From hard spheres dynamics to the linearized Boltzmann equation

Thierry Bodineau

Joint work with Isabelle Gallagher, Laure Saint-Raymond

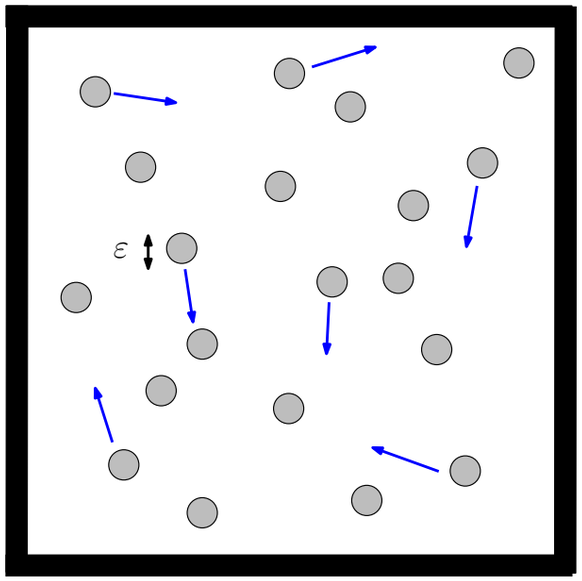
Outline.

- Linearized Boltzmann equation & acoustic equations
- $\mathbb{L}^2$ - approach & a mild version of local equilibrium
- Lanford's strategy & pruning procedure
- Coupling with the Boltzmann hierarchy

Goal. Fluctuating Boltzmann equation

Microscopic scale : Newtonian dynamics

$$Z_N(t) = (x_i(t), v_i(t))_{i \leq N}$$



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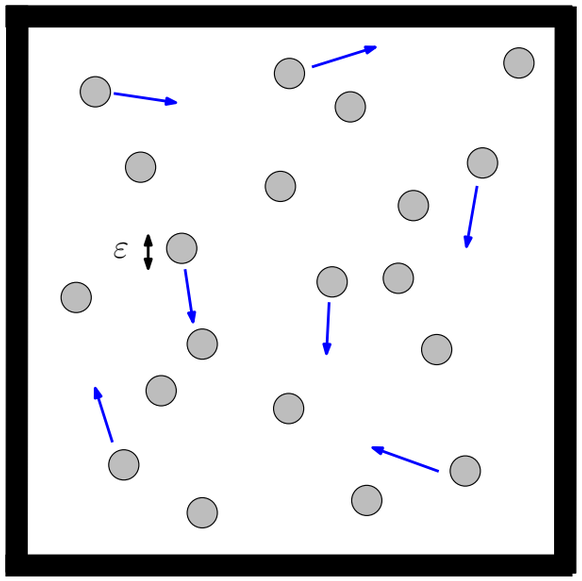

Starting at equilibrium

$$\frac{1}{N}\sum_{i=1}^{N} h\big(z_i(t)\big) \underset{N\to\infty}{\longrightarrow} \mathbb{E}(h)$$

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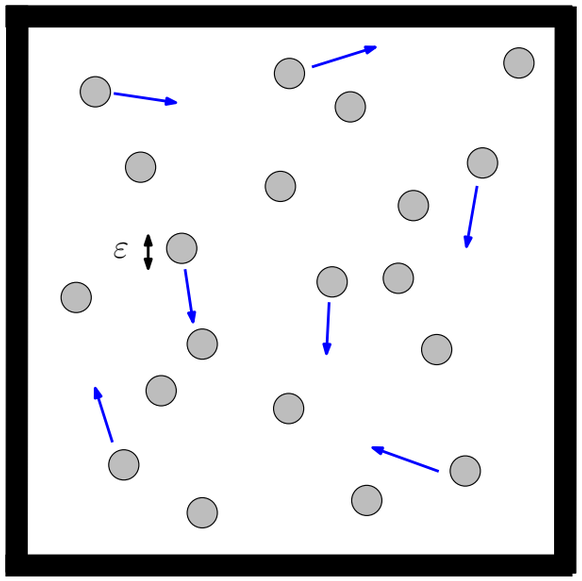

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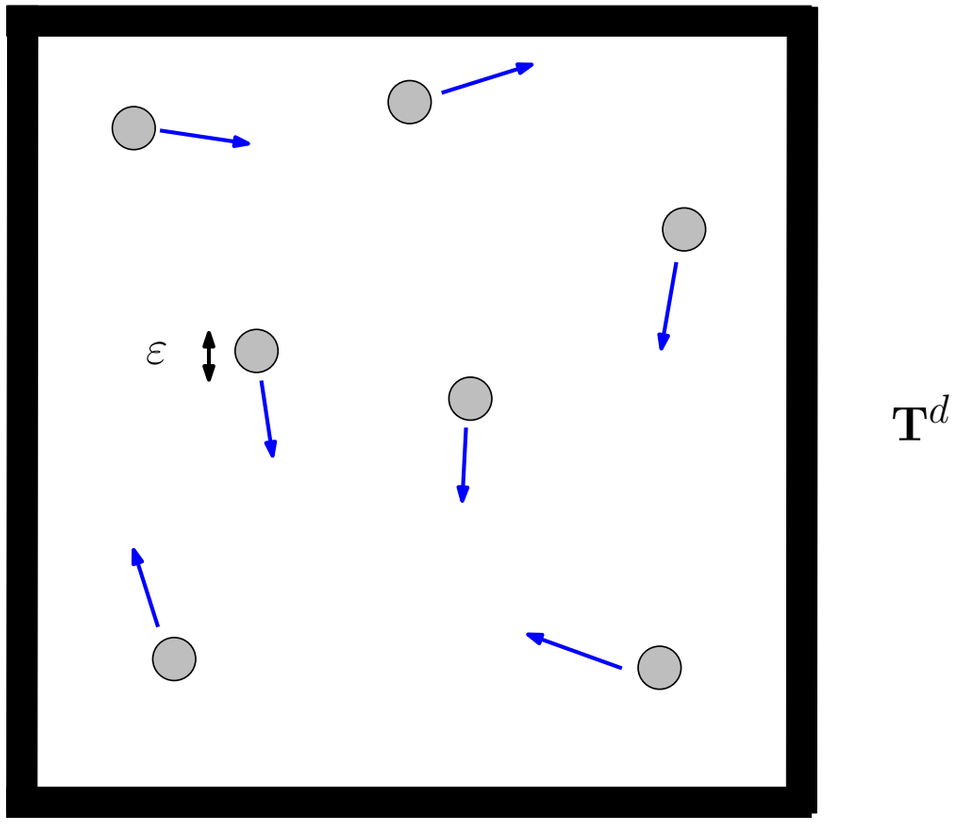
Fluctuation field.

$$\zeta^N(h, Z_N(t)) = \frac{1}{\sqrt{N}}\sum_{i=1}^{N} h(z_i(t)) \qquad \text{with} \qquad \int h(x,v) M_\beta(v) dx dv = 0$$

Question. Dynamical fluctuations at equilibrium ?

Diluted Gas of hard spheres

Gas of N hard spheres with deterministic Newtonian dynamics (elastic collisions).



Dimension : $d \geq 2$

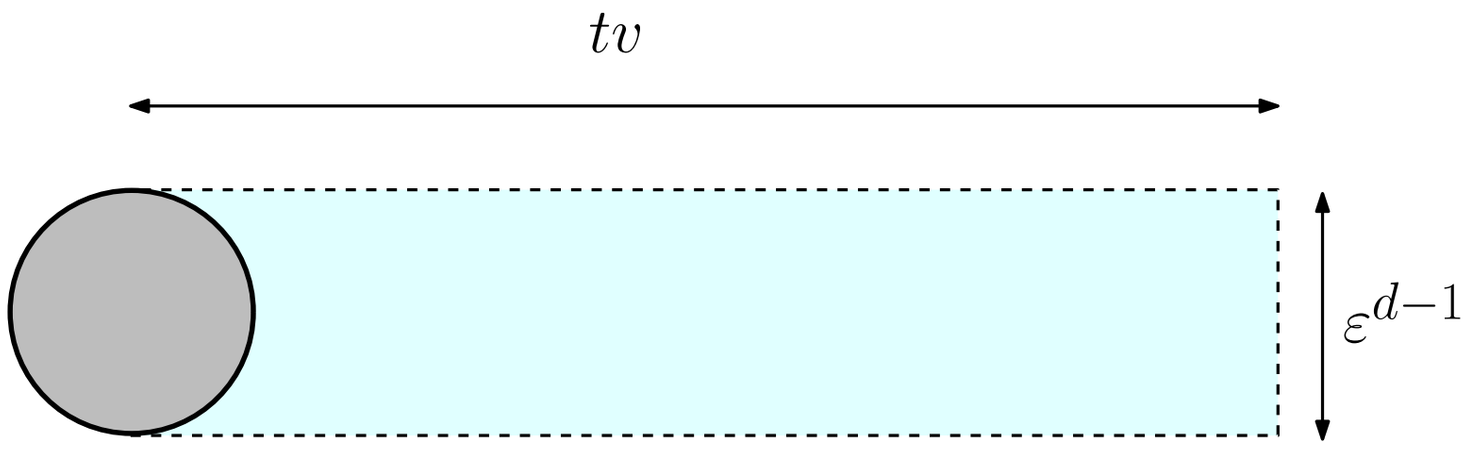
Periodic domain: $T^d = [0,1]^d$

Sphere radius = ε

Boltzmann-Grad scaling

$$N\varepsilon^{d-1} = \alpha$$

Boltzmann-Grad scaling



- Volume covered by a particle $= tv\varepsilon^{d-1}$
- On average N particles per unit volume

On average, a particle has α collisions per unit of time

$$N \times \varepsilon^{d-1} \equiv \alpha$$

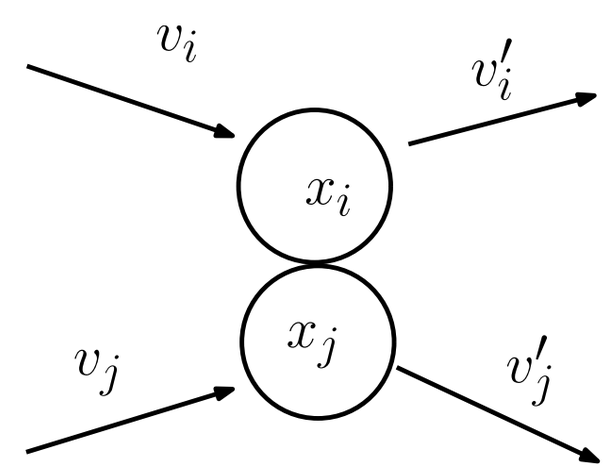
Hard Sphere dynamics

Gas of N hard spheres : $Z_N = \{(x_i(t), v_i(t)\}_{i \leq N}$

$$\frac{dx_i}{dt} = v_i, \qquad \frac{dv_i}{dt} = 0 \qquad \text{as long as} \quad |x_i(t) - x_j(t)| > \varepsilon\,,$$

and elastic collisions if $|x_i(t) - x_j(t)| = \varepsilon$

$$\begin{cases} v_i' + v_j' = v_i + v_j \\ |v_j'|^2 + |v_j'|^2 = |v_i|^2 + |v_j|^2 \end{cases}$$



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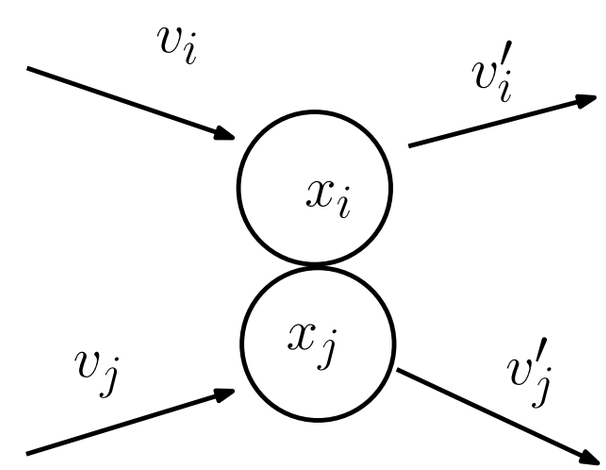

Liouville equation for the particle density $f_N(t, Z_N)$

$$\partial_t f_N + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} f_N = 0$$

in the phase space

$$\mathcal{D}_\varepsilon^N := \left\{ Z_N \in \mathbf{T}^{dN} \times \mathbb{R}^{dN} \,/\, \forall i \neq j\,, \quad |x_i - x_j| > \varepsilon \right\}$$

with specular reflection on the boundary $\partial\, \mathcal{D}_\varepsilon^N$.

