Hydrodynamics of the N-BBM process

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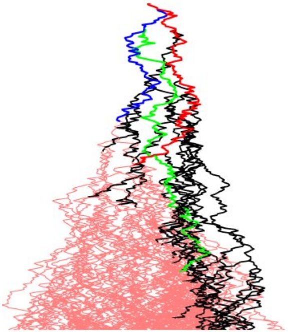


Illustration by Eric Brunet

Institut Henri Poincaré, June 2017

Brunet and Derrida N branching particles in $\mathbb{R}$ with selection:

discrete time.

Particle at x dies and creates random offsprings around x.

Select the rightmost N particles.

iterate

Pascal Maillard studied the N-BBM process.

N particles move as independent Brownian motions in $\mathbb{R}$,

each particle, at rate 1, creates a new particle at its current position.

At each branching time, the left-most particle is removed.

The number N of particles is then conserved.

Brunet Derrida (1997) Shift in the velocity of a front due to a cutoff PRE

Brunet, Derrida, Mueller, Munier (2006). Noisy traveling waves: effect of selection on genealogies. EPL + (06) + (07)

Bérard, Gouéré (2010) Brunet-Derrida behavior of branching-selection particle systems on the line CMP.

Bérard, Maillard (2014) The limiting process of N-BRW with polynomial tails EJP.

Durrett, Remenik (2011) Brunet-Derrida particle systems, free boundary problems and Wiener-Hopf equations AOP.

Derrida, Shi (2017) Large deviations for the BBM in presence of selection or coalescence Preprint.

Julien Berestycki, Brunet, Derrida (2017) Exact solution and precise asymptotics of a Fisher-KPP type front ArXiv

Hydrodynamics

Density ρ with left boundary $L_0 = \arg\max_a \int_a^\infty \rho(x)dx > -\infty$

Time zero: iid continuous random variables with density ρ.

$X_t :=$ set of positions of N-BBM particles at time t.

Theorem 1. *[Existence]*

For every $t \geq 0$, there is a density function $\psi(\cdot, t) : \mathbb{R} \to \mathbb{R}^+$ such that,

$$\lim_{N\to\infty} \frac{\left|X_t \cap [a, \infty)\right|}{N} = \int_a^\infty \psi(r,t)dr, \quad \textit{a.s. and in } L^1.$$

for any $a \in \mathbb{R}$.

Free boundary problem.

Density ρ with left boundary $L_0 = \arg\max_a \int_a^\infty \rho(x)dx > -\infty$

Find $((u(r,t), L_t) : r \in \mathbb{R}, t \in [0,T])$ such that:

$$\begin{aligned} &u_t = \tfrac{1}{2}u_{rr} + u, \qquad \text{in } \ (L_t, +\infty); \\ &u(r,0) = \rho(r); \\ &u(L_t,t) = 0, \qquad \int_{L_t}^\infty u(r,t)dr = 1. \end{aligned}$$

If one finds a continuous function L_t such that

$$e^t P\big(L_s \le B_s^\rho, 0 \le s \le t\big) = 1, \qquad t \ge 0.$$

where B_s^ρ is BM with random initial position $B_0^\rho \sim \rho$, then

$$\int \varphi(r)u(r,t)dr = e^t E\big(\varphi(B_t^\rho)\,\mathbf{1}\{L_s \le B_s^\rho, 0 \le s \le t\}\big)$$

Theorem 2. *If L_t is a continuous function* such that

$$((u(r,t), L_t) : t \in [0,T]$$

is a solution of the free boundary problem, then the hydrodynamic limit ψ coincides with u:

$$\psi(\cdot,t) = u(\cdot,t), \quad t \in [0,T]. \tag{1}$$

Lee (2017) proved that if $\rho \in C_c^2([L_0,\infty))$ and $\rho'_{L_0} = 2$ then there exist $T > 0$ and a solution (u, L) of the free boundary problem with the following properties:

- $\{L_t : t \in [0,T]\}$ is in $C^1[0,T]$, $L_{t=0} = L_0$
- $u \in C(D_{L,T}) \cap C^{2,1}(D_{L,T})$,
  where $D_{L,T} = \{(r,t) : L_t < r, 0 < t < T\}$.

