The asymmetric KMP model

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Outline

1) Symmetric case: how KMP is connected to other processes SIP, BEP (via so-called thermalization), and how this leads immediately to *a one-parameter family KMP*(α).
2) Duality of *KMP* (and all other dualities between the models of this family) *follows from self-duality of SIP*(α).
3) Self-duality of SIP(α) in turn follows *from its algebraic structure and consequent symmetries (commuting operators)*: the generator of SIP is the co-product of the Casimir in $\mathscr{U}(SU(1,1))$ (in a discrete representation).

4) To find the "correct asymmetric SIP(α) (ASIP(q,α)), *this algebraic construction has now to be performed in* $\mathscr{U}_q(SU(1,1))$. Built in the construction are symmetries and self-duality (comparable to Schütz self-duality of ASEP).

5) From ASIP(q,α) with weak asymmetry $q = 1 - \frac{\sigma}{N}$, we find a "diffusion limit" (many particle limit) called ABEP(q,α).

6) This ABEP(q,α) process then yields AKMP(σ,α) via thermalization. The AKMP(σ,α) has *the same dual as KMP*(α), all the asymmetry is put into *the duality function*.

The KMP process

The KMP (Kipnis, Marchioro, Presutti, J. Stat. Phys. 1982) process on a (finite) graph (S, E) is a Markov process $\{X(t), t \geq 0\}$ on $[0, \infty)^S$ (energies associated to vertices) described as follows

1. Every edge is selected with rate 1 (independently for different edges)
2. If the edge $e = (ij), i, j \in S$ is selected, then the energies x_i, x_j associated to the vertices of the edge are replaced by

$$\epsilon(x_i + x_j), (1 - \epsilon)(x_i + x_j)$$

with ϵ uniformly distributed on $[0, 1]$ (every time of updating independently chosen).

The discrete KMP process

The dKMP process on a (finite) graph (S, E) is a Markov process $\{\eta(t), t \geq 0\}$ on $[0, \infty)^S$ (particle numbers associated to vertices) described as follows

1. Every edge is selected with rate 1 (independently for different edges)
2. If the edge $e = (ij), i, j \in S$ is selected, then the particle numbers η_i, η_j associated to the vertices of the edge are replaced by

$$k_e, \eta_i + \eta_j - k_e$$

   where k_e is (discrete) uniformly distributed on $\{0, 1, 2, \ldots, \eta_i + \eta_j\}$ (every time of updating independently chosen).

Duality of KMP and dKMP

Putting

$$D(\eta, x) = \prod_{i \in S} \frac{x_i^{\eta_i}}{\eta_i!}$$

We have the duality

$$\mathbb{E}_\eta^{dKMP} D(\eta(t), x) = \mathbb{E}_x^{KMP} D(\eta, X(t))$$

Which implies e.g.

$$\mathbb{E}_x^{KMP}(X_i(t)) = \sum_j p_t(i,j) x_j$$

where $p_t(i,j)$ is the transition probability for continuous-time rate 1 simple random walk on (S, E).

Self-duality of dKMP

The dKMP is *self-dual*: putting

$$D(\xi,\eta) = \prod_{i\in S} \frac{\eta_i!\Gamma(1)}{(\eta_i - \xi_i)!\Gamma(1+\xi_i)} = \prod_i \binom{\eta_i}{\xi_i}$$

We have

$$\mathbb{E}^{dKMP}_\xi D(\xi(t),\eta) = \mathbb{E}^{dKMP}_\eta D(\xi,\eta(t))$$

Further relations between dKMP and KMP

- *The KMP is the "many particle limit" of dKMP.* Taking $\eta_i = \lfloor x_i N \rfloor$ in dKMP and denoting η_t^N its time-evolution under dKMP, we have, when $N \to \infty$

$$\frac{\eta^N(t)}{N} \to X(t)$$

  with $X_i(0) = x_i$
- The duality between dKMP and KMP can thus be derived from *the self-duality of dKMP* via

$$\lim_{N\to\infty} \frac{1}{N^{\xi_i}} \binom{\lfloor x_i N \rfloor}{\xi_i} = \frac{x_i^{\xi_i}}{\xi_i!}$$

A one-parameter family of KMP models

Given $\alpha > 0$ we define KMP(α) as the Markov process $\{X(t), t \geq 0\}$ on $[0, \infty)^S$ (energies associated to vertices) described as follows

1. Every edge is selected with rate 1 (independently for different edges)
2. If the edge $e = (ij), i, j \in S$ is selected, then the energies x_i, x_j associated to the vertices of the edge are replaced by

   $$\epsilon(x_i + x_j), (1 - \epsilon)(x_i + x_j)$$

   with ϵ $Beta(\alpha, \alpha)$ distributed on $[0, 1]$ (every time of updating independently chosen).

