Friction? & Reversibility? and Chaotic Hypothesis

Basic question: $\Rightarrow$ microscopic dynamics is reversible but macroscopic equations are not; dissipation is the reason.

Dissipation is phenomenologically introduced: for instance in Navier-Stokes fluids it is the viscosity $\nu > 0$:

$$\partial_t \vec{u} = -(\mathbf{u} \cdot \boldsymbol{\partial})\, \vec{u} + \nu\, \Delta\, \vec{u} + \vec{f} - \vec{\partial} p$$

In the heat equation it is the thermal conduction coefficient, in Lorenz61 atmospheric turbulence model it is in the linear part just as in the Lorenz96 model:

$$\dot{x}_j = x_{j-1}(x_{j+1} - x_{j-2}) + F - \nu\, x_j, \qquad j = 0, \ldots, N-1$$

Question: can the fundamental time reversal symmetry be preserved also in models of macroscopic phenomena? **or**

or is the essence of dissipation represented by the phenomenologic coefficients?

This does not mean questioning that the macroscopic equations arise when suitably rescaled observables are imagined studied at singular rescaling values.

It is convenient to examine concrete cases in the attempt to formulate a conjecture of equivalence between reversible and irreversible models.

In the '980s reversible equations of motion were used successfully in simulations to study simple fluids in stationary states: energy could not be kept constant (not even bounded) unless external forces work was balanced.

Artificial forces, like equally artificial stochastic noise or phenomenological friction , forbade energy build-up,[1].

The justification was that, although somewhat simpler than stochastic forces, the new equations were equivalent.

Idea: every (even if macroscopic) dissipative evolution can be equivalent to a reversible one, provided motions are sufficiently chaotic, (as they usually are under strong forcing or large N). [2, 3, 4, 5]

“In microscopically reversible (chaotic) systems time reversal symmetry cannot be spontaneously broken, but only phenomenologically so”,[6].

Mechanism proposed: “same” as that for equilibrium ensembles, *i.e.* for collections of stationary states.

e.g. (recall) microcanonical ensemble $\boldsymbol{\mu}_E^M$ with energy E and density fixed is a probability distribution on phase space very different from canonical ensemble $\boldsymbol{\mu}_\beta^C$ with same density and inverse temperature β;

Yet they are equivalent provided the average energy $E = \boldsymbol{\mu}_\beta^C(E(x))$ in the canonical ensemble coincides with the microcanonical energy E (as $V \to \infty$). Or reciprocally: if $\beta^{-1} = \frac{2}{3}\boldsymbol{\mu}_E^C(K(x))$.

Of course not "everything" is the same: just "local observables have the same average values"

Can this be done for stationary nonequilibrium?

Start from a work of V. Lucarini and G., [7].

Consider the special case of Lorenz96 chain (periodic b.c.)

$$\dot{x}_j = f_j(x) + F - \nu\, x_j, \qquad \nu > 0,\ j = 1, \ldots, N \qquad (Eq)$$

$f_j \equiv x_{j-1}(x_{j+1} - x_{j-2})$ (so that $f(x) = f(-x) \Rightarrow$ time reversal)

Chaotic hypothesis: "think of it as an Anosov system" (Cohen,G, if F is large), [8, 9, 10]

analogue of the periodicity≡ergodicity hypothesis of Boltzmann, Clausius, Maxwell, and possibly as unintuitive), [11, 12, 13].

Consider two "ensembles", *i.e.* collections of stationary distributions

(1): Vary ν and let $\boldsymbol{\mu}^C_\nu$ stationary distrib. for (Eq)
Let $E = \boldsymbol{\mu}^C_\nu \sum_j x_i^2$:
this is an "ensemble" (viscosity ensemble), ∼canonical.

Next consider a new equation:

Replacing ν by $\qquad \alpha(x) = \frac{\sum_i F x_i}{\sum_i x_i^2}$

New (Eq) has $E(x) = \sum_i x_i^2$ as exact constant of motion

$$\dot{x}_j = x_{j-1}(x_{j+1} - x_{j-2}) + F - \alpha(x) x_j, \qquad (Eqnew)$$

(2): Vary E and let $\boldsymbol{\mu}^M_E$ station. distrib.: this is the (energy ensemble), ∼microcanonical.

Volume contracts by the divergence $\sigma(x) = \sum \partial_j (\alpha(x) x_j)$

$$\sigma(x) = (N-1)\alpha(x), \quad p = \tau^{-1} \int_0^\tau \sigma(x(t)) dt / \langle \sigma \rangle$$

State $\boldsymbol{\mu}_E^M$ labeled by E corresponds to state $\boldsymbol{\mu}_\nu^C$ labeled by $\nu \Rightarrow$ equivalent if $\boldsymbol{\mu}_E(\alpha(x)) = \boldsymbol{\mu}_\nu(E(x))$

$$\boldsymbol{\mu}_\nu^C \sim \boldsymbol{\mu}_E^M \longleftrightarrow E = \boldsymbol{\mu}_\nu^C(E(\cdot)) \longleftrightarrow \nu = \boldsymbol{\mu}_E^M(\alpha(\cdot))$$

Give the same statistics in the limit of large $R = \frac{F}{\nu^2}$.

