Discrete interface dynamics and hydrodynamic limits

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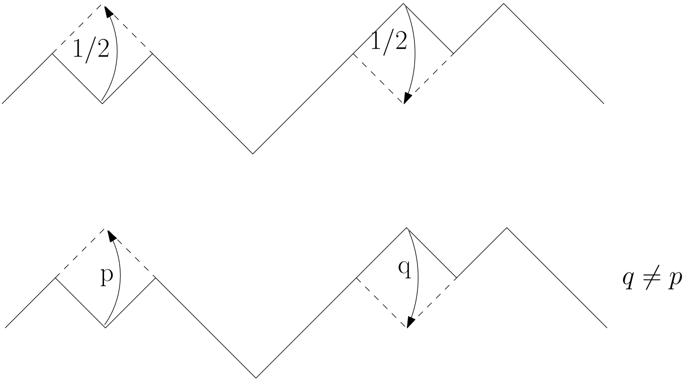
Framework: stochastic interface dynamics

Interface dynamics modeled by (reversible or irreversible) Markov chains with local update rules.
Typical questions:

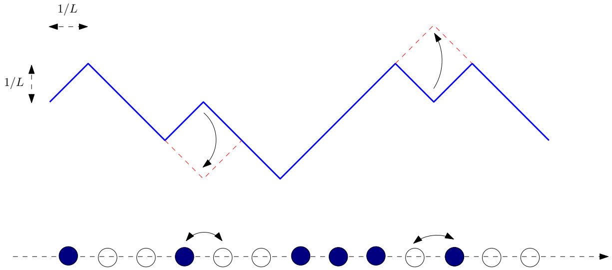
- stationary states (for *interface gradients*)
- space-time correlations of height fluctuations
- hydrodynamic limit
- formation of shocks
- ...

Main object of this talk: $(2+1)$-dimensional models (related to lozenge tilings) where these questions can be (partly) answered

Symmetric vs. asymmetric random dynamics

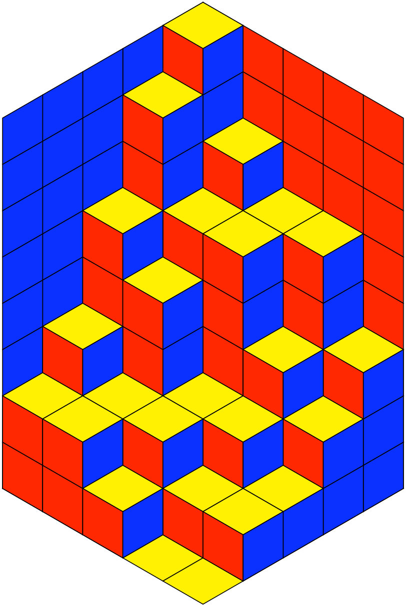


For $d = 1$: Symmetric vs. Asymmetric Simple Exclusion Process

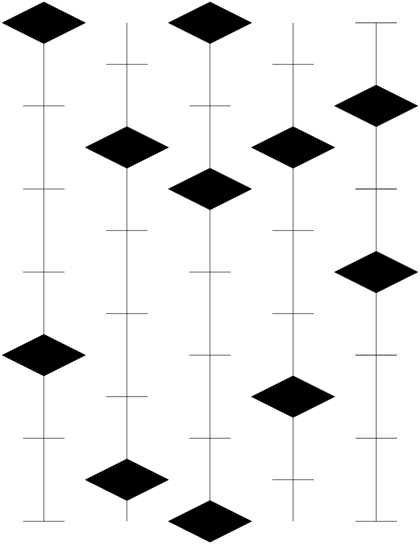


In both SSEP/ASEP, Bernoulli(ρ) are invariant.
For $p \neq q$, irreversibility (particle flux).