General strategy

We use a kind of Trotter-Kato approximation as upper and lower bounds.

Durrett and Remenik upperbound for the Brunet-Derrida model. Leftmost particle motion is *increasing*: natural lower bounds.

Upper and lower bounds method was used in several papers:

- De Masi, F and Presutti (2015) Symmetric simple exclusion process with free boundaries. PTRF
- Carinci, De Masi, Giardinà, and Presutti (2016) Free boundary problems in PDEs and particle systems. SpringerBriefs in Mathematical Physics.

We introduce labelled versions of the processes and a coupling of trajectories to prove the lowerbound.

Ranked BBM, a tool Let $(Z_0^1, \ldots, Z_0^N)$ BBM initial positions.

$B_0^{i,1} = Z_0^i$, iid with density ρ.

N_t^i: is the size of the ith BBM family.

$B_t^{i,j}$: is the j-th member of the i-th family at time t, $1 \le j \le N_t^i$.

birth-time order.

$$\text{BBM:} \quad Z_t = \{B_t^{i,j} : 1 \le j \le N_t^i, 1 \le i \le N\}$$

$B_{[0,t]}^{i,j}$ trajectory coincides with ancestors before birth.

(i, j) is the rank of the jth particle of i-family

N-BBM as subset of BBM

Let $X_0 = Z_0$, $\tau_0 = 0$

τ_n branching times of BBM.

$$X_t := \{B_t^{i,j} \; : B_{\tau_n}^{i,j} \geq L_{\tau_n}, \text{ for all } \tau_n \leq t\}$$

$$L_{\tau_n} := \text{ defined iteratively such that } |X_t| = N \text{ for all } t$$

X_t has the law of N-BBM.

Stochastic barriers.

Fix $\delta > 0$

$X_0^{\delta,\pm} = Z_0$.

The upper barrier. Post-selection at time $k\delta$.

$$X_{k\delta}^{\delta,+} := N \text{ rightmost } \{B_{k\delta}^{i,j} : B_{(k-1)\delta}^{i,j} \in X_{(k-1)\delta}^{\delta,+}\}$$

$$L_{k\delta}^{\delta,+} := \min X_{k\delta}^{\delta,+}$$

The number of particles in $X_{k\delta}^{\delta,+}$ is exactly N for all k.

The lower barrier.

Pre selection at time $(k-1)\delta$.

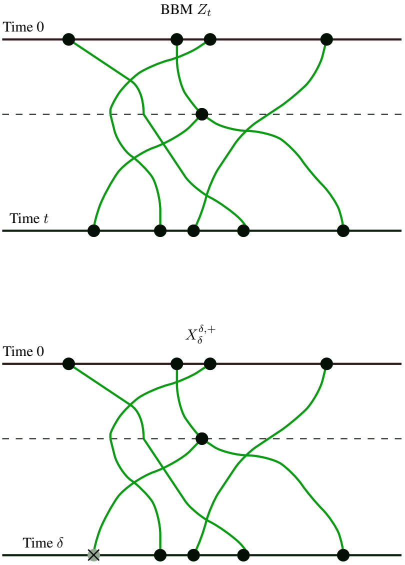
Select maximal number of rightmost particles at time $(k-1)\delta$ keeping no more than N particles at time $k\delta$.

$$L^{\delta,-}_{(k-1)\delta} := \text{cutting point at time } (k-1)\delta$$

$$X^{\delta,-}_{k\delta} := \{B^{i,j}_{k\delta} : B^{i,j}_{(k-1)\delta} \in X^{\delta,-}_{(k-1)\delta} \cap [L^{\delta,-}_{(k-1)\delta}, \infty)\}$$

Only entire families of particles at time $(k-1)\delta$ are kept at time $k\delta$.

The number of particles in $X^{\delta,-}_{k\delta}$ is $N - O(1)$.