The discrete KMP process

The dKMP(α) process on a (finite) graph (S, E) is a Markov process $\{\eta(t), t \geq 0\}$ on $[0, \infty)^S$ (particle numbers associated to vertices) described as follows

1. Every edge is selected with rate 1 (independently for different edges)
2. If the edge $e = (ij), i, j \in S$ is selected, then the particle numbers η_i, η_j associated to the vertices of the edge are replaced by

$$k_e, \eta_i + \eta_j - k_e$$

   where k_e is (discrete) Beta(α, α) binomial distributed on $\{0, 1, 2, \ldots, \eta_i + \eta_j\}$ (every time of updating independently chosen). Beta Binomial is defined via

$$P(k_e = n) = \binom{\eta_i + \eta_j}{n} \mathbb{E}(p^n (1-p)^{\eta_i + \eta_j - n})$$

   where $\mathbb{E}$ is w.r.t. p according to $Beta(\alpha, \alpha)$ distribution.

Self-duality of $d\text{KMP}(\alpha)$

The $d\text{KMP}(\alpha)$ is also self-dual: putting

$$D(\xi,\eta)=\prod_{i\in S}\frac{\eta_i!\Gamma(\alpha)}{(\eta_i-\xi_i)!\Gamma(\alpha+\xi_i)}$$

then we have

$$\mathbb{E}_\xi^{d\text{KMP}(\alpha)}D(\xi(t),\eta)=\mathbb{E}_\eta^{d\text{KMP}(\alpha)}D(\xi,\eta(t))$$

from this we can derive, as before, duality of $\text{KMP}(\alpha)$ and $d\text{KMP}(\alpha)$ with

$$D(\eta,x)=\prod_i\frac{x_i^{\eta_i}\Gamma(\alpha)}{\Gamma(\alpha+\eta_i)}$$

Thermalization

For a process on X^S ($X = [0, \infty)$ or $X = \mathbb{N}$) with generator of type

$$L = \sum_{e \in E} L_e$$

we define its thermalization as

$$\mathscr{T}(L) := \mathscr{L} = \sum_{e \in E} \mathscr{L}_e$$

with

$$\mathscr{L}_e f = \lim_{t \to \infty} (e^{tL_e} - I) f$$

Notice that this is a kind of projection, i.e.,

$$\mathscr{T}(\mathscr{T}(L)) = \mathscr{T}(L)$$

Relation between $SIP(\alpha)$ and $dKMP(\alpha)$

In the SIP(α) only one particle jumps at a time and a particle hops from i to j (if $ij \in E$) at rate

$$r(\eta_i, \eta_j) = \eta_i(\alpha + \eta_j)$$

So the generator reads

$$L^{\mathrm{SIP}(\alpha)} = \sum_{e=ij\in E} \left[r(\eta_i, \eta_j)(f(\eta^{ij}) - f(\eta)) + r(\eta_j, \eta_i)(f(\eta^{ji}) - f(\eta)) \right]$$

We then have

$$L^{d\mathrm{KMP}(\alpha)} = \mathscr{T}(L^{\mathrm{SIP}(\alpha)})$$

i.e., *$dKMP(\alpha)$ is the thermalization of $SIP(\alpha)$*.