Initial Data

Equilibrium distribution

$$M_{N,\beta}(Z_N) = \frac{1}{\mathcal{Z}_{N,\beta}} \exp\Big(-\frac{\beta}{2} \sum_{i=1}^{N} |v_i|^2 \Big) \prod_{i \neq j} 1_{|x_i - x_j| > \varepsilon}$$

Initial data :

$$f^0_{N,\beta}(Z_N) = \left(\prod_{i=1}^{N} f^0(z_i) \right) M_{N,\beta}(Z_N)$$

Density of a particle at time t :

$$f_N^{(1)}(t, z_1) = \int dz_2 \dots dz_N \, f_N(t, z_1, z_2, \dots, z_N)$$

Question. Convergence $\quad f_N^{(1)}(t, z_1) \xrightarrow[\substack{N \to \infty \\ N\varepsilon^{d-1} = \alpha}]{?} f(t, z_1)$

Boltzmann equation

Theorem.

For chaotic initial data $f_N^0(Z_N) \simeq \prod_{i=1}^N f^0(z_i)$ the density of the particle system converges up to a time t >0 to the solution of the Boltzmann equation when $N \to \infty$, $N\varepsilon^{d-1} = \alpha$

$$\partial_t f + v \cdot \nabla_x f$$
$$= \alpha \iint_{\mathbf{S}^{d-1} \times \mathbb{R}^d} [f(v')f(v_1') - f(v)f(v_1)] \big((v - v_1) \cdot \nu\big)_+ \, dv_1 d\nu$$

with $v' = v + \nu \cdot (v_1 - v)\,\nu\,, \quad v_1' = v_1 - \nu \cdot (v_1 - v)\,\nu$

[Lanford], [King], [Alexander], [Uchiyama], [Cercignani, Illner, Pulvirenti], [Simonella], [Gallagher, Saint-Raymond, Texier], [Pulvirenti, Saffirio, Simonella] ...

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Lanford's strategy leads to a short time convergence which depends on f^0. The convergence time remains short even if initially the system starts from equilibrium !!!

Large time asymptotics

Equilibrium distribution

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Initial data for Lanford's theorem

$$f^0_{N,\beta}(Z_N) = \left(\prod_{i=1}^{N} f^0(z_i) \right) M_{N,\beta}(Z_N) \longleftarrow \simeq \exp(N)$$

Question.

Perturbation of the equilibrium distribution of order N

Fluctuation field.

$$\zeta^N(h, Z_N(t)) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} h(z_i(t)) \qquad \text{with} \qquad \int h(x,v) M_\beta(v) dx dv = 0$$

Covariance of the fluctuation field :

$$\mathbb{E}\Big(\zeta^N\big(g, Z_N(0)\big)\ \zeta^N\big(h, Z_N(t)\big)\Big)$$

$$= \frac{1}{N} \int M_{N,\beta}(Z_N) \left(\sum_{i=1}^{N} g\big(z_i\big)\right) \left(\sum_{i=1}^{N} h\big(z_i(t)\big)\right)$$

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Symmetry

$$= \int M_{N,\beta}(Z_N) \left(\sum_{i=1}^{N} g(z_i)\right)\ h(z_1(t))$$

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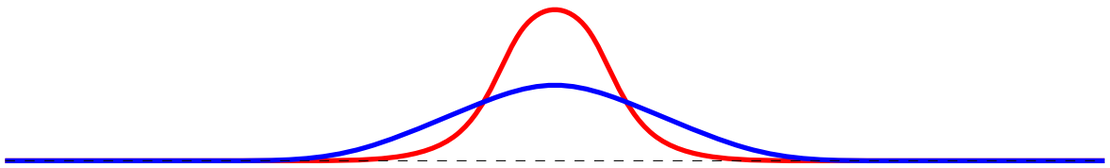
$$= \int \boxed{M_{N,\beta}(Z_N) \left(\sum_{i=1}^{N} g(z_i)\right)}\ h(z_1(t))$$

New initial data

Linearized Boltzmann equation

Response to a small perturbation



$$(\partial_t + v \cdot \nabla_x) g = -\alpha \, \mathcal{L} \, g,$$

$$\mathcal{L} \, g(v) := \int M_\beta(v_1) \Big(g(v) + g(v_1) - g(v') - g(v_1') \Big) \Big((v_1 - v) \cdot \nu \Big)_+ d\nu dv_1$$

Linearized Boltzmann equation

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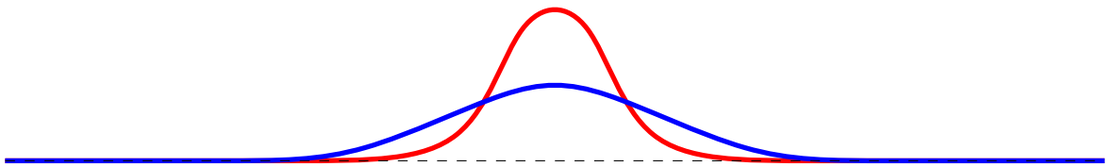

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Background

Linearized Boltzmann equation

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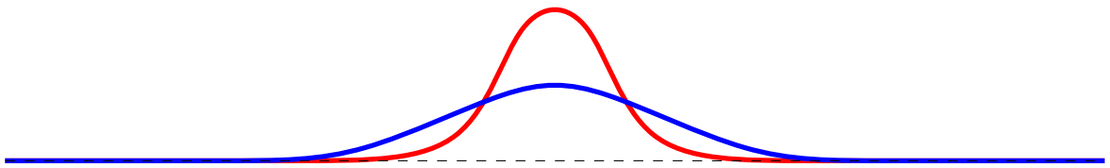

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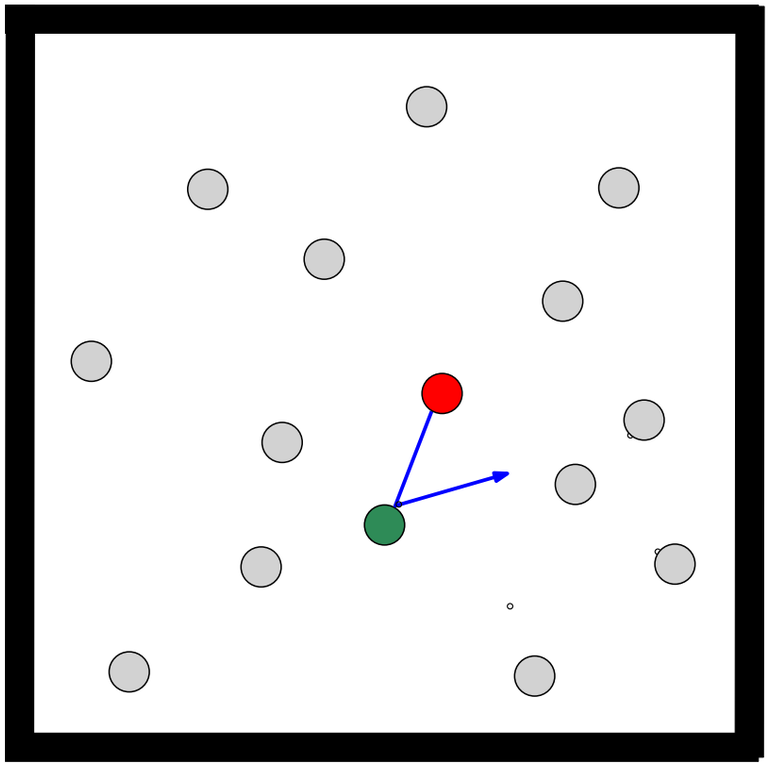
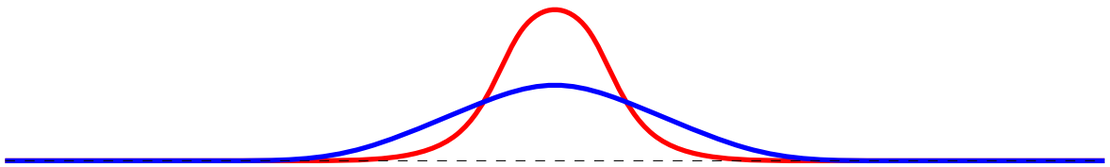
Tagged particle



- perturbation of the tagged particle

Linearized Boltzmann equation

Response to a small perturbation



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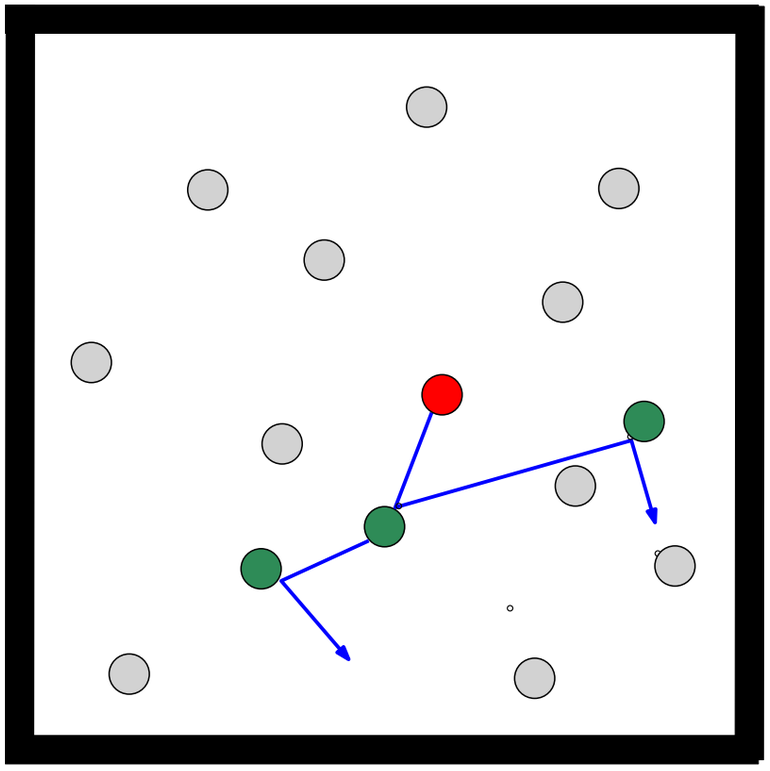
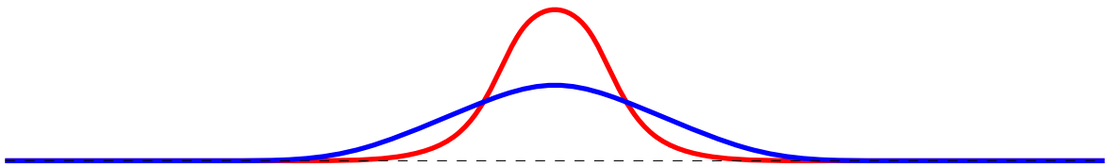