BBM Z_t
Time 0
Time t
N-BBM X_t
Time 0
Time t
$X_\delta^{\delta,+}$
Time 0
Time δ
$X_\delta^{\delta,-}$
Time 0
Time δ

Mass transport partial order

$$X \preccurlyeq Y \quad \text{if and only if} \quad |X \cap [a,\infty)| \le |Y \cap [a,\infty)| \quad \forall a \in \mathbb{R}.$$

Proposition 3. *Coupling* $\big((\hat{X}^{\delta,-}_{k\delta}, \hat{X}_{k\delta}, \hat{X}^{\delta,+}_{k\delta}) : k \ge 0\big)$ *such that*

$$\hat{X}^{\delta,-}_{k\delta} \preccurlyeq \hat{X}_{k\delta} \preccurlyeq \hat{X}^{\delta,+}_{k\delta}, \quad k \ge 0.$$

$\hat{X}^{\delta,-}_t$ is a subset of $\hat{Z}_t$, a BBM with the same law as Z_t.

Deterministic barriers. $u \in L^1(\mathbb{R}, \mathbb{R}_+)$.

Gaussian kernel: $$G_t u(a) := \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-(a-r)^2/2t} u(r)\, dr.$$

$e^t G_t \rho$ solves $u_t = \frac{1}{2} u_{rr} + u$ with initial ρ.

Cut operator C_m is defined by

$$C_m u(a) := u(a) \mathbf{1}\Big\{ \int_a^{\infty} u(r) dr \leq m \Big\},$$

so that $C_m u$ has total mass $m \wedge \|u\|_1$.

For $\delta > 0$ and $k \in \mathbb{N}$, define the *upper and lower barriers*:

$$S_0^{\delta,\pm}\rho := \rho \quad \text{Initial condition}$$

$$S_{k\delta}^{\delta,+}\rho := \left(C_1\left(e^{\delta}G_\delta\right)\right)^k\rho \quad \text{(diffuse \& grow) + cut;}$$

$$S_{k\delta}^{\delta,-}\rho := \left(\left(e^{\delta}G_\delta\right)C_{e^{-\delta}}\right)^k\rho \quad \text{cut + (diffuse \& grow)}$$

We have $\left\|S_{k\delta}^{\delta,\pm}\rho\right\|_1 = \|\rho\|_1 = 1$ for all k.

Hydrodynamics of δ-barriers

We prove that for fixed δ

the stochastic barriers converge to the macroscopic barriers:

Theorem 4. *Conditions of Theorem 1 and fixed δ:*

$$\lim_{N\to\infty} \frac{|X_{k\delta}^{\delta,\pm} \cap [r,\infty)|}{N} = \int_a^\infty S_{k\delta}^{\delta,\pm}\rho, \quad \textit{a.s. and in } L^1.$$

The same is true for the coupling marginals $\hat{X}_{k\delta}^{\delta,\pm}$.

Convergence of macroscopic barriers

Partial order: Take $u, v : \mathbb{R} \to \mathbb{R}^+$ and denote

$$u \preccurlyeq v \quad \text{iff} \quad \int_a^\infty u \le \int_a^\infty v \quad \forall a \in \mathbb{R}.$$

Fix t and take diadic $\delta = t2^{-n}$. We prove

- $S_t^{\delta,-}\rho$ is increasing and $S_t^{\delta,+}\rho$ decreasing in n (diadics).
- $\left\|S_t^{\delta,+}\rho - S_t^{\delta,-}\rho\right\|_1 \le c\delta$.
- There exists a continuous function ψ such that for any $t > 0$,

$$\lim_{n\to\infty} \|S_t^{\delta,\pm}\rho - \psi(\cdot,t)\|_1 = 0.$$

Sketch of proof of Theorem 1

By coupling $\hat{X}_t^{\delta,-} \preccurlyeq \hat{X}_t \preccurlyeq \hat{X}_t^{\delta,+}$.

Convergences in the sense of the Theorem 1:

$N \to \infty$:
The stochastic barriers $\hat{X}_t^{\delta,\pm}$ converge to the macroscopic barriers $S_t^{\delta,\pm}$.

$\delta \to 0$:
The macroscopic barriers converge to a function ψ, along diadics $\delta \to 0$.

Corollary:
N-BBM $\hat{X}_t$ converge to ψ as $N \to \infty$.

This is Theorem 1.

Sketch of proof of Theorem 2

We show that for continuous L_t, the solution u of the free boundary problem is in between the barriers:

$$S^{\delta,-}_{k\delta}\rho \preccurlyeq u(\cdot, k\delta) \preccurlyeq S^{\delta,+}_{k\delta}\rho.$$

Here we use the Brownian representation of the solution.

