n \alpha_i \alpha_j \langle v_i, v_j\rangle_H = \langle \alpha_1 v_1 + \dots + \alpha_n v_n, \alpha_1 v_1 + \dots + \alpha_n v_n\rangle_H$

$= |\alpha_1 v_1 + \dots + \alpha_n v_n|_H^2$.

So $\forall p > 0$, $\|\alpha_1 G_1 + \dots + \alpha_n G_n\|_p = \gamma_p \cdot \left|\sum_{i=1}^n \alpha_i v_i\right|_H$

$\varepsilon_1, \dots, \varepsilon_n$ ± 1 symmetric;

$\|\varepsilon_1 v_1 + \dots + \varepsilon_n v_n\|_p = (\mathbb{E}_\varepsilon|\varepsilon_1 v_1 + \dots + \varepsilon_n v_n|_H^p)^{1/p}$

$= \|\,|\varepsilon_1 v_1 + \dots + \varepsilon_n v_n|_H\|_{p,\varepsilon}$

$= \left\|\frac{1}{\gamma_q}\|\varepsilon_1 G_1 + \dots + \varepsilon_n G_n\|_{q,G}\right\|_{p,\varepsilon}$ inside! $G_1, \dots, G_n$ fixed

$\overset{\text{greedy hasty}}{\le} \frac{1}{\gamma_q}\|\|\varepsilon_1 G_1 + \dots + \varepsilon_n G_n\|_{p,\varepsilon}\|_{q,G}$.

$\overset{\text{Khintchine}}{\le} \|C_{p,q} \cdot \|\varepsilon_1 G_1 + \dots + \varepsilon_n G_n\|_{q,\varepsilon}\|_{q,G}$

$= \frac{C_{p,q}}{\gamma_q}(\mathbb{E}_G \overset{\text{Fubini}}{\mathbb{E}_\varepsilon}\|\varepsilon_1 G_1 + \dots + \varepsilon_n G_n\|^q)^{1/q}$

$$= \frac{C_{p,q}}{\gamma_q}\left(\gamma_q^q \left|\sum_i \varepsilon_i v_i\right|_H^q\right)^{1/q} = C_{p,q}\left(\mathbb{E}\left|\sum_i \varepsilon_i v_i\right|^{q}\right.$$

$$= C_{p,q}\, \|S\|_q . \qquad \boxtimes$$

↳ Other ≠ follows b/c embedding real line into Hilbert space.