- perturbation of the tagged particle
- perturbation of the background

Linearized Boltzmann equation

Response to a small perturbation



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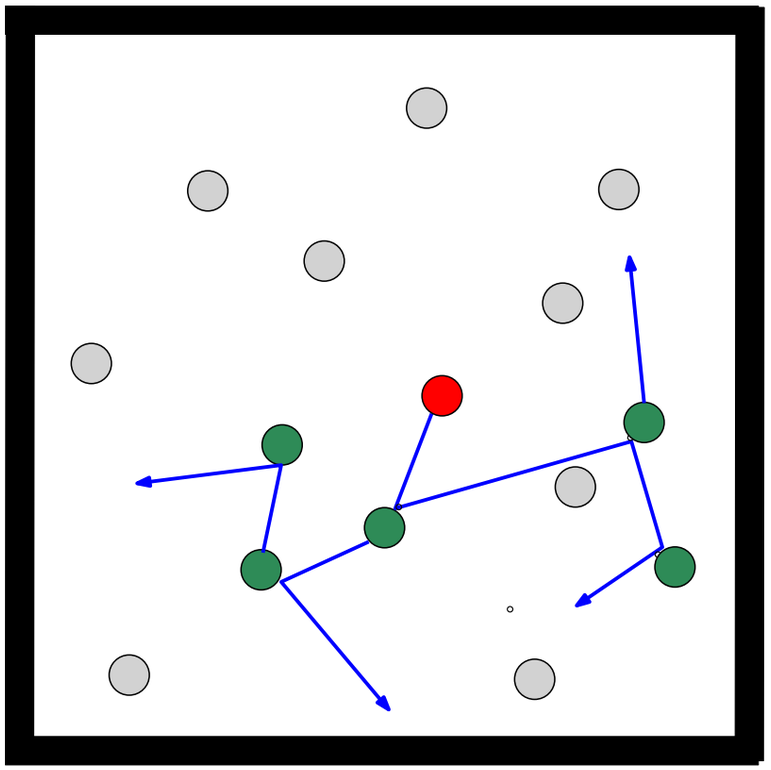
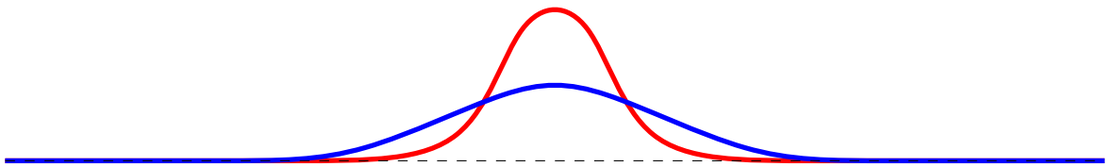


A cloud of particles is modified.

On averaged the distribution of each background particle changes by an order : $O\Big(\frac{\alpha t}{N}\Big)$

Linearized Boltzmann equation

Response to a small perturbation



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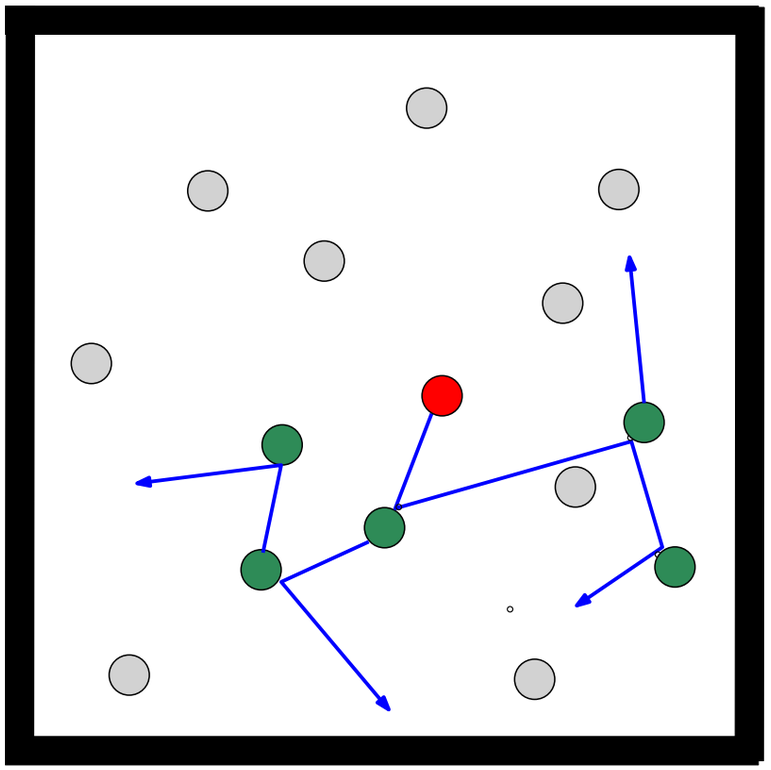

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Goal: Capture corrections $\simeq \frac{1}{N}$

Linearized Boltzmann equation

Perturbation of order 1 (tagged particle)

$$f_N^0(Z_N) = M_{N,\beta}(Z_N)\, g_0\big(z_1\big) \longrightarrow \text{corrections of order} \simeq \frac{1}{N}$$

Perturbation of order N (symmetric version)

$$f_N^0(Z_N) = M_{N,\beta}(Z_N) \left(\sum_{i=1}^{N} g_0\big(z_i\big) \right) \longrightarrow \text{corrections of order} \simeq 1$$

with $\displaystyle\int M_\beta(v) g_0(z) dz = 0$

Question. Large time behavior of $f_N^{(1)}(t, z_1)$

N particle system

$f_N^{(1)}(x_1, v_1, t)$

$$\alpha = N\varepsilon^{d-1}$$

$$N \to \infty$$

Linearized Boltzmann equation

$g_\alpha(x_1, v_1, t)$

[van Beijeren, Lanford, Lebowitz, Spohn] (short time)

N particle system

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$N \to \infty$

Linearized Boltzmann equation

$g_\alpha(x_1, v_1, t)$

$\alpha \to \infty$

[Bardos,Golse,Levermore]

Initially :

$$g(0, x, v) := \rho_0(x) + u_0(x) \cdot v + \frac{\beta|v|^2 - d}{2}\,\theta_0(x)$$

density fluctuation

momentum fluctuation

energy fluctuation

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$$g(t, x, v) := \rho(t, x) + u(t, x) \cdot v + \frac{\beta|v|^2 - d}{2}\theta(t, x)$$

Acoustic equations

$$\begin{cases} \partial_t \rho + \nabla_x \cdot u = 0 \\ \partial_t u + \nabla_x(\rho + \theta) = 0 \\ \partial_t \theta + \nabla_x \cdot u = 0 \end{cases}$$

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$f_N^{(1)}(x_1, v_1, t)$

$\alpha = N\varepsilon$

$N \to \infty$

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Theorem [BGSR]

For d = 2, convergence for any $t > 0$

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$N \to \infty$

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$g_\alpha(x_1, v_1, t)$

Theorem [BGSR]

For d = 2, convergence for any $t > 0$

$\alpha \ll \sqrt{\log\log\log N}$

$\to \infty$

Initially :

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Consequence of the previous Theorem

Covariance of the fluctuation field :

$$\lim_{N\to\infty} \mathbb{E}_{M_{N,\beta}} \Big(\zeta^N(h, Z_N(0)) \zeta^N(\tilde{h}, Z_N(t)) \Big)$$

$$= \int dz\, M_\beta(v) \exp\big(- t(v \cdot \nabla_x + \alpha \mathcal{L}) \big) h(z)\, \tilde{h}(z)$$

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Question.

Convergence of the field to the Ornstein-Uhlenbeck process ?

[Spohn], [Rezakhanlou]

Derivation of the linearized Boltzmann equation

Step 1. Control of the collision operators

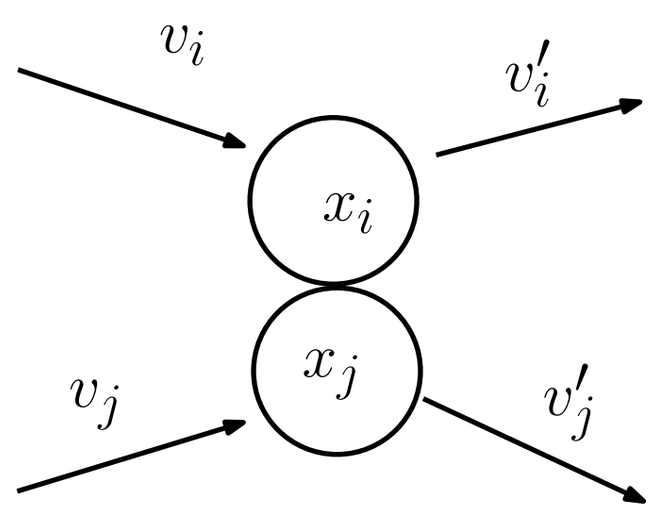
BBGKY hierarchy for the marginals

Evolution of the first marginal

$$(\partial_t + v_1 \cdot \nabla_{x_1}) f_N^{(1)}(t, z_1) = \alpha \big(C_{1,2} f_N^{(2)}\big)(t, z_1)$$

Collision operator

$$\big(C_{1,2} f_N^{(2)}\big)(z_1) := \int_{\mathbf{S}^{d-1} \times \mathbb{R}^d} f_N^{(2)}(x_1, v_1', x_1 + \varepsilon\nu, v_2') \Big((v_2 - v_1) \cdot \nu\Big)_+ d\nu dv_2$$
$$- \int_{\mathbf{S}^{d-1} \times \mathbb{R}^d} f_N^{(2)}(x_1, v_1, x_1 + \varepsilon\nu, v_2) \Big((v_2 - v_1) \cdot \nu\Big)_- d\nu dv_2$$



BBGKY hierarchy for the marginals

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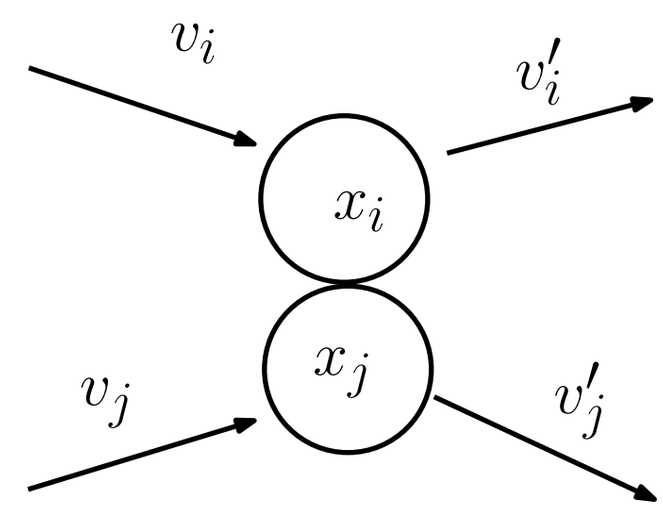

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Collision operator

$$\big(C_{1,2} f_N^{(2)}\big)(z_1) := \int_{\mathbf{S}^{d-1} \times \mathbb{R}^d} f_N^{(2)}(x_1, v_1', x_1 + \varepsilon\nu, v_2') \Big((v_2 - v_1) \cdot \nu\Big)_+ d\nu dv_2$$
$$- \int_{\mathbf{S}^{d-1} \times \mathbb{R}^d} f_N^{(2)}(x_1, v_1, x_1 + \varepsilon\nu, v_2) \Big((v_2 - v_1) \cdot \nu\Big)_- d\nu dv_2$$

Hope : Propagation of chaos

$$f_N^{(2)}(x_1, v_1, x_1 + \varepsilon\nu, v_2) \simeq f_N^{(1)}(x_1, v_1) f_N^{(1)}(x_1 + \varepsilon\nu, v_2)$$

Consequence: Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f = \iint \left[f(v') f(v_1') - f(v) f(v_1)\right] \big((v - v_1) \cdot \nu\big)_+ dv_1 d\nu$$

BBGKY hierarchy for the marginals

For $s < N$ and on $\mathcal{D}_\varepsilon^s = \left\{ Z_s = (x_i, v_i)_{i \le s} \,\middle|\, \quad i \neq j, \quad |x_i - x_j| > \varepsilon \right\}$

$$(\partial_t + \sum_{i=1}^{s} v_i \cdot \nabla_{x_i}) f_N^{(s)}(t, Z_s) = \alpha \big(C_{s,s+1} f_N^{(s+1)} \big)(t, Z_s)$$

where the collision term is defined by

$$\begin{aligned}
&\big(C_{s,s+1} f_N^{(s+1)} \big)(Z_s) \\
&:= \frac{(N-s)\varepsilon^{d-1}}{\alpha} \sum_{i=1}^{s} \int_{\mathbf{S}^{d-1} \times \mathbb{R}^d} f_N^{(s+1)}(\ldots, x_i, v_i', \ldots, x_i + \varepsilon\nu, v_{s+1}') \Big((v_{s+1} - v_i) \cdot \nu \Big)_+ d\nu dv_{s+1} \\
&- \frac{(N-s)\varepsilon^{d-1}}{\alpha} \sum_{i=1}^{s} \int_{\mathbf{S}^{d-1} \times \mathbb{R}^d} f_N^{(s+1)}(\ldots, x_i, v_i, \ldots, x_i + \varepsilon\nu, v_{s+1}) \Big((v_{s+1} - v_i) \cdot \nu \Big)_- d\nu dv_{s+1}
\end{aligned}$$

where $\mathbf{S}^{d-1}$ denotes the unit sphere in $\mathbb{R}^d$.