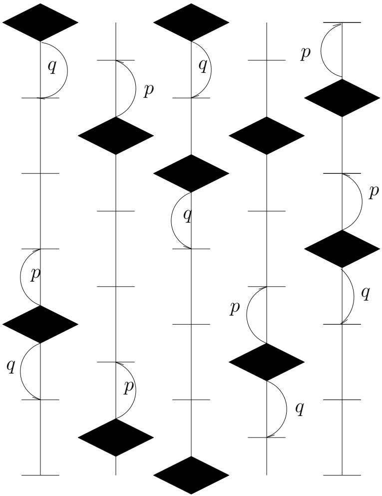
Generalization to $(2+1)$ dimensions



Interlaced particle configurations



The "single-flip dynamics"



“Analog” of Bernoulli measures: Ergodic Gibbs measures

- Choose $\rho = (\rho_1, \rho_2, \rho_3)$ with $\rho_i \in (0,1), \rho_1 + \rho_2 + \rho_3 = 1$. There exists a unique translation invariant, ergodic Gibbs measure π_ρ s.t. the density of horizontal, NW and NE lozenges are ρ_1, ρ_2, ρ_3.

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- lozenge densities $\rho \Leftrightarrow$ average interface slope $s_\rho \in \mathcal{P}$.
- height function $\sim$ massless Gaussian field: if $\int_{\mathbb{R}^2} \varphi(x)dx = 0$,

$$\epsilon^2 \sum_x \varphi(\epsilon x) h_x \stackrel{\epsilon \to 0}{\longrightarrow} \int \varphi(x) X(x) dx$$

with $\langle X(x) X(y) \rangle = -\frac{1}{2\pi^2} \log |x - y|$.

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  [P. Caputo, F. Martinelli, F. T., CMP '12, B. Laslier, F. T., CMP '15]
- Unknown: convergence to hydrodynamic limit after diffusive space-time rescaling: $t = \tau L^2, x = \xi L$

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- Stationary states: unknown. Presumably very different from π_ρ. Numerical simulations [Forrest-Tang-Wolf Phys Rev A 1992] show $t^{0.24\dots}$ growth of height fluctuations.
- *non-explicit* hydrodynamic limit (hyperbolic rescaling):

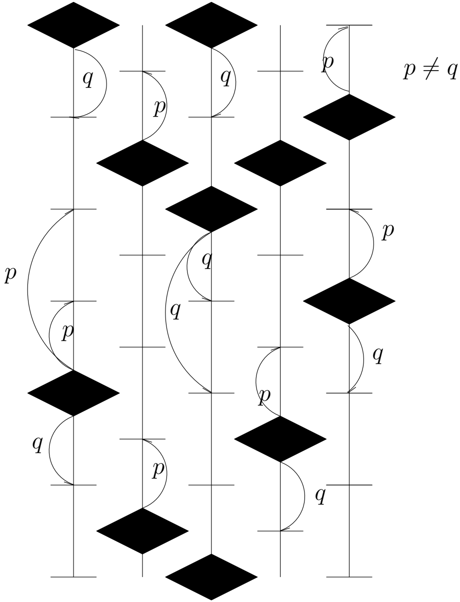
$$\lim_{L\to\infty} \frac{1}{L} h(xL, tL) = \phi(x,t) \quad \text{almost surely},$$

where ϕ is Hopf-Lax solution of

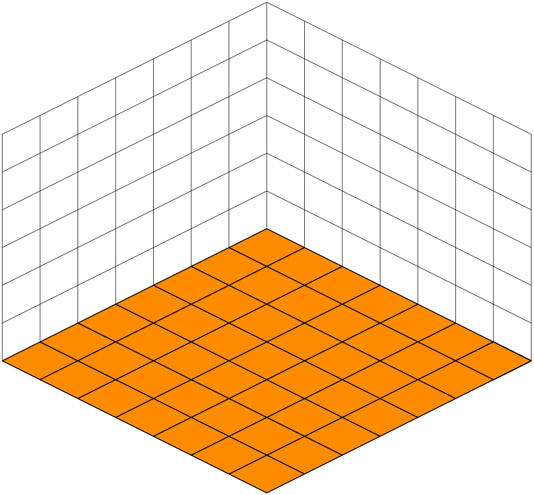
$$\partial_t \phi + V(\nabla\phi) = 0$$

for some convex and unknown $V(\cdot)$.
Super-addivity method [Seppäläinen, Rezakhanlou]

Part I: A growth process with longer jumps



Dynamics well defined?