Proof of Pre-selection inequalities.

Rank order

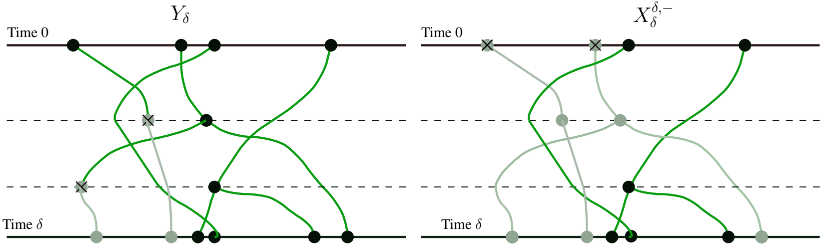
$$(i, j) \prec (i', j') \text{ if and only if } B_0^{i,1} < B_0^{i',1} \text{ or } i = i' \text{ and } j < j'. \tag{2}$$

N rank-selected BBM:

$$Y_t := \{B_t^{i,j} : \left|\{B_t^{i',j'} : (i,j) \prec (i',j')\}\right| < N\},$$

We have $X_\delta^{\delta,-} \subset Y_\delta$, which in turn implies

$$X_\delta^{\delta,-} \preceq Y_\delta.$$



Labeled N-BBM.

$$(X_t^1, \ldots, X_t^N) \in \mathbb{R}^N$$

X_t^ℓ is just a labelling of N-BBM as function of $(B^{i,j}_{[0,t]} : i, j)$:

When one of the Brownian particles branches at time s, identify

$X^n_{s-} :=$ the branching particle

$X^m_{s-} :=$ the position of the leftmost particle (to be erased)

At time s put

$X^m_s = X^n_{s-}$

X^m_t will follow the newborn Brownian particle until next branching.

Labeled rank-selected N-BBM.

$$((Y_t^1,\sigma_t^1),\ldots,(Y_t^N,\sigma_t^N)) \in \left(\mathbb{R}\times\mathbb{N}^2\right)^N$$

Y_t^ℓ is a labelling of the rank-selected N-BBM Y_t.

σ_t^ℓ tracks the rank of the Y^ℓ-particles in the Y-tree.

When one of the Brownian particles branches at time s, identify

$Y_{s-}^n :=$ the branching particle, $\sigma_{s-}^n = (i,j)$

$Y_{s-}^h :=$ lowest ranked Y-particle (to be erased)

At time s put

$Y_s^h = Y_{s-}^n$ and this particle will follow now the newborn Brownian particle

$\sigma_s^h = (i, M_{s-}^i + 1)$ (youngest new element of the Y^i branching family)

Coupling. $(X_t^1, \dots, X_t^N), ((Y_t^1, \sigma_t^1) \dots (Y_t^N, \sigma_t^N))$

Between branchings $X_t^\ell - Y_t^\ell$ and σ_t^ℓ are constant.

s branching time for X process.

X_{s-}^n and Y_{s-}^n branching particles.

$\sigma_{s-}^n = (i, j)$ rank of Y-branching particle

$X_{s-}^m :=$ leftmost X-particle (to be erased).

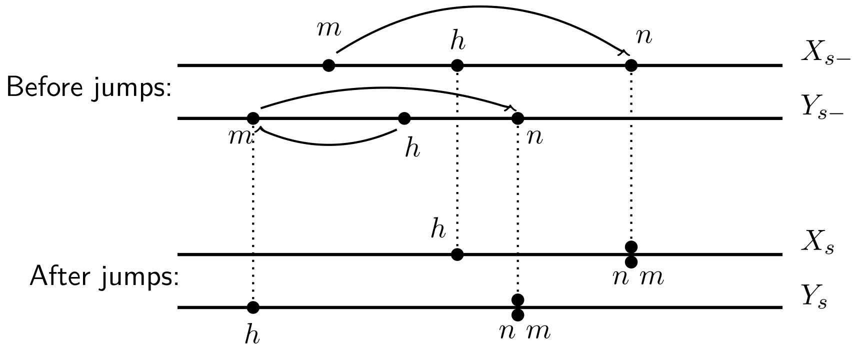
$Y_{s-}^h :=$ lowest-rank Y-particle (to be erased).