Duhamel formula

Denote by $\mathbf{S}_s$ the semi-group associated to free transport in $\mathcal{D}_\varepsilon^s$

Duhamel Formula

$$f_N^{(1)}(t) = \mathbf{S}_1(t) f_N^{(1)}(0) + \alpha \int_0^t \mathbf{S}_1(t-t_1) C_{1,2} f_N^{(2)}(t_1)\, dt_1 \,,$$

Iterated Duhamel formula

$$f_N^{(1)}(t) = \sum_{n=0}^{N-1} \alpha^n \, Q_{1,1+n}(t) f_N^{(1+n)}(0)$$

Idea : Use the initial randomness

with

$$Q_{s,s+n}(t) := \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} dt_n \dots dt_1 \; \mathbf{S}_s(t-t_1) C_{s,s+1} \mathbf{S}_{s+1}(t_1-t_2) C_{s+1,s+2} \dots \mathbf{S}_{s+n}(t_n)$$

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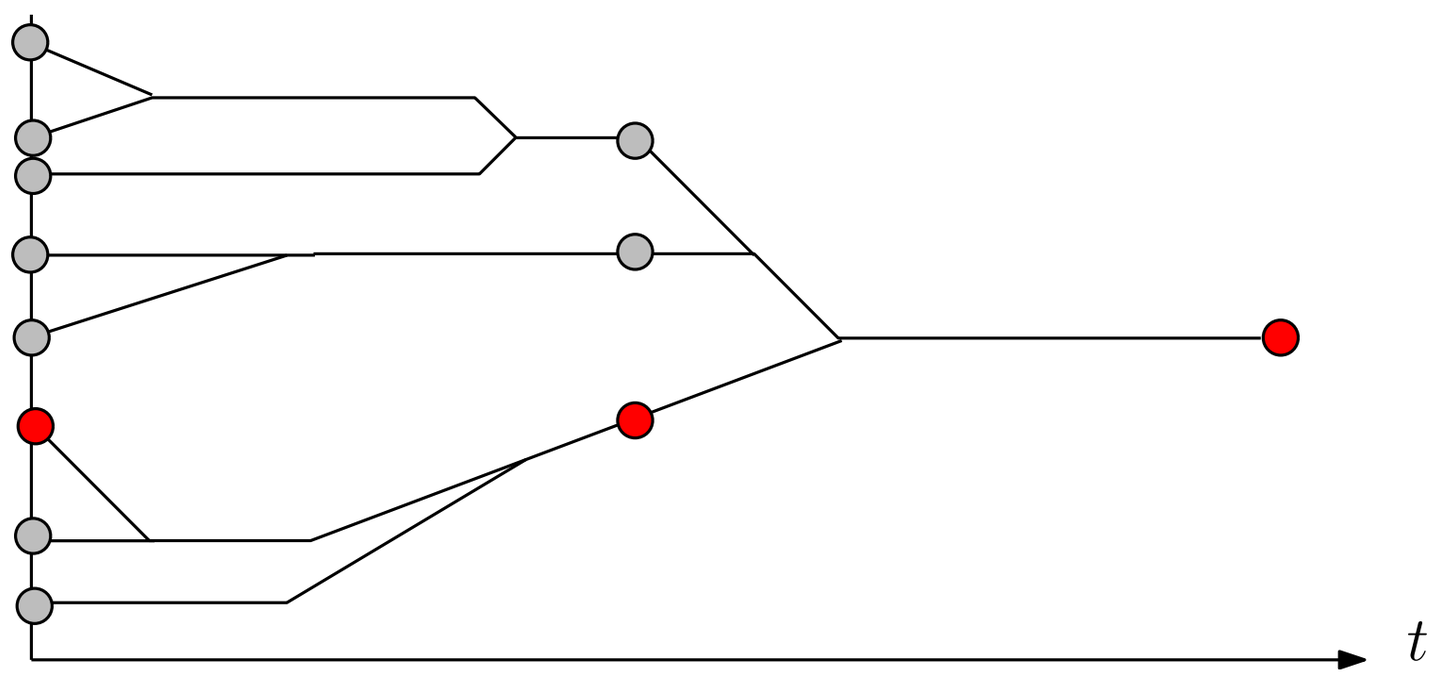
Interpretation as a collision tree

- Transport operator
- Addition of a particle to the tree after each collision



Issue : convergence of the series when N diverges

$$f_N^{(1)}(t) = \sum_{n=0}^{N-1} \alpha^n Q_{1,1+n}(t) f_N^{(1+n)}(0)$$

Continuity estimates for the collision operators

Weighted norms

$\|f_k\|_{\varepsilon,k,\beta} := \sup_{Z_k \in \mathcal{D}_\varepsilon^k} \left| f_k(Z_k) \ \exp\left(\frac{\beta}{2} \sum_{i=1}^k |v_i|^2\right) \right| < \infty$

Collision operators estimates

$$\left\| Q_{s,s+n}(t) f_{s+n} \right\|_{\varepsilon,s,\beta/2} \leq e^{s-1} (C_d(\beta) t)^n \|f_{s+n}\|_{\varepsilon,s+n,\beta}$$

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Series is only controlled for short times t and small α

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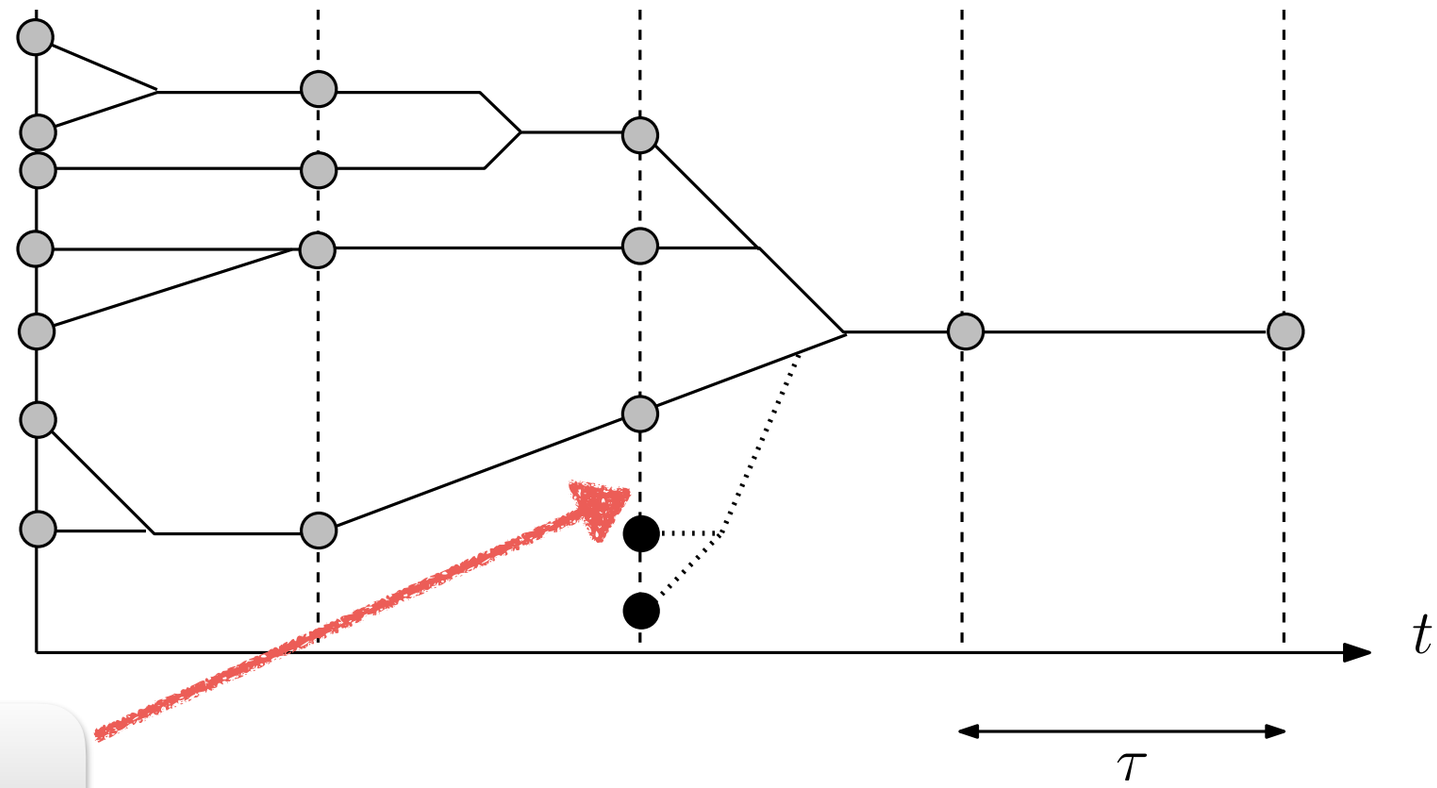
Removing large collision trees ?



Stopping the series at intermediate times

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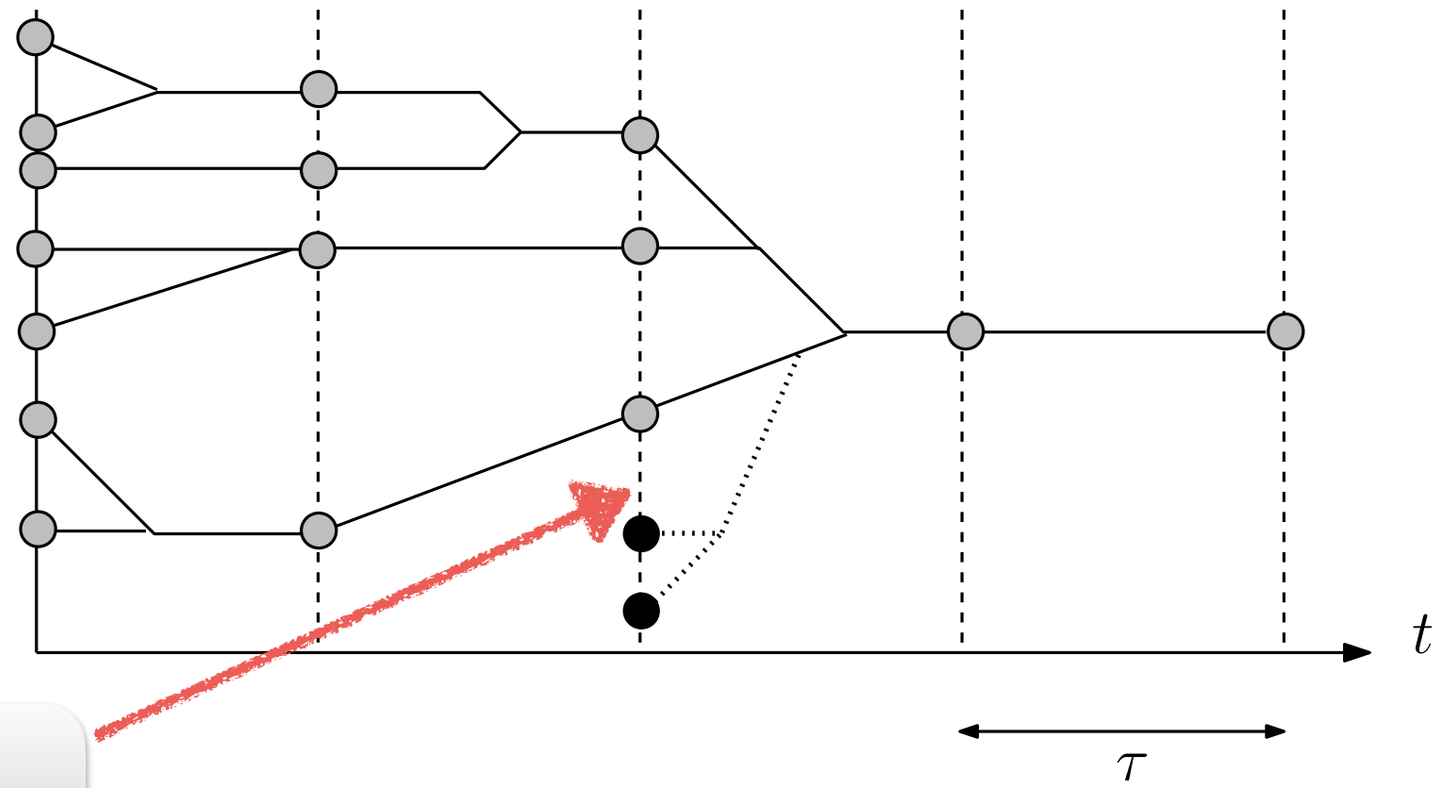

Stopping the series at intermediate times

Questions :

- Deriving uniform controls in time
- A priori estimates on the particle system

Step 2.