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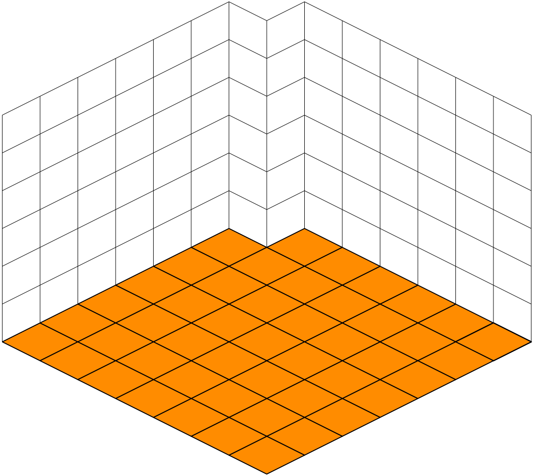

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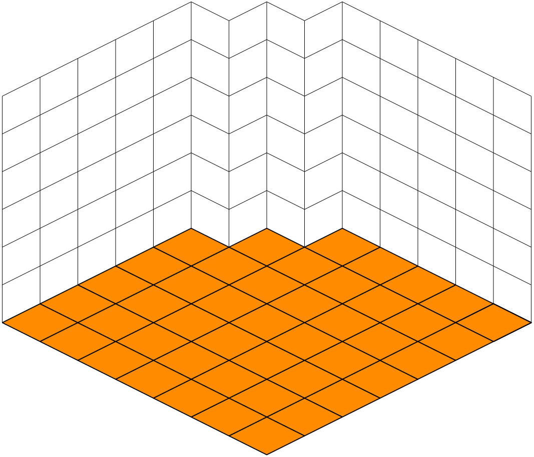

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An "integrable" growth process

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For a special deterministic initial condition, certain space-time correlations of particle occupations given by determinants:

$$\mathbb{P}(\text{ particle at } (x_i, t_i), i \leq N) = N \times N \text{ determinant} \tag{1}$$

An "integrable" growth process

This allowed Borodin-Ferrari to obtain various results:

- hydrodynamic limit:

$$\lim_{L\to\infty} \frac{1}{L} h(xL, \tau L) = \phi(x, \tau),$$

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We want to treat "generic" initial conditions.

The stationary process

Theorem 1 [F. T., Ann. Probab. 2017+]

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  (simplified/improved result in [S. Chhita, P. L. Ferrari, F.T. '17])

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$$v(\nabla\phi) = -\frac{1}{\pi}\frac{\sin(\pi\partial_{x_1}\phi)\sin(\pi\partial_{x_2}\phi)}{\sin(\pi(1-\partial_{x_1}\phi-\partial_{x_2}\phi))}$$

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- Explicit computation shows that the Hessian of $v(\rho)$ has signature $(+,-)$.
- Theorem 1 extends to a growth model on domino tilings of the plane [S. Chhita, P. L. Ferrari '15, Chhita-Ferrari-F.T. '17]

A hydrodynamic limit

Theorem 2 [M. Legras, F. T., arXiv '17]
Totally asymmetric case: $p = 0, q = 1$.
• If the initial condition approximates a smooth profile:

$$\lim_{L} \frac{1}{L} h(xL) = \phi_0(x)$$

with $\nabla \phi_0(x) \in \mathring{\mathcal{P}}$, then

$$\lim_{L} \frac{1}{L} h(xL, tL) = \phi(x,t), \quad t \leq T_{shocks}$$

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• convergence to viscosity solution for $t > T_{shocks}$ if initial profile is convex.

Remarks on the hydrodynamic limit

- $v(\cdot)$ has singularities

$$\text{(Recall: } v(\nabla\phi) = -\frac{1}{\pi}\frac{\sin(\pi\partial_{x_1}\phi)\sin(\pi\partial_{x_2}\phi)}{\sin(\pi(1-\partial_{x_1}\phi-\partial_{x_2}\phi))}\text{)}$$

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- $v(\cdot)$ *neither concave nor convex*. Theorem cannot be obtained by sub/super-additivity
- Borodin-Ferrari initial condition: characteristics do not cross, classical solution for all times. General initial condition: singularities appear in finite time.