At time s put

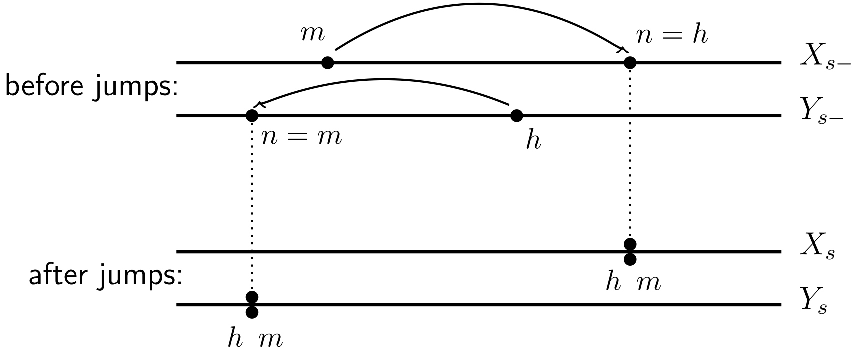
$X_s^m = X_{s-}^n$, $Y_s^h = Y_{s-}^m$, $Y_s^m = Y_{s-}^n$

X_s^m and Y_s^m will follow now the (same) newborn Brownian particle

$\sigma_s^h = \sigma_{s-}^n$, $\sigma_s^m = (i, M_{s-}^i + 1)$ (youngest new element of the branching family)



Relative positions of particles at branching time s.



Coupling between $\underline{x}(t)$ and $\Big(\underline{y}(t), \underline{\sigma}(t)\Big)$. When $n = m$ only the h-th Y-particle jumps to Y^n_{s-}.
When $n = h$ only the m-th X-particle jumps to X^h_{s-}.
Perform two cases simultaneously.

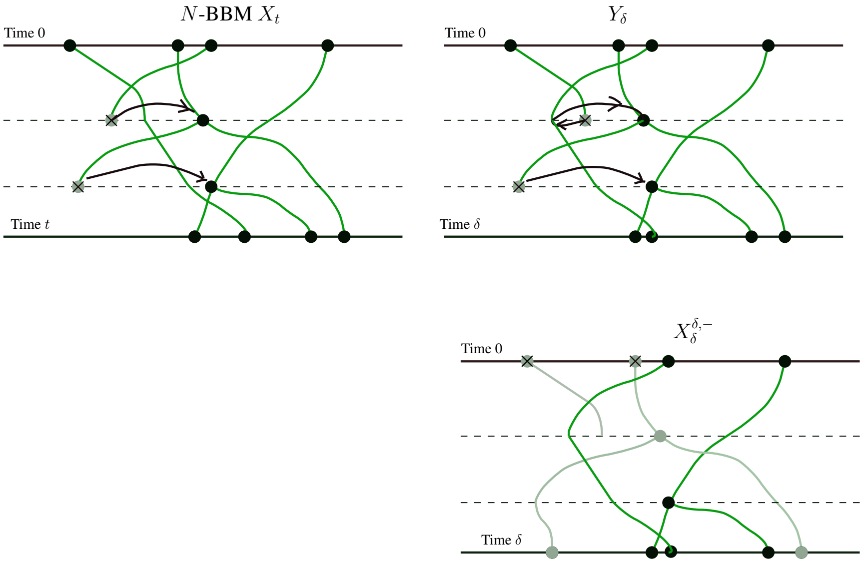
The coupling satisfies

$$Y_t^\ell \leq X_t^\ell, \quad \text{for all } t, \ell.$$

Hence

$$\hat{X}_\delta^{\delta,-} \preceq \hat{Y}_t \preceq X_t, \quad \text{a.s.}$$

$M_t^i :=$ size of Y_0^i family at time t.


N-BBM X_t
Time 0
Time t
Y_δ
Time 0
Time δ
$X_\delta^{\delta,-}$
Time 0
Time δ

The post-selection process

N-BBM X_t is a subset of the BBM Z_t.