$\mathbb{L}^2$ *estimates and*
a mild version of local equilibrium

A priori estimates

Initial data of order N : $$f_N^0(Z_N) = M_{N,\beta}(Z_N)\left(\sum_{i=1}^{N} g_0(z_i)\right)$$

No uniform bonds in L^∞: $$\left|f_N^{(s)}(t, Z_s)\right| \leq N\, C^s M_\beta^{\otimes s}(Z_s)\, \|g_0\|_{L^\infty}$$

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L^2 bounds are preserved in time

$$\int dZ_N M_{N,\beta}(Z_N)\left(\frac{f_N^0(Z_N)}{M_{N,\beta}(Z_N)}\right)^2 \leq CN$$

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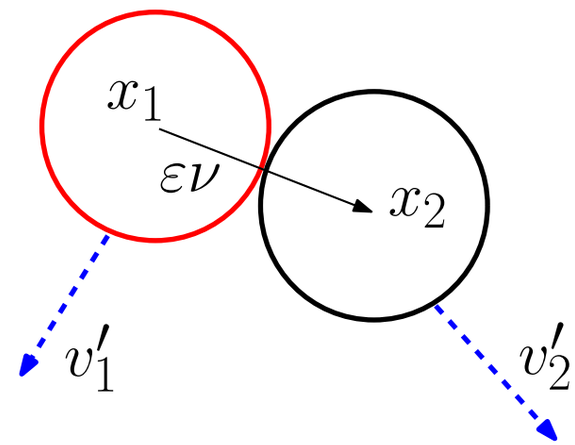
$$\Rightarrow \quad \int dZ_N M_{N,\beta}(Z_N)\left(\frac{f_N(t, Z_N)}{M_{N,\beta}(Z_N)}\right)^2 \leq CN$$

L^2 estimates are more natural for the linearized operator

New strategy L^2 estimates on the collision kernel

$$C^+_{1,2} f_N^{(2)}(z_1) = \int f_N^{(2)}(x_1, v_1', x_1 + \varepsilon\nu, v_2') \Big((v_2 - v_1) \cdot \nu \Big)_+ d\nu dv_2$$

A dimension is missing
for L^2 estimates



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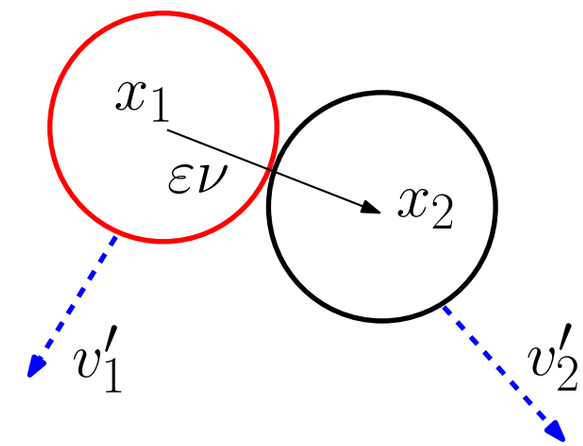
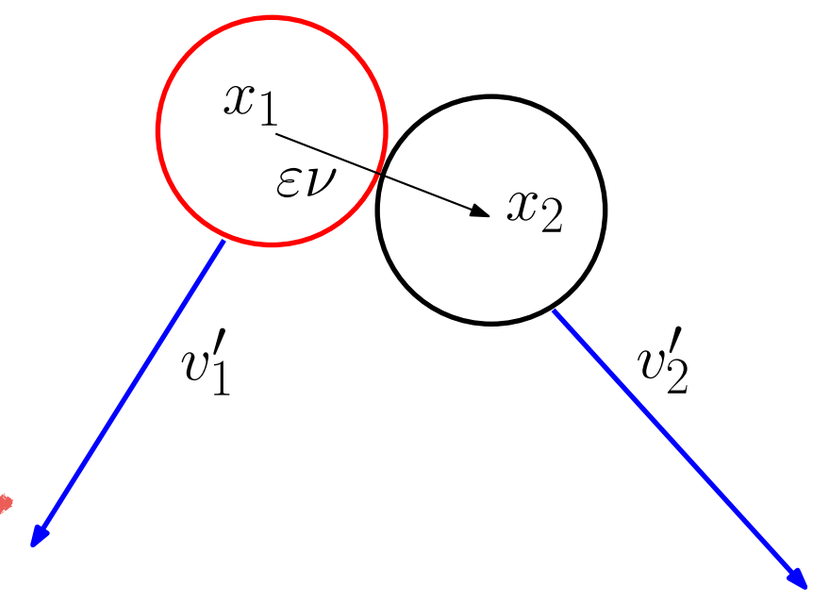


$$\Big| \int dz_1 \int_0^T d\tau\, C^+_{1,2}\, \mathbf{S}_2(\tau) f_N^{(2)} \Big| \leq C\sqrt{\frac{T}{\varepsilon}} \| f_N^{(2)} \|_{L^2}$$

bad estimate

Additional time dimension



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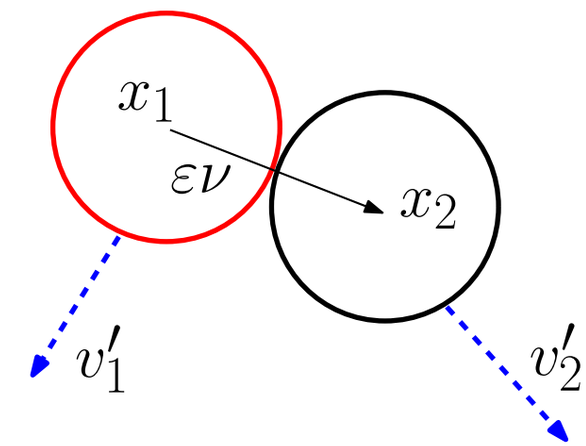

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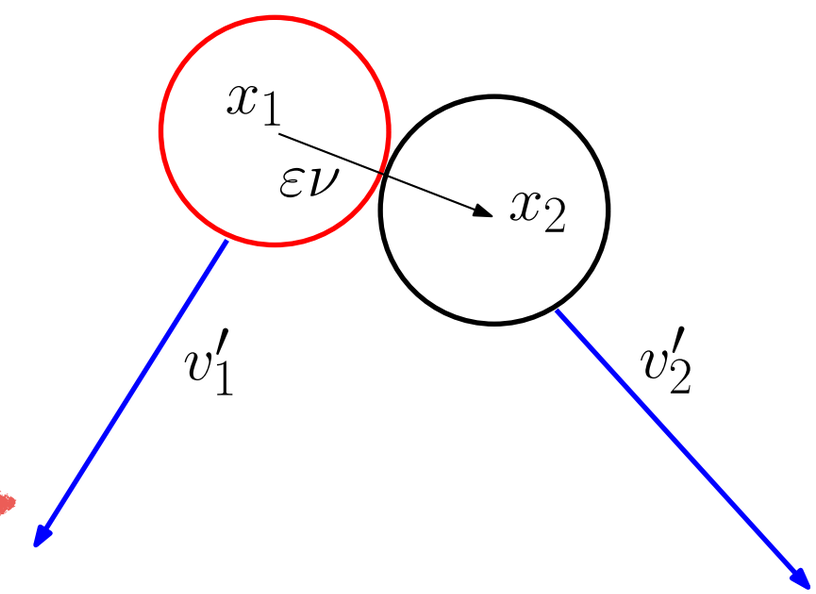

$$\frac{1}{\varepsilon} \int_0^\epsilon dr \, \varphi(r) \leq \begin{cases} \|\varphi\|_{L^\infty} \\ \frac{1}{\sqrt{\varepsilon}} \|\varphi\|_{L^2} \end{cases}$$

Singular domain of integration

Difficulties to estimate the collision kernel

1/ *Divergence of the L^2 estimates*

$$\Big| \int dz_1 \int_0^T d\tau C_{1,2}^{+} \mathbf{S}_2(\tau) f_N^{(2)} \Big| \leq C\sqrt{TN} \| f_N^{(2)} \|_{L^2}$$

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Difficulty to control multiple collisions.

$$Q_{s,s+n}(t) := \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} dt_n \dots dt_1 \, \mathbf{S}_s(t-t_1) C_{s,s+1} \\ \mathbf{S}_{s+1}(t_1 - t_2) C_{s+1,s+2} \dots \mathbf{S}_{s+n}(t_n)$$

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Disaster ! even for short time

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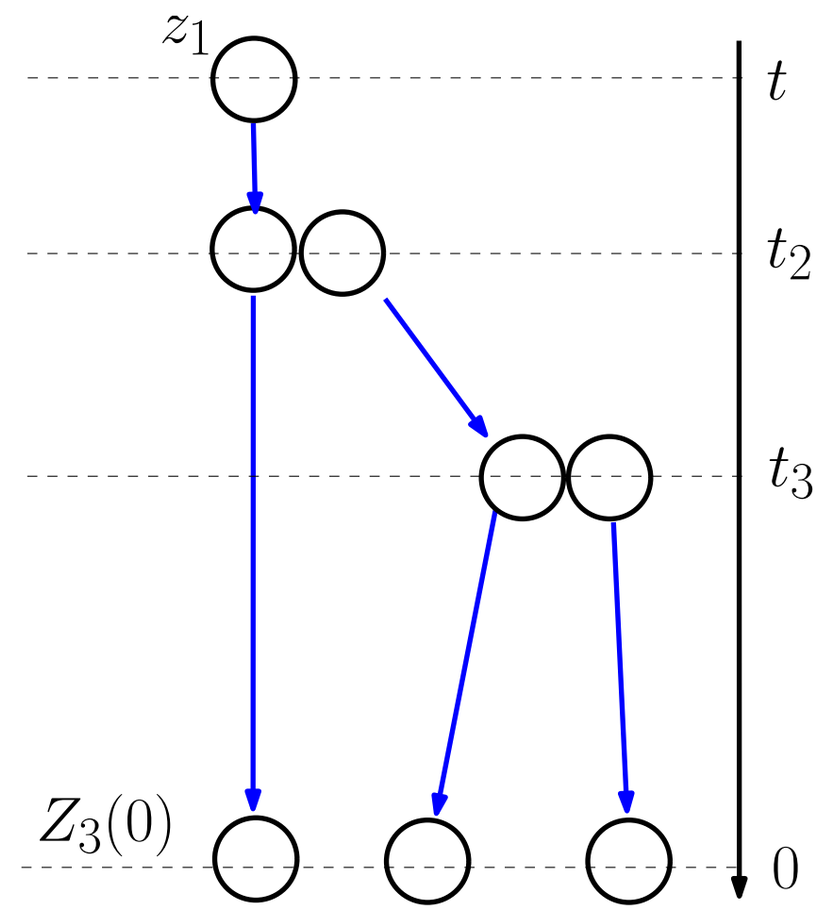
Difficulties to estimate the collision kernel