A heuristic link with 2D KPZ equation

One expects (in some sense) height fluctuations in stationary state π_ρ to be described by

$$\partial_t h(t,x) = \Delta h(t,x) + \nabla h(t,x) \cdot Q_\rho \nabla h(t,x) + \dot{W}(t,x)$$

with $\dot{W}$ a space-time noise and Q_ρ the Hessian of $v(\rho)$.

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A heuristic link with 2D KPZ equation

Recall:

- for the single-flip dynamics, $v(\cdot)$ unknown but convex: signature of Q_ρ is $(+,+)$. "Isotropic KPZ equation"
- B-F dynamics. From explicit form of $v(\cdot)$, signature of Q_ρ is $(+,-)$. "Anisotropic KPZ equation"

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Wolf [PRL '91] predicted:

- Anisotropic case: non-linearity irrelevant, fluctuations grow $\sim \sqrt{\log t}$ as if $Q_\rho = 0$ (Stochastic Heat Equation).
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  simulations: $\nu \approx 0.24$
- Joint work with A. Borodin and I. Corwin [CMP 2017+]: a variant of the $(2+1)$-d growth process in the AKPZ class for which convergence to the stochastic heat equation can be proven

Part II: Back to the reversible process

We expect: if the initial condition approximates smooth profile,

$$\lim_L \frac{1}{L} h(xL) = \phi_0(x)$$

then

$$\lim_L \frac{1}{L} h(xL, tL^2) = \phi(x, t)$$

with

$$\partial_t \phi = \mu(\nabla \phi) \sum_{i,j=1}^{2} \sigma_{i,j}(\nabla \phi) \partial^2_{x_i, x_j} \phi.$$

$\mu > 0$: mobility. $\{\sigma_{i,j}\}$: positive symmetric matrix, Hessian of surface tension.

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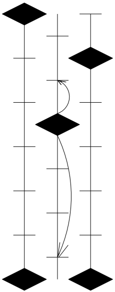
One exception (for $d > 1$ interfaces): Ginzburg-Landau model with symmetric convex potential.
Funaki-Spohn '97: hydrodynamic limit with $\mu(\nabla\phi) \equiv 1$

A reversible process with longer jumps



rate $= \frac{1}{\text{length of jump}}$

[Luby-Randall-Sinclair, SIAM J. Comput. '01, D. Wilson, Ann. Appl. Probab. '04]

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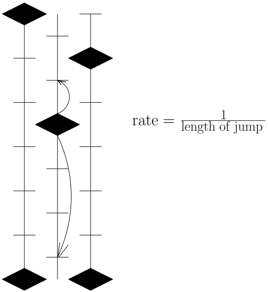

[Luby-Randall-Sinclair, SIAM J. Comput. '01, D. Wilson, Ann. Appl. Probab. '04]

Linear response theory:

$$\mu(\rho) = \pi_\rho(f(\eta)) - \int_0^\infty dt\, \pi_\rho(g(\eta(t))g(\eta(0))$$

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$$\mu(\rho) = \frac{1}{\pi} \frac{\sin(\pi\rho_1)\sin(\pi\rho_2)}{\sin(\pi(1-\rho_1-\rho_2))}$$

[B. Laslier, F. T., Ann. H. Poincaré 2017+]

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NB same as velocity function $v(\cdot)$ of the long-jump growth process: Einstein relation

A (diffusive) hydrodynamic limit

Theorem 3 [B. Laslier, F. T., arXiv '17]
On the torus, convergence to the limit PDE:

$$\left\| \frac{h(\cdot, L^2 t)}{L} - \phi(\cdot, t) \right\|_2^2 := \frac{1}{L^2} \mathbb{E} \sum_x \left| \frac{h(xL, tL^2)}{L} - \phi(x, t) \right|^2 \stackrel{L\to\infty}{\to} 0$$

with ϕ solution of

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Proof via H^{-1} method (Yau, Funaki-Spohn).