Analogy: "canonical" $\boldsymbol{\mu}_\beta^C$ = "microcanonical" $\boldsymbol{\mu}_E^M$.

Why? several reasons. Eg. chaoticity implies self averaging for the observable $\alpha(x)$ which replaces viscosity in (Eqnew):

$$\alpha(x(t)) = \frac{\sum_i F x_i}{\sum_i x_i^2} \qquad \text{"self - averaging"}$$

"and other reasons" (??). **But** of course this also means that $\boldsymbol{\mu}_E^M(\alpha) \equiv \nu = \frac{1}{R}$ (non trivial first check).

In the work with V. Lucarini,[7], tests were performed at $N = 32$ (with checks up to $N = 512$) and high R (at $R > 8$, system is very chaotic with > 20 Lyap.s exponents and at larger R it has $\sim \frac{1}{2}N$ Lyap.exp. > 0).

1) $\boldsymbol{\mu}_{\overline{E}}(\alpha) = \nu \longleftrightarrow \boldsymbol{\mu}_{\nu}(E) = \overline{E}$ which is clearly a key selfconsitency test.

2) If g is reasonable ("local") observable $\frac{1}{T}\int_0^T g(S_t x)dt$ has same statistics in both

3) Found its N-independence and ensemble independence of the Lyapunov spectrum (and check of the interpolation as $N \uparrow$, [14])

4) In so doing found several scaling and pairing rules for Lyapunov exponents (somewhat surprisingly), continuimg the list of scaling properties found by Lorenz.

5) The "fluctuation Relation" holds for the fluctuations of phase space vol. (reversible case): reflecting the chaotic hypothesis: last but not least as it is a rather stringent test of the chaotic hypothesis for Lorenz96, and checked a local version of the F.R.

A list of some scaling relations (irreversible model):

$$E = \sum_i x_i^2, \qquad M = \sum_i x_i$$

$$\frac{\overline{E}^i_R}{N} \sim c_E R^{4/3}, \quad \frac{\overline{M}^i_R}{N} \sim 2 c_E R^{1/3} \quad c_E = 0.59 \pm 0.01$$

$$\frac{std(E)^i_R}{N} = \frac{\left(\overline{E^2}^i_R - (\overline{E}^i_R)^2\right)^{1/2}}{N} = \tilde{c}_E R^{4/3}, \quad \tilde{c}_E \sim 0.2 c_E$$

$$\frac{std(M)^i_R}{N} = \tilde{c}_M R^{2/3} \quad \tilde{c}_E \sim 0.046 \pm 0.001$$

The first two confirm Lorenz96, the 3d,4th "new", and the 5th gives the "decorrelation" time of $\langle\, M(t)M(0)\,\rangle$

$$t_{dec}^{i,M} \sim c_M R^{-2/3} \quad c_M = 1.28 \pm 0.01$$

it is important because it sets the time scale to probe in testing the equivalence conj.

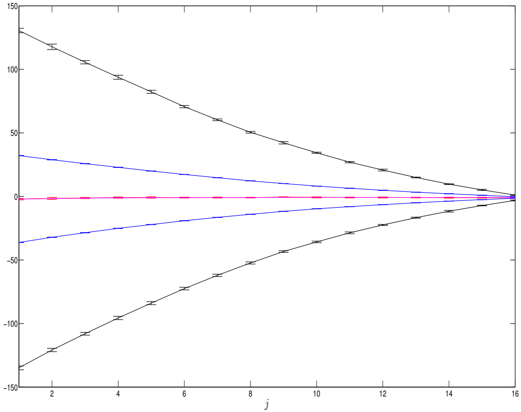
Before showing main results on the F.R. some graphs illustrate other other aspects of the model.