$X_\delta^{\delta,+} = N$ right-most Z-particles at time δ. Hence,

$$X_\delta \preceq X_\delta^{\delta,+}.$$

Domination We have proven the dominations

$$\hat{X}_{k\delta}^{\delta,-} \preceq X_{k\delta} \preceq \hat{X}_{k\delta}^{\delta,+}.$$

for $k = 1$. Iterate to obtain the same for all k.

Construct the coupling for each time interval and then the Brownian tree $\hat{B}$ containing $\hat{Y}_{k\delta} \supset \hat{X}_{k\delta}^{\delta,-}$.

Similarly construct Brownian tree containing $\hat{X}_{k\delta}^{\delta,+}$.

Construct new BBM process $\hat{B}_{[0,t]}$ by

Attaching independent BBM to loose branches of $\hat{Y}_t$.

Proposition 5. $\hat{B}_{[0,t]}$ *has the same law as the BBM* $B_{[0,t]}$ *and*

$$\hat{Y}_t := \{\hat{B}_t^{i,j} : \left|\{\hat{B}_t^{i',j'} : (i,j) \prec (i',j')\}\right| < N\}$$

is the rank selected process associated to $\hat{B}_{[0,t]}$.

The rightmost families with up to N *total particles coincide*

$$\hat{N}_t^i = M_t^i \text{ if } \sum_j N_t^j \mathbf{1}\{B_0^{j,1} \geq B_0^{i,1}\} \leq N$$

Hydrodynamic limit for the barriers

Macroscopic left boundaries

For $\delta > 0$ and $\ell \le k$ denote

$$L_{\ell\delta}^{\delta,+} := \sup_r \Big\{ \int_{-\infty}^{r} S_{\ell\delta}^{\delta,+} \rho(r') dr' = 0 \Big\};$$
$$L_{\ell\delta}^{\delta,-} := \sup_r \Big\{ \int_{-\infty}^{r} S_{\ell\delta}^{\delta,-} \rho(r') dr' < 1 - e^{-\delta} \Big\}. \quad (1)$$

Brownian representation of macroscopic barriers:

$B_{[0,t]} = (B_s : s \in [0,t])$ Brownian motion with

B_0, *random* variable with density ρ.

Lemma 6. *For test function $\varphi \in L^\infty(\mathbb{R})$ and $t > 0$,*

$$\int \varphi S^{\delta,+}_{k\delta} \rho \,=\, e^{k\delta} E\big[\varphi(B_{k\delta}) \mathbf{1}\{B_{\ell\delta} > L^{\delta,+}_{\ell\delta} : 1 \le \ell \le k\}\big].$$

$$\int \varphi S^{\delta,-}_{k\delta} \rho \,=\, e^{k\delta} E\big[\varphi(B_{k\delta}) \mathbf{1}\{B_{\ell\delta} > L^{\delta,-}_{\ell\delta} : 0 \le \ell \le k-1\}\big].$$

Generic LLN over trajectories of BBM

Let $B_0^{i,1}$ iid with density ρ.

N_t^i size at time t of the i-th BBM family. $EN_t^i = e^t$.

Proposition 7. *Let g be bounded. Then*

$$\mu_t^N g := \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N_t^i} g(B_{[0,t]}^{i,j}) \underset{N\to\infty}{\longrightarrow} e^t Eg(B_{[0,t]}), \quad \textit{a.s. and in } L^1. \qquad (0)$$

a.s. and in L^1.

Proof. By the many-to-one Lemma we have

$$E\mu_t^N g = EN_t\, Eg(B_{[0,t]}) = e^t Eg(B_{[0,t]}), \qquad (1)$$

The variance of $\mu_t^N g$ is order $1/N$, by family independence. □

Corollary 8 (Hydrodynamics of the BBM)**.**

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N_t^i} \varphi(B_t^{i,j}) = e^t E\varphi(B_t) \qquad \textit{a.s. and in } L^1.$$

$$= e^t \int \varphi(r) G_t \rho(r) dr, \tag{2}$$

Proof of Hydrodynamics for barriers

Proof of Theorem 4 BBM representation of stochastic barriers:

$$\pi_{k\delta}^{N,\delta,+}\varphi = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N_{k\delta}^{i}} \varphi(B_{k\delta}^{i,j})\mathbf{1}\{B_{\ell\delta}^{i,j} \geq L_{\ell\delta}^{N,\delta,+} : 1 \leq \ell \leq k\}$$

$$\pi_{k\delta}^{N,\delta,-}\varphi = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N_{k\delta}^{i}} \varphi(B_{k\delta}^{i,j})\mathbf{1}\{B_{\ell\delta}^{i,j} \geq L_{\ell\delta}^{N,\delta,-} : 0 \leq \ell \leq k-1\}.$$

We show that as $N \to \infty$

$L_{\ell\delta}^{N,\delta,\pm}$ can be replaced by $L_{\ell\delta}^{\delta,\pm}$, and get the result by the generic LLN.