Non-trivial fact: PDE contracts $\mathbb{L}^2$ distance between solutions (would be trivial if $\mu(\cdot) \equiv 1$).

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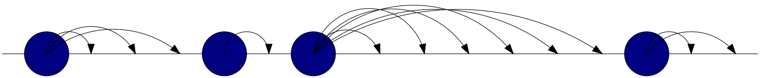
Conclusions

- single-flip version of both reversible and irreversible process are too hard (no gradient condition/no known stationary measures)...
- ...but "natural" longer-jump versions can be analyzed in detail (some "integrable structure" behind)
- Caveat: for the growth process, long-jump and single-flip versions are in two different universality classes (AKPZ/KPZ)
- presumably, results and methods extend to a class of 2d growth processes introduced by Borodin-Ferrari, Petrov, Borodin-Bufetov-Olshanski...

Thanks!

Ideas I: Comparison with the Hammersley process (HP)



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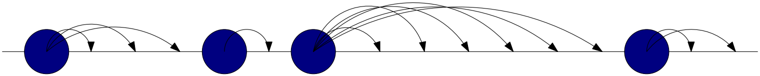

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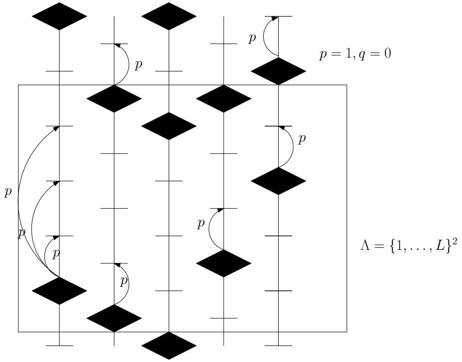
Lozenge dynamics $\sim$ infinite set of coupled Hammersley processes. Comparison: lozenges move less than HP particles

Ideas II: Fluctuations



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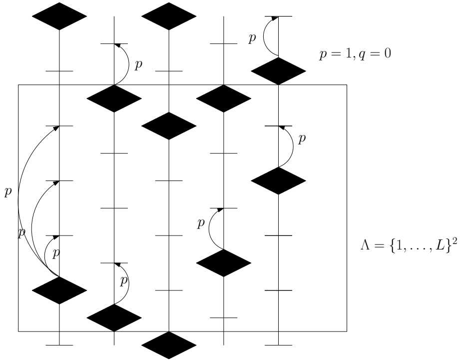


Let $Q_\Lambda(t) = \sum_{x \in \Lambda}(h_x(t) - h_x(0))$.

$$\frac{d}{dt}\langle Q_\Lambda(t) \rangle = \langle \sum_x |V(x, \uparrow) \cap \Lambda| \rangle, \quad \langle \cdot \rangle := \mathbb{E}_{\pi_\rho}.$$

Ideas II: Fluctuations

Similarly, one can prove

$$\frac{d}{dt}\langle (Q_\Lambda(t)-\langle Q_\Lambda(t)\rangle)^2\rangle \leq \sqrt{\langle (Q_\Lambda(t)-\langle Q_\Lambda(t)\rangle)^2\rangle}\,L\sqrt{\log L}+O(L^2)$$

so that

$$\langle (Q_\Lambda(T)-\langle Q_\Lambda(T)\rangle)^2\rangle = O(T^2L^2\log L).$$

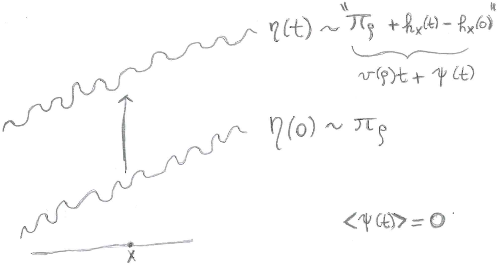
Ideas II: Fluctuations

Similarly, one can prove

$$\frac{d}{dt}\langle (Q_\Lambda(t) - \langle Q_\Lambda(t)\rangle)^2\rangle \leq \sqrt{\langle (Q_\Lambda(t) - \langle Q_\Lambda(t)\rangle)^2\rangle}\, L\sqrt{\log L} + O(L^2)$$

so that

$$\langle (Q_\Lambda(T) - \langle Q_\Lambda(T)\rangle)^2\rangle = O(T^2 L^2 \log L).$$