2/ *Recollisions*

Given a collision tree :

$$\int dz_1 \int_0^t dt_2 \int_0^{t_2} dt_3\, \mathbf{S}_1(t-t_1) C^+_{1,2}\, \mathbf{S}_2(t_2-t_3) C^+_{1,2}\, \mathbf{S}_3(t_3) f_N^{(3)}\big(Z_3(0)\big)$$



Use the change of variables

$$\Big(z_1, (t_2, \nu_2, v_2), (t_3, \nu_3, v_3)\Big) \to Z_3(0)$$

to recover $\|f_N^{(3)}\|_{L_1}$

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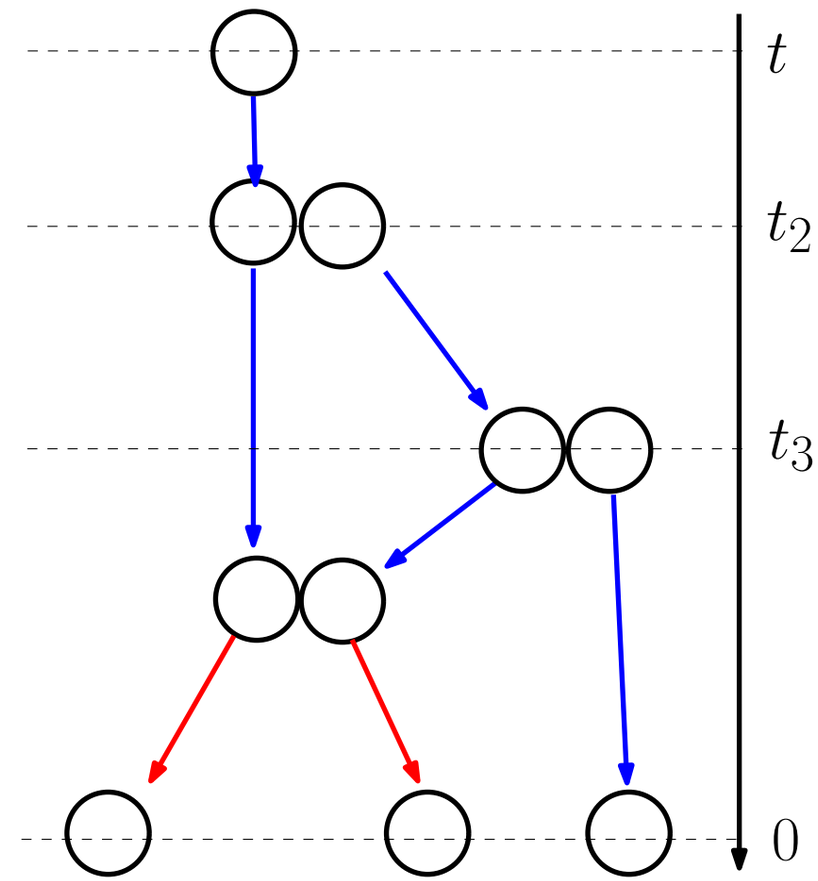

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Problem. This mapping is not bijective

One has to control the recollisions.

A mild version of local equilibrium

L^2 estimates would be fine if

$$f_N^{(s)}(t, Z_s) = M_\beta^{\otimes s}(V_s) \sum_{i=1}^{s} g(t, z_i)$$

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Key : L^2 control of the higher order correlations at **any** time

$$f_N^{(s)}(t, Z_s) = M_\beta^{\otimes s}(V_s) \sum_{m=1}^{s} \sum_{\sigma \in \mathfrak{S}_s^m} g_N^m(t, Z_\sigma)$$

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Consequence of the L^2 a priori bound

Proof : exchangeability of the measure.

Local equilibrium & the $\mathbb{L}^2$ estimates

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exact balance

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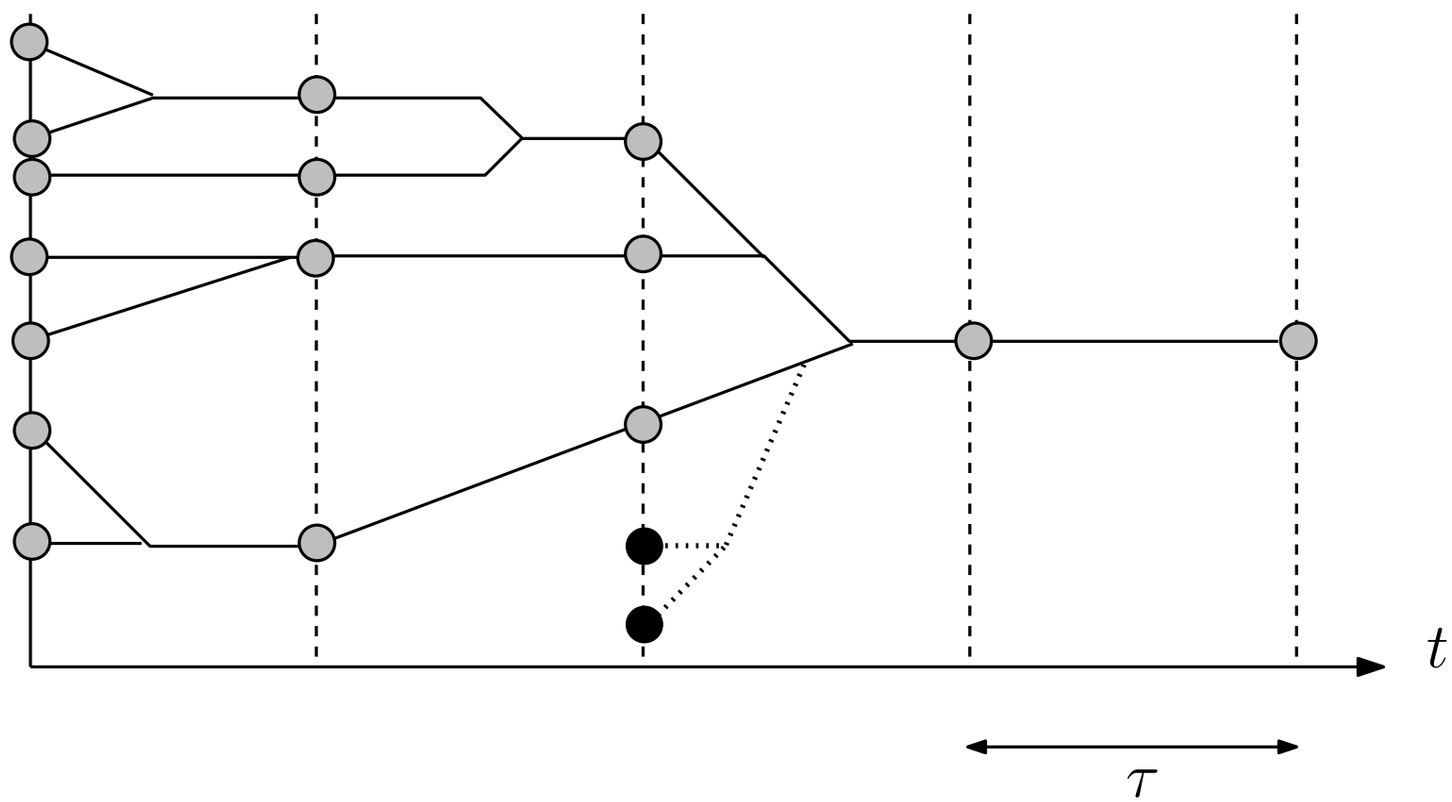
The $\mathbb{L}^2$ estimates can be used to truncate the series at any time thanks to the decomposition of the measure.

Pruning procedure

Decompose : $[0, t] = \bigcup_{k=1}^{K} [(k-1)\tau, k\tau]$ for some $\tau > 0$

Good collision trees.

Less than $n_k = 2^k$ collisions during $[(K-k)\tau, (K-k+1)\tau]$



In each time interval $[(K-k)\tau, (K-k+1)\tau]$ the $\mathbb{L}^2$ decomposition is used to estimate the cost of too many collisions

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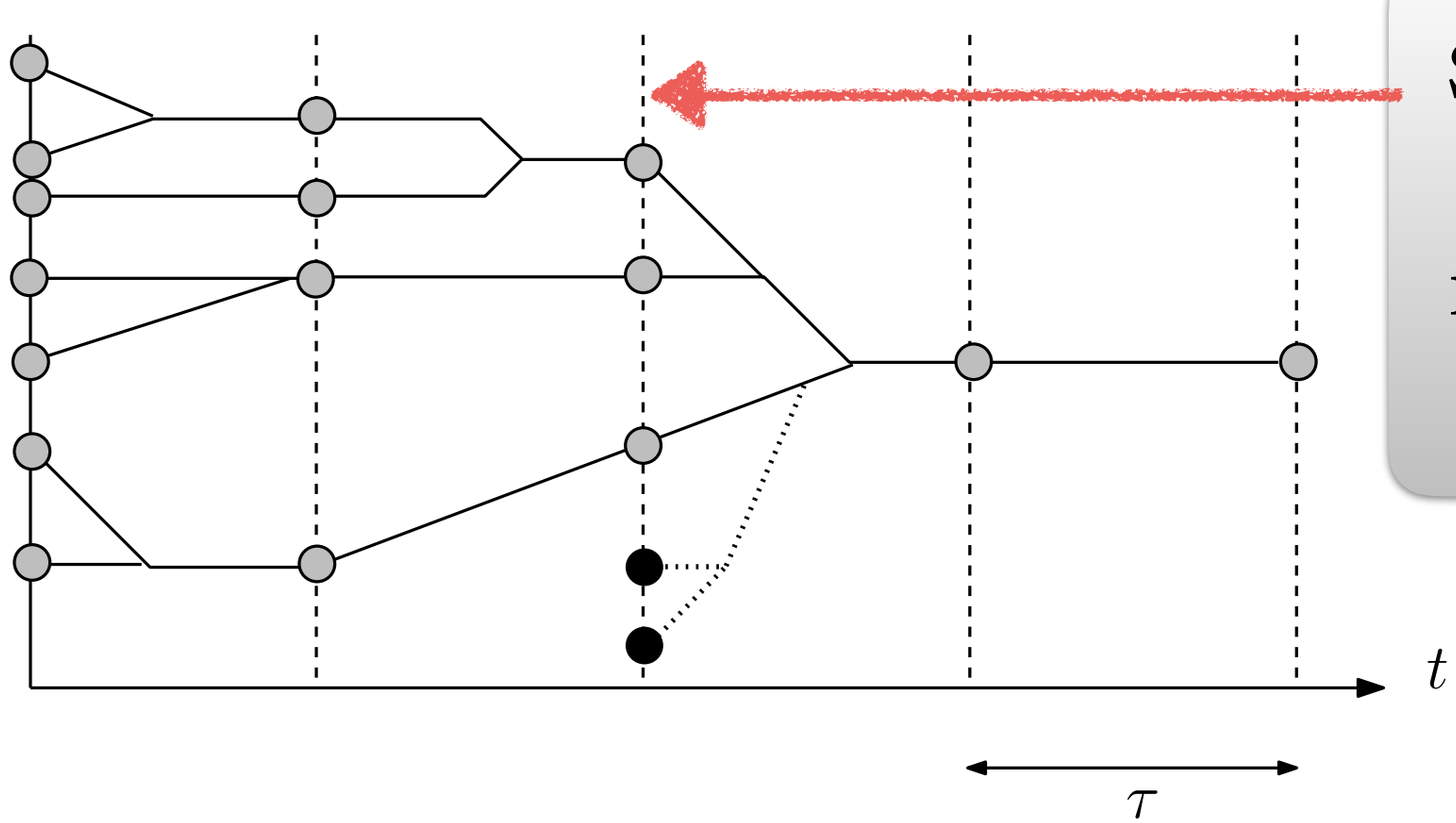

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Pruning procedure

$$f_N^{(1)}(t) = \sum_{j_1=0}^{2} \dots \sum_{j_K=0}^{2^K} \alpha^{J_k-1} Q_{1,J_1}(\tau) Q_{J_1,J_2}(\tau) \dots Q_{J_{K-1},J_K}(\tau)\, f_N^{0(J_K)} + R_N^K(t)$$

with $\quad J_\ell = 1 + j_1 + \cdots + j_\ell$

- The main contribution is given by the good collision trees with $j_k \leq 2^k$ during the time interval $[(K-k)\tau, (K-k+1)\tau]$
- The contribution of the large trees $R_N^K(t)$ is controlled in $\mathbb{L}^2$

$$\|R_N^K(t)\|_{\mathbb{L}^2} \leq C_\alpha \sqrt{\frac{t^4}{K}}$$

➪ If t is large, then K has to be very large and τ very small.