Use the fact that the random left boundaries are exact quantiles of 1. □

Proof of Theorem 2 *The limit function ψ is the solution of the free boundary problem.*

The local solution of the free boundary problem is in between the barriers:

Theorem 9. *Let $t \in (0, T]$, $\delta \in \{2^{-n}t, n \in \mathbb{N}\}$. Then*

$$S_t^{\delta,-}\rho \preccurlyeq u(\cdot, t) \preccurlyeq S_t^{\delta,+}\rho, \qquad t = k\delta$$

The upperbound is immediate. The lower bound reduces to show the following stochastic order between conditioned probability measures:

$$P_{u_0}(B_t \geq r | \tau^L \leq \delta) \leq P_{u_1}(B_t \geq r | \tau^L > \delta) \tag{3}$$

where $u_1 = C_{e^{-\delta}} u$, $u_0 = u - u_1$,

$$P_{u_i}(B_t \in A) := \frac{1}{\|u_i\|_1} \int u_i(x) P_x(B_t \in A) dx. \tag{4}$$

and τ^L is the hitting time of the boundary.

Stationary N-BBM X_t be N-BBM. Process as seen from leftmost particle:

$$X'_t := \{x - \min X_t : x \in X_t\}$$

Durrett and Remenik for a related Brunet-Derrida process proved:

Theorem 10. *N-BBM as seen from leftmost particle is Harris recurrent.*

ν_N unique invariant measure. Under ν_N asymptotic speed

$$\alpha_N = (N-1)\,\nu_N\big[\min(X \setminus \{0\})\big],$$

X'_t starting with anything converges in distribution to ν_N and

$$\lim_{t\to\infty} \frac{\min X_t}{t} = \alpha_N.$$

α_N converges to asymtotic speed of the first particle in BBM:

$$\lim_{N\to\infty} \alpha_N = \sqrt{2}.$$ *Berard and Gouéré*

Travelling wave solutions $u(r,t) = w(r - \alpha t)$, where w must satisfy

$$\frac{1}{2}w'' + \alpha w' + w = 0, \quad w(0) = 0, \quad \int_0^\infty w(r)dr = 1.$$

Groisman and Jonckheere (2013): for each speed $\alpha \geq \sqrt{2}$ there is a solution w_α

$$w_\alpha(r) = \begin{cases} M_\alpha \, r e^{-\alpha r} & \text{if } \alpha = \sqrt{2} \\ M_\alpha \, e^{-\alpha r} \sinh\left(r\sqrt{\alpha^2 - 2}\right) & \text{if } \alpha > \sqrt{2} \end{cases} \tag{4}$$

where M_α is a normalization constant.

w_α is the unique qsd for Brownian motion with drift $-\alpha$ and absorption rate 1; see Martínez and San Martín (1994).

Open problems. (1) Let X_t be the N-BBM process with initial configuration sampled from the stationary measure ν^N.

Show that the empirical distribution of X_t converges to a measure with density $w_{\sqrt{2}}(t\sqrt{2}+\cdot)$, as $N \to \infty$. This would be a *strong selection principle* for N-BBM.

Problem: we do not control the particle-particle correlations in the ν_N distributed initial configuration.

If we start with independent particles with distribution $w_{\sqrt{2}}$, then Theorem 1 and $w_{\sqrt{2}}(t\sqrt{2}+\cdot)$ strong solution of FBP imply convergence of the empirical measure to this solution.

(2) "Yaglom limit"? Does $u(\cdot - L_t, t)$ converges to w_α for some $\alpha \geq \sqrt{2}$? Fix α, which conditions must satisfy ρ to converge to w_α?

(3) Give a simple proof of existence of the solution for the FBP.