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If $L = 1$, we get the (useless) bound $\sqrt{\langle \psi(T)^2\rangle} = O(T)$.
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If we choose $L = T$ we get then $\sqrt{\langle \psi(T)^2 \rangle} = O(\sqrt{\log T})$ as wished.

Ideas III: Invariance on the torus

For simplicity, $p = 1, q = 0$.

Stationary measure π_ρ^L: uniform measure with fraction ρ_i of lozenges of type $i = 1, 2, 3$.

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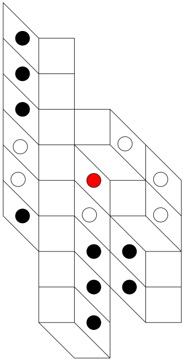
Key fact:

Lemma: The probability of seeing an inter-particle gap $\geq \log R$ within distance R from the origin before time 1 is $O(R^{-K})$ for every K.

Towards the Stochastic Heat Equation

One can generalize the model: rates depend on a parameter $r \in [0,1)$ and (in a special way) on the distances between a particle and its six neighbors



$$\text{rate} = \frac{(1-r^{B-1})(1-r^{D})}{1-r^{C+1}}$$

$r = 0$: back to Borodin – Ferrari dyn.

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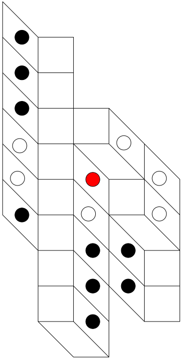

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Theorem 3 [Corwin-Toninelli, ECP 2016]: explicit stationary measure of Gibbs type.

Towards the Stochastic Heat Equation

For $r = e^{-\varepsilon} \to 1$, with $1/\varepsilon$ rescaling of time and particle distances, particle positions z_p have Gaussian fluctuations.

Theorem 4 [Borodin-Corwin-Toninelli, CMP 2016+]:

$$\varepsilon(z_p(t/\varepsilon) - z_p(0)) \to Vt$$

and

$$\sqrt{\varepsilon}(z_p(t/\varepsilon) - z_p(0) - \varepsilon^{-1}Vt) \to \xi_p(t)$$

and $\xi_p(t)$ ($\Leftrightarrow$ height fluctuations w.r.t. deterministically growing profile) solve a linear system of SDEs.

Towards the Stochastic Heat Equation

In that limit, space-time correlations can be computed:

$$\mathbb{E}\left[\xi_{x,t}\ \xi_{y,s}\right] - \mathbb{E}\left[\xi_{x,t}\right]\mathbb{E}\left[\xi_{y,s}\right]$$

Towards the Stochastic Heat Equation

Along a special direction $U \in \mathbb{R}^2$ ("characteristics")

$$\mathbb{E}\left[\xi_{\frac{tU}{\delta}+\frac{x}{\sqrt{\delta}},\frac{t}{\delta}}\ \xi_{\frac{sU}{\delta}+\frac{y}{\sqrt{\delta}},\frac{s}{\delta}}\right]-\mathbb{E}\left[\xi_{\frac{tU}{\delta}+\frac{x}{\sqrt{\delta}},\frac{t}{\delta}}\right]\mathbb{E}\left[\xi_{\frac{sU}{\delta}+\frac{y}{\sqrt{\delta}},\frac{s}{\delta}}\right]$$

tends as $\delta \to 0$ to $C(s,t,x-y)$, the space-time correlation of the 2d SHE

$$\partial_t h = \Delta h + \dot{W}, \quad h(0,x) = 0.$$

For all other directions U', correlations ≈ 0 if $t - s \gg \sqrt{t}$.

Remark: A similar behavior expected for growth models in the Anisotropic KPZ class. E.g. the Borodin-Ferrari dynamics.