Self-duality of SIP(α)

$$D(\xi, \eta) = \prod_{i \in S} \frac{\eta_i! \Gamma(\alpha)}{(\eta_i - \xi_i)! \Gamma(\alpha + \xi_i)}$$

then we have

$$\mathbb{E}_\xi^{\mathrm{SIP}(\alpha)} D(\xi(t), \eta) = \mathbb{E}_\eta^{\mathrm{SIP}(\alpha)} D(\xi, \eta(t))$$

This self-duality is the "source" duality from which all the others follow (by taking many particle limits or thermalizations)

Brownian energy process BEP(α)

If one takes the many particle limit $\eta_i = \lfloor N x_i \rfloor$ in the SIP(α) we obtain a process of diffusion type with generator

$$L^{\mathrm{BEP}(\alpha)} = \sum_{ij=e\in E} \left[x_i x_j (\partial_i - \partial_j)^2 - 2\alpha (x_i - x_j)(\partial_i - \partial_j) \right]$$

From self-duality of SIP(α), one infers duality of this process with SIP(α) with

$$D(\eta, x) = \prod_{i\in S} \frac{x_i^{\eta_i} \Gamma(\alpha)}{\Gamma(\alpha + \eta_i)}$$

Moreover, the thermalization of this process is the process KMP(α).

Self-duality and symmetries

The self-duality of SIP(α) follows from its algebraic structure. The self-duality of a process with generator L can (in most cases) be summarized via

$$L_{\text{left}} D(\xi, \eta) = L_{\text{right}} D(\xi, \eta)$$

We denote this by $L \longrightarrow^{D} L$ In the finite state space case this relation reads in matrix form

$$LD = DL^T$$

The following fact connects symmetries with self-duality functions: if S commutes with L, i.e., if

$$[S, L] = SL - LS = 0$$

then

$$L \longrightarrow^{D} L$$

implies

$$L \longrightarrow^{S_{\text{left}} D} L$$

i.e., from a given self-duality function and a symmetry one can produce a new self-duality function.

A "cheap" self-duality function is given by

$$D_{\text{cheap}}(\xi, \eta) = \frac{1}{\mu(\xi)} \delta_{\xi,\eta}$$

where μ is a reversible measure. Other, more useful self-dualities can then be made by acting with symmetries on this one (provided we have symmetries). In this sense, self-duality can be viewed as a generalization of reversibility (from diagonal to non-diagonal D).

Symmetries of the SIP generator

The single edge generator of SIP(α) is

$$\left[r(\eta_i, \eta_j)(f(\eta^{ij}) - f(\eta)) + r(\eta_j, \eta_i)(f(\eta^{ji}) - f(\eta))\right]$$

where we remind $r(k, n) = k(\alpha + n)$. In order to discover its symmetries, we have to go to its algebraic structure

We introduce the following operators working on functions $f : \mathbb{N} \to \mathbb{R}$.

$$\begin{aligned} K^+ f(n) &= (\alpha + n) f(n+1) \\ K^- f(n) &= n f(n-1) \\ K^0 f(n) &= \left(\frac{\alpha}{2} + n\right) f(n) \end{aligned} \tag{1}$$

These operators K^+, K^-, K^0 satisfy

$$[K^{\pm}, K^0] = \pm K^{\pm}, \ [K^+, K^-] = 2K^0 \tag{2}$$

These are *the commutation relations of the dual algebra of* $\mathscr{U}(SU(1,1))$ (the commutation relations of $\mathscr{U}(SU(1,1))$ being the same with opposite signs, i.e. $[K^0, K^{\pm}] = \pm K^{\pm}, [K^-, K^+] = 2K^0$).

In terms of these operators the single edge generator of SIP(α) reads

$$L_{12} = K_1^+ K_2^- + K_1^- K_2^+ - 2K_1^0 K_2^0 + \frac{\alpha^2}{2} \tag{3}$$

This operator L_{12} is naturally related to a distinguished central element of $\mathscr{U}(SU(1,1)$,

$$C = (K^0)^2 - \frac{1}{2}(K^+ K^- + K^- K^+) \tag{4}$$

the so-called Casimir element. This is the reason that this operator has many symmetries.