Pruning procedure

$$f_N^{(1)}(t) = \sum_{j_1=0}^{2} \dots \sum_{j_K=0}^{2^K} \alpha^{J_k-1} Q_{1,J_1}(\tau) Q_{J_1,J_2}(\tau) \dots Q_{J_{K-1},J_K}(\tau)\, f_N^{0(J_K)} + R_N^K(t)$$

with $\quad J_\ell = 1 + j_1 + \cdots + j_\ell$

- The main contribution is given by the good collision trees with $j_k \leq 2^k$ during the time interval $[(K-k)\tau, (K-k+1)\tau]$
- The contribution of the large trees $R_N^K(t)$ is controlled in $\mathbb{L}^2$

$$\|R_N^K(t)\|_{\mathbb{L}^2} \leq C_\alpha \sqrt{\frac{t^4}{K}}$$

➪ If t is large, then K has to be very large and τ very small.

$\mathbb{L}^\infty$ controls are required for multiple recollisions ...

Derivation of the linearized Boltzmann equation

Step 3. Comparison with the Boltzmann hierarchy

Boltzmann hierarchy

For $s \geq 1$ and $Z_s \in \mathbf{T}^{ds} \times \mathbb{R}^{ds}$

$$(\partial_t + \sum_{i=1}^{s} v_i \cdot \nabla_{x_i}) g^{(s)}(t, Z_s) = \alpha\big(C^0_{s,s+1}\big) g^{(s+1)}(t, Z_s)$$

where the collision term is defined by

$$\begin{aligned}
&\big(C^0_{s,s+1} g^{(s+1)}\big)(Z_s) \\
:= &\cancel{(N/(N-s))\varepsilon^{d-1}/\alpha} \sum_{i=1}^{s} \int_{\mathbf{S}^{d-1}\times\mathbb{R}^d} g^{(s+1)}(\ldots, x_i, v_i^*, \ldots, x_i \cancel{+\varepsilon\nu}, v_{s+1}^*)\Big((v_{s+1} - v_i)\cdot\nu\Big)_+ d\nu dv_{s+1} \\
&- \cancel{(N/(N-s))\varepsilon^{d-1}/\alpha} \sum_{i=1}^{s} \int_{\mathbf{S}^{d-1}\times\mathbb{R}^d} g^{(s+1)}(\ldots, x_i, v_i, \ldots, x_i \cancel{+\varepsilon\nu}, v_{s+1})\Big((v_{s+1} - v_i)\cdot\nu\Big)_- d\nu dv_{s+1}
\end{aligned}$$

This is the **limit** hierarchy when $\varepsilon \to 0$ and $N \to \infty$.

Boltzmann hierarchy

For $s \geq 1$ and $Z_s \in \mathbf{T}^{ds} \times \mathbb{R}^{ds}$

$$(\partial_t + \sum_{i=1}^{s} v_i \cdot \nabla_{x_i}) g^{(s)}(t, Z_s) = \alpha\big(C^0_{s,s+1}\big) g^{(s+1)}(t, Z_s)$$

Iterated Duhamel formula

$$g^{(1)}(t) = \sum_{n=0}^{\infty} \alpha^n Q^0_{1,1+n}(t) g^{(1+n)}(0)$$

Explicit solution :

$$g^{(s)}(t) = \left(\sum_{i=1}^{s} g_\alpha(t, z_1) \right) \prod_{i=2}^{s} M_\beta(v_i)$$

with $g_\alpha(t, z_1) M_\beta(v_1)$ solution of the **Linearized Boltzman equation**

Comparing the BBGKY and Boltzmann hierarchies

As $N \to \infty$ in the scaling $N\varepsilon^{d-1} = \alpha$,

$$\left| \left(f_N^{0(s)} - g^{0(s)} \right) \prod_{i \neq j} 1_{|x_i - x_j| > \varepsilon} \right| \leq C^s \varepsilon \alpha \, \mu \, M_\beta^{\otimes s}$$

for the **initial distributions**

$$f_N^0(Z_N) = M_{N,\beta}(Z_N) \left(\sum_{i=1}^N g_0(z_i) \right), \qquad \text{Microscopic dynamics}$$

$$g^{0(s)}(Z_s) = \left(\prod_{i=1}^s M_\beta(v_i) \right) \left(\sum_{i=1}^N g_0(z_i) \right), \qquad \text{Boltzmann hierarchy}$$

Main Goal

$$\| f_N^{(1)} - g^{(1)} \|_{L^2([0,t] \times \mathbf{T}^d \times \mathbb{R}^d)} \to 0 \,, \quad \text{as } N \to \infty$$

Comparing the truncated hierarchies

$$f_N^{(1,K)}(t) = \sum_{j_1=0}^{2} \cdots \sum_{j_K=0}^{2^K} \alpha^{J_K} Q_{1,J_1}(\tau) Q_{J_1,J_2}(\tau) \dots Q_{J_{K-1},J_K}(\tau)\, f_N^{0(J_K)}$$

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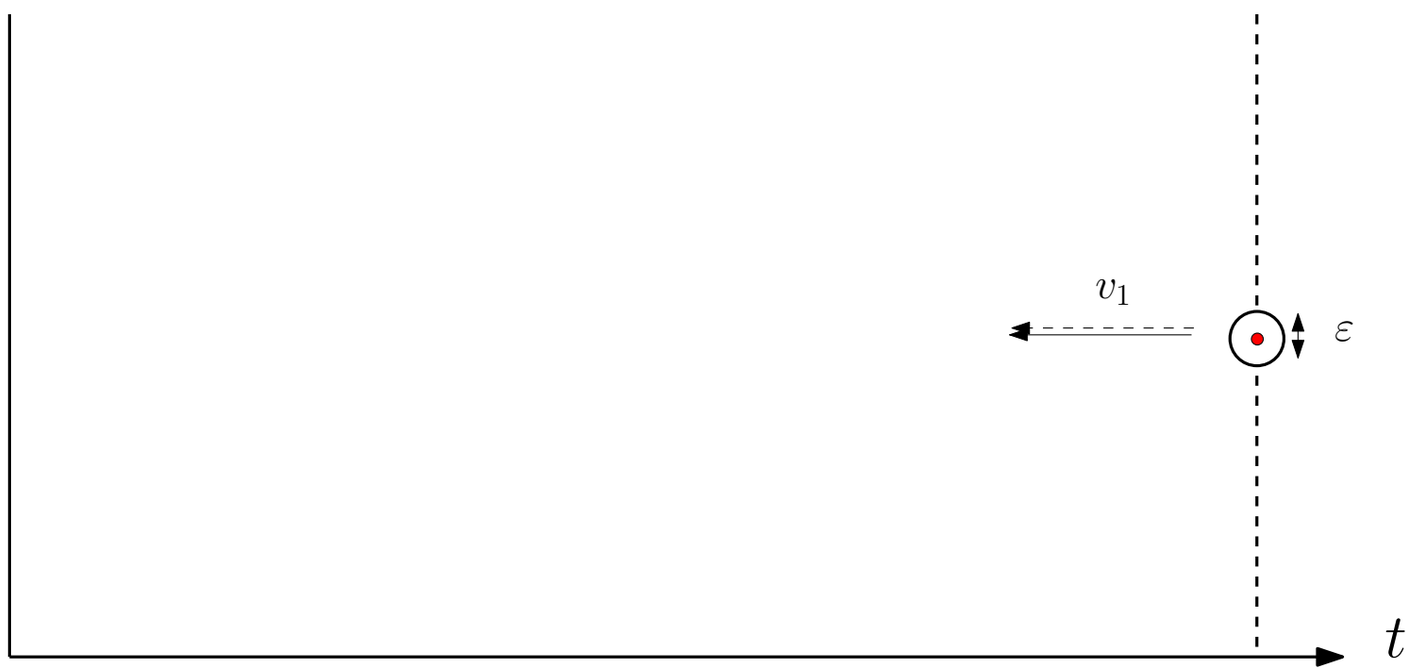
Geometric interpretation of the collisions operators:

Backward dynamics

Coupling the hierarchies



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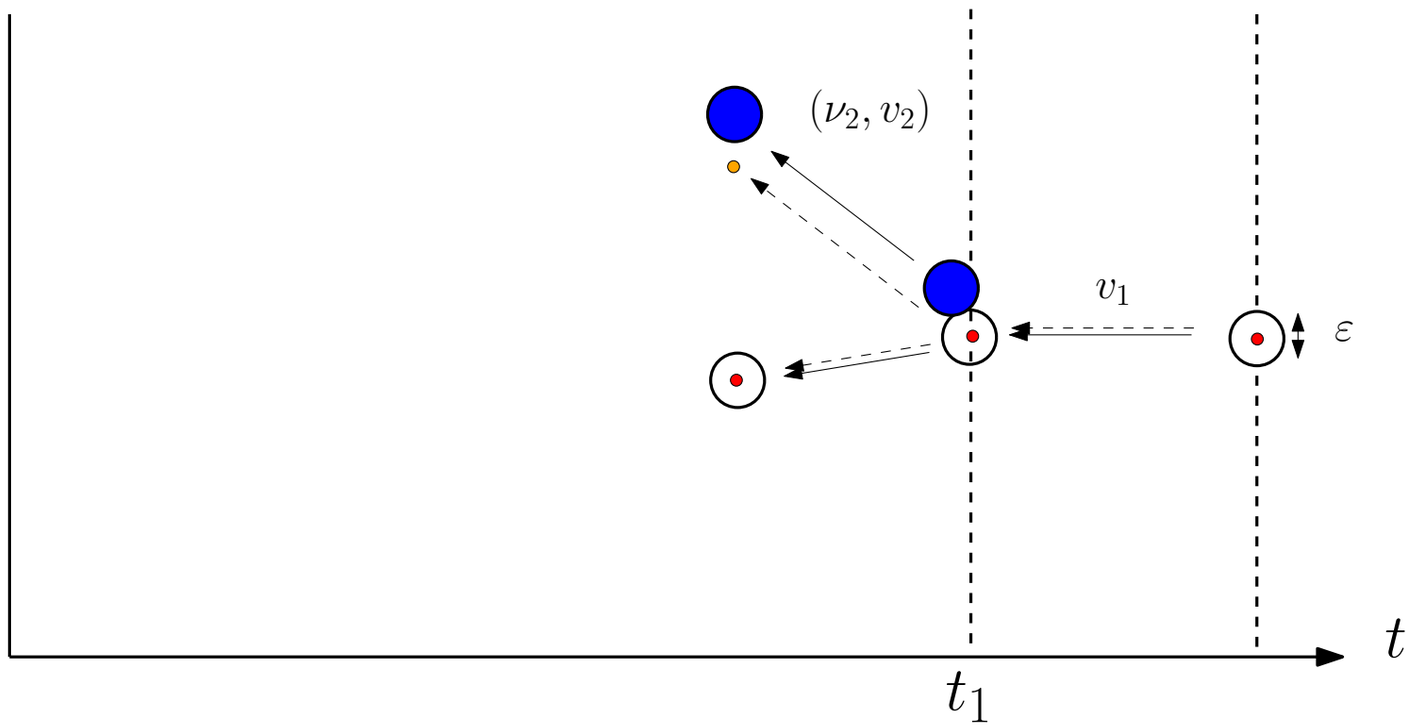

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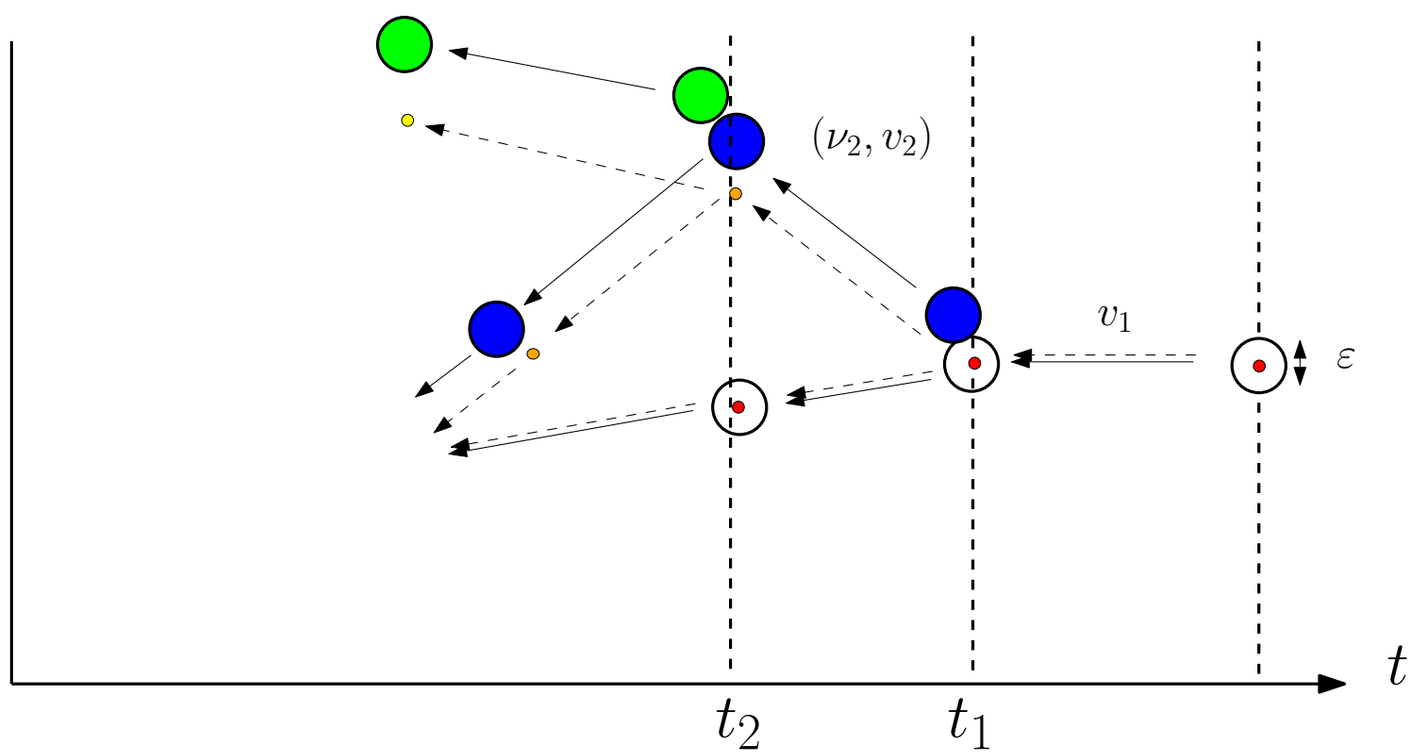

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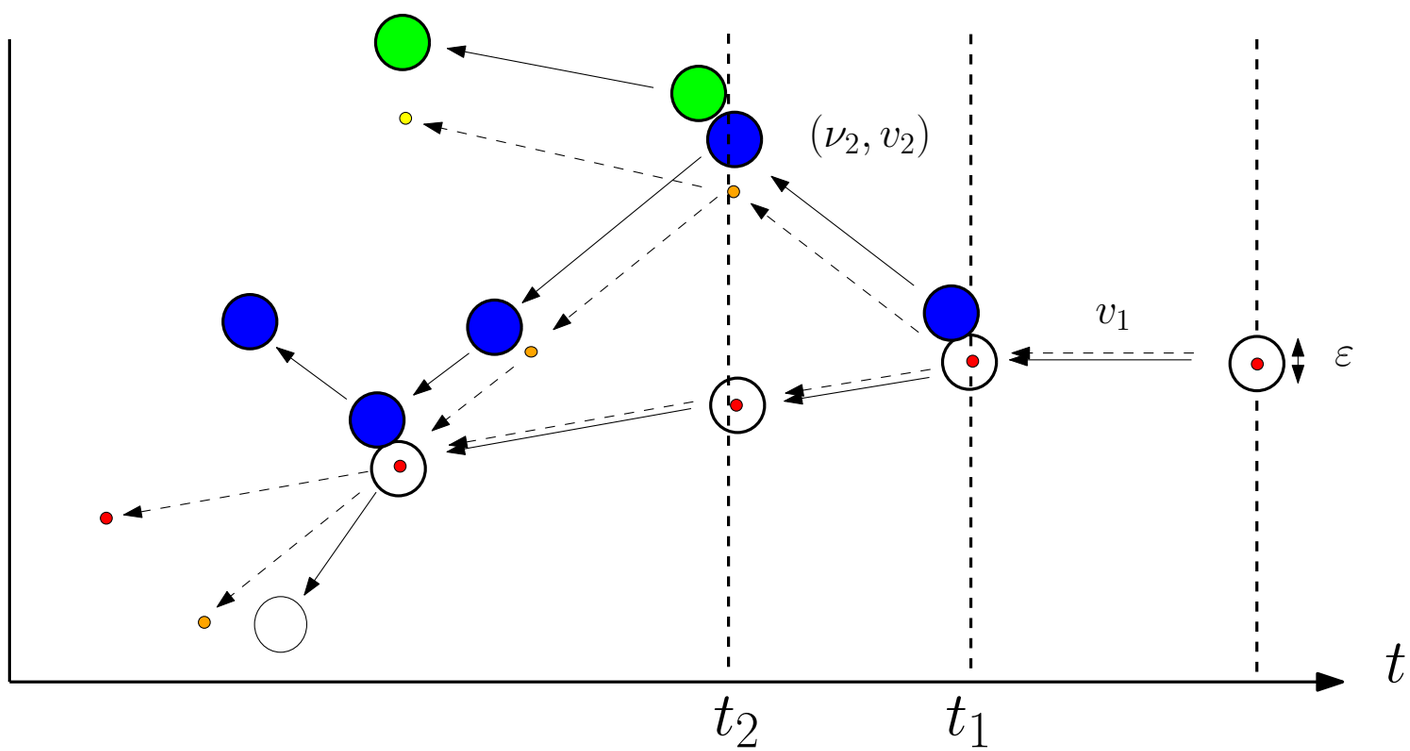
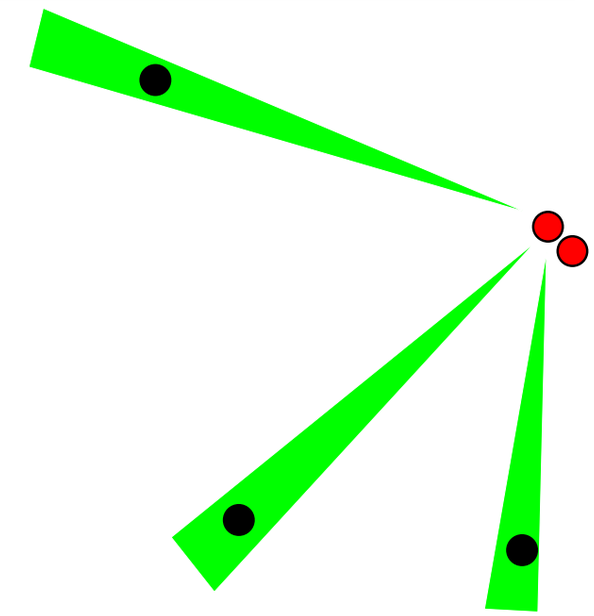


Removing the recollisions

BBGKY and Boltzmann trajectories can be coupled if there are no recollisions

Up to a small set of velocities, the pseudo-trajectories have no recollisions.

The cost of 1 recollision is bounded by $\varepsilon|\log\varepsilon|^3$



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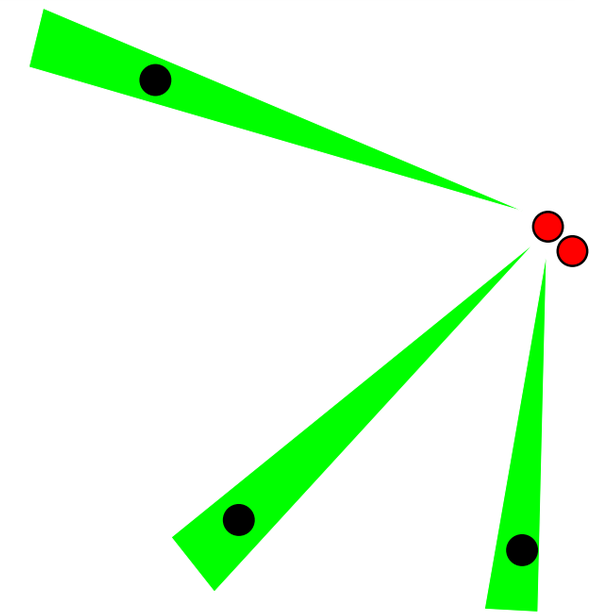

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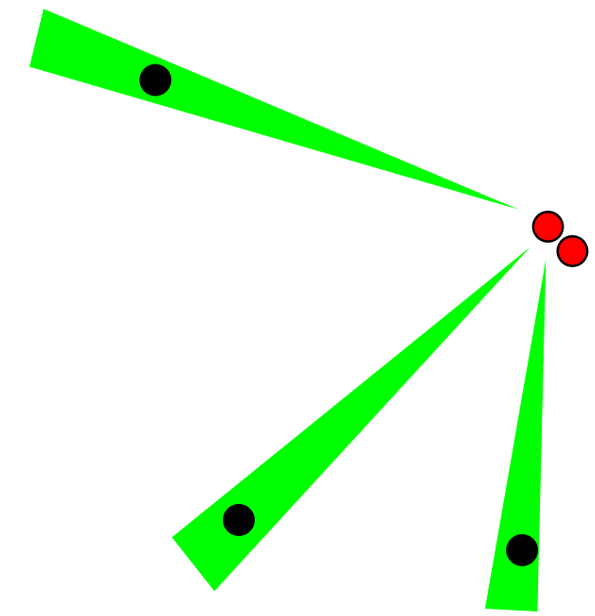

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Main Difficulty

The cost of observing at least 2 recollisions is less than ε.

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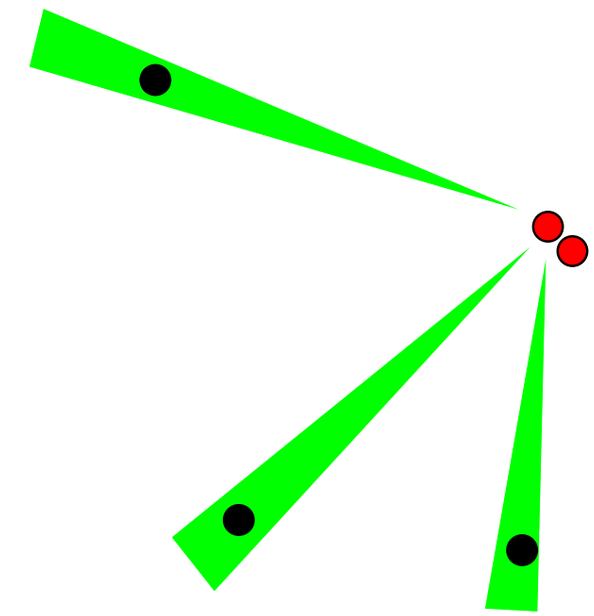

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Conclusion

Deterministic dynamics of a diluted hard-sphere gas :

- Covariance of the fluctuation field at large times
- Linearized Boltzmann equation & acoustic equations

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Open problems.

- Linearized Boltzmann equation in dimension 3
- Fluctuating Boltzmann equation [Spohn]