First we define the co-product on the generating elements: for $u \in \{+, -, 0\}$

$$\Delta(K^u) = K^u \otimes I + I \otimes K^u = K_1^u + K_2^u \tag{5}$$

and extend Δ to a homomorphism between the algebras $\mathscr{A}$ and $\mathscr{A} \otimes \mathscr{A}$. $\Delta : \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$ is then called *coproduct*. It has the property (co-associativity)

$$(\Delta \otimes I)\Delta = (I \otimes \Delta)\Delta$$

which allows to consider iterated coproducts, e.g., $\Delta^2 : \mathscr{A} \to \mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A}$

$$\Delta^2(K^u) = (\Delta \otimes I)\Delta(K^u) = K_1^u + K_2^u + K_3^u, \ u \in \{-, +, 0\}$$

We have

$$\Delta(-C) = (K_1^+ K_2^- + K_2^+ K_1^-) - 2K_1^0 K_2^0 - C_1 - C_2 \qquad (6)$$

As a consequence, the generator L_{12} commutes with $\Delta(A)$ for every algebra-element (because C is central and Δ preserves commutators). In particular L_{12} commutes with

$$K_1^u + K_2^u, u \in \{0, +, -\}$$

These symmetries are responsible for the self-duality of SIP(α):

$$D = e^{K_1^+ + K_2^+} D_{\text{cheap}}$$

Taking the exponential is natural because we want factorized (over vertices) self-dualities.

Summary so far

- The generator (on two edges) of the SIP(α) is the coproduct applied to the Casimir operator (in the discrete representation).
- As a consequence, the generator (on two edges) of the SIP(α) has many commuting elements (symmetries).
- The self-duality of SIP(α) follows immediately from the application of a symmetry ($e^{K_1^+ + K_2^+}$) on a trivial self-duality function coming from the reversible product measure.
- All dualities and self-dualities of processes related to SIP(α) (BEP(α), dKMP(α), KMP(α)) follow from this self-duality of SIP(α), and taking limits and or thermalizations.

The asymmetric inclusion process

Now we *start from deformed algebra* $\mathscr{U}_q(SU(1,1))$ with commutation relations

$$[K^+, K^-] = -[2K^0]_q, [K^0, K^\pm] = \pm K^\pm$$

$0 < q < 1$ is the parameter tuning the asymmetry. q-numbers are defined via

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

The Casimir element of $\mathscr{U}_q(SU(1,1))$ is given by

$$C = [K^0]_q[K^0 - 1]_q - K^+K^-$$

and the coproduct on the generating elements is given by

$$\begin{aligned} \Delta(K^\pm) &= K^\pm \otimes q^{-K^0} + q^{K^0} \otimes K^\pm \\ \Delta(K^0) &= K^0 \otimes I + I \otimes K^0 \end{aligned}$$

iterated coproducts via

$$\Delta^n = (\Delta \otimes I)(\Delta^{n-1})$$

We now have to start from the q-deformed version $\mathscr{U}_q(SU(1,1))$, and apply the same strategy:

- Copy the coproduct of the Casimir along the edges $(i, i+1)$ of the finite graph $\{1, 2, \ldots, L\}$. This gives an operator of the form

$$H = \sum_{i=1}^{L-1} h_{i,i+1}$$

which is not yet a Markov generator, but of the form

$$Hg = Lg - \varphi g$$

i.e., a Markov generator minus a multiplication operator.

- Turn the Hamiltonian operator thus obtained into a generator via a "ground-state transformation": if $He^f = 0$ (positive groundstate) then

$$\mathscr{L} g = e^{-f} H(e^f g)$$

  is a Markov generator. The symmetries of H are in one-to-one correspondence with the symmetries of $\mathscr{L}$.
- The analogue of the "exponential symmetries" $e^{\sum_i K_i^u}$ are a well-chosen q-deformed exponential of $\Delta^{(L-1)}(K^u)$. These symmetries then yield the self-dualities of the process with generator $\mathscr{L}$

Explicitly, we have the generator of SIP(q,α) is given by

$$\mathscr{L} = \sum_i \mathscr{L}_{i,i+1}$$

with

$$\begin{aligned} \mathscr{L}_{i,i+1} f(\eta) &= q^{\eta_i - \eta_{i+1} + (\alpha - 1)} [\eta_i]_q [\alpha + \eta_{i+1}]_q (f(\eta^{i,i+1}) - f(\eta)) \\ &+ q^{\eta_i - \eta_{i+1} - (\alpha - 1)} [\eta_{i+1}]_q [\alpha + \eta_i]_q (f(\eta^{i+1,i}) - f(\eta)) \end{aligned}$$

This process is self-dual with self-duality functions

$$D(\xi^{l_1,\ldots,l_n}, \eta) = \frac{q^{-2\alpha \sum_{m=1}^n l_m - n^2}}{q^\alpha - q^{-\alpha}} \prod_{m=1}^n (q^{2N_{l_m}(\eta)} - q^{2N_{l_{m+1}}(\eta)})$$

where $\xi^{l_1,\ldots,l_n}$ denotes the configuration with particles at the n *different location* $l_1, \ldots, l_m$, and

$$N_i(\eta) = \sum_{j=i}^L \eta_j$$

is the number of particles to the right of i.

The ABEP(σ, α)

Now we take the limit $q = 1 - \frac{\sigma}{N}$ (weak asymmetry), $\eta_i = \lfloor N x_i \rfloor$ (many particles) in the ASIP(q, α) and we find a diffusion (in limit $N \to \infty$) process called ABEP(σ, α) with generator

$$\mathscr{L} = \sum_{i=1}^{L-1} \mathscr{L}_{i,i+1}$$

$$\begin{aligned} \mathscr{L}_{i,i+1} &= \frac{1}{4\sigma^2} \left(1 - e^{-2\sigma x_i}\right) \left(e^{2\sigma x_{i+1}} - 1\right) (\partial_i - \partial_{i+1})^2 \\ &- \frac{1}{2\sigma} \left(\left(1 - e^{-2\sigma x_i}\right) \left(e^{2\sigma x_{i+1}} - 1\right) + 2\alpha(2 - e^{-2\sigma x_i} - e^{2\sigma x_{i+1}})\right) \\ &\quad \times (\partial_i - \partial_{i+1}) \end{aligned}$$

Duality of ABEP(σ, α)

- From the self-duality of ASIP(q, α) we obtain *duality between ABEP(σ, α) and SIP(α)* with duality functions

$$D^\sigma(\xi, x) = \prod_{i=1}^{L} \frac{\Gamma(\alpha)}{\Gamma(\alpha+\xi_i)} \left(\frac{e^{-2\sigma E_{i+1}(x)} - e^{-2\sigma E_i(x)}}{2\sigma} \right)^{\xi_i}$$

  with $E_i(x) = \sum_{j=i}^{L} x_i$ is the total energy to the right of i.
- i.e., in the dual process, the asymmetry is disappearing, and the only trace of the asymmetry is in the duality function.
- So this is an example of a truly bulk-asymmetric process dual to a symmetric process.

Example

$$\mathbb{E}_x^{\text{ABEP}(\sigma,\alpha)}(e^{-2\sigma J_i(x(t))}) = \sum_k p_t(i,k) e^{-2\sigma(E_k(x) - E_i(x))}$$

The AKMP(σ, α)

The AKMP(σ, α) is then defined as the *thermalization of ABEP*(σ, α) This gives the following process: the energies of every edge are (at rate 1) updated according to

$$(x_i, x_{i+1}) \to (B_\sigma^{(x_i+x_{i+1})}(x_i + x_{i+1}), (1 - B_\sigma^{(x_i+x_{i+1})})(x_i + x_{i+1}))$$

with B_σ^E a random variable on $[0, 1]$ with probability density

$$f_{B_\sigma^E} = C_{E,\sigma,\alpha}^{-1} e^{2\sigma E w}((e^{2\sigma E w} - 1)(1 - e^{-2\sigma E(1-w)}))^{\alpha - 1}$$

$$C_{E,\sigma,\alpha} = \int_0^1 e^{2\sigma E w}((e^{2\sigma E w} - 1)(1 - e^{-2\sigma E(1-w)}))^{\alpha - 1} dw$$

which is the asymmetric analogue of the *Beta*(α, α) distribution in KMP(α).

This AKMP(σ, α) is dual to *the dKMP*(α) with the duality functions

$$D^{\sigma}(\xi, x) = \prod_{i=1}^{L} \frac{\Gamma(\alpha)}{\Gamma(\alpha + \xi_i)} \left(\frac{e^{-2\sigma E_{i+1}(x)} - e^{-2\sigma E_i(x)}}{2\sigma} \right)^{\xi_i}$$

Two open questions

- For SIP(α) we can characterize *all self-duality functions* among which there are also orthogonal polynomials (Franceschini, Giardinà; R., Sau). Can this be done also in the asymmetric case?
- Are there "correct" reservoirs for ABEP(σ, α) (or AKMP(σ, α)) such that the dual has absorbing boundaries?

Thanks for your attention !