6) A "pairing rule" is found: setting $x = \frac{i}{N}$ the $i-th$ Lyap. exp. $\lambda_i = \lambda(x)$ satisfy

$$|\lambda(x)+1| \sim |2x-1|^{\frac{5}{3}} R^{\frac{2}{3}}$$

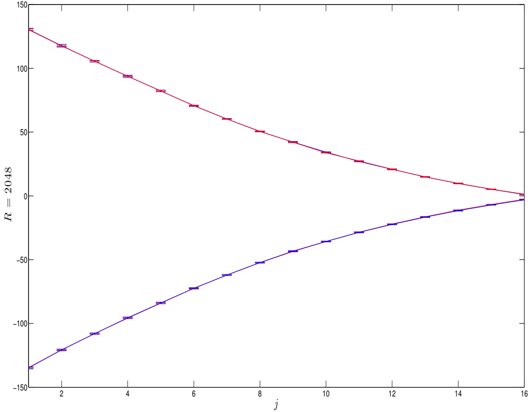
(we fix $\nu = 1$ and call R the size of the forcing)

(Irreversible) model Lyapunov exponents arranged pairwise



Black: Lyap. exp.s $R = 2048$
Magenta: $\pi(j) = (\lambda_j + \lambda_{N-j+1})/2$.
Blue: Lyap. exp.s $R = 256$
value of $\pi(j)$ at $R = 252$ (invisible below magenta).

Lyapunov exp. reversible $\equiv$ irrev



Red: Lyap exps $R = 2048$.

Dimension of Attractor

The $|\lambda(x)+1| \sim c_\lambda |2x-1|^{5/3} R^{2/3}$ yields the full spectrum: hence

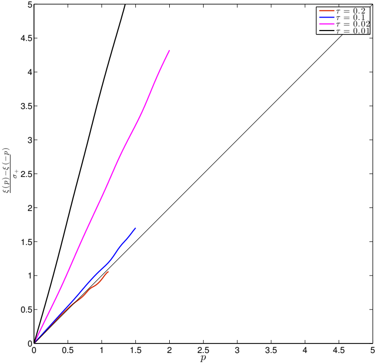
From the asymtotic expressions for the Lyap. exp. the KY dimension of the attractor turns out:

$$N - d_{KY} = \frac{N}{1 + c_\lambda R^{\frac{2}{3}}} \xrightarrow[R\to\infty]{} 0, \qquad \forall\ N$$

i.e. attractor has a dimension virtually indistinguishable from that of the full phase space.

However SRB distribution deeply different from equidistribution: as it can be made clear by the equivalence (if holding). Therefore validity of the Fluctuation Relation becomes a key test

Check Fluctuation Relation (FR) (*i.e.* check of Chaotic Hypothesis)



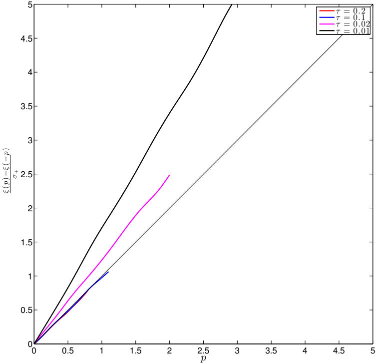
$$p = \frac{1}{\tau} \frac{\int_0^\tau \sigma(x(t))dt}{\langle \sigma \rangle_{srb}}$$

$$\frac{1}{\tau \overline{\sigma}_{srb}} \log \frac{P_\tau^R(p)}{P_\tau^R(-p)} = p \quad ???$$

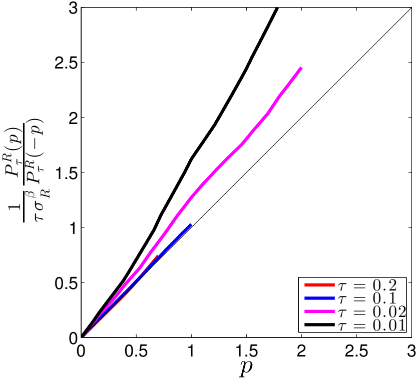
F.R. slope $c(\tau) \xrightarrow[R\to\infty]{} 1,\ R = 512$

$$c(\tau) = 1 + \left(\frac{t^{r,\sigma}_{dec,R}}{\tau}\right)^{4/3} = 1 + \left(\frac{c_\sigma}{\tau}\right)^{4/3} R^{-8/9}$$

Check Fluctuation Relation



F.R. $R = 2048$, approach 1 as $\tau \uparrow$ beyond decorrelation time



Local F.R. for $R = 2048$

$$\frac{1}{\tau} \log \frac{P^R_\tau(p)}{P^R_\tau(-p)} = \overline{\sigma^\beta}_R p + O(\tau^{-1}) = \beta \overline{\sigma}_R p + O(\tau^{-1})$$

Other examples: NS equation (periodic container $\mathcal{O}$) with viscosity ν (this was the origin of my conjecture ('996))

$$\dot{\vec{u}} + (\vec{u}\cdot\boldsymbol{\partial})\vec{u} = -\boldsymbol{\partial} p + \vec{g} + \nu\Delta\vec{u} = 0, \quad \boldsymbol{\partial}\cdot\vec{u} = 0$$

with forcing "on large scale" ($e.g.$ $g_{\mathbf{k}} \neq 0$ only for $\mathbf{k} = \pm(2,-1)$) and the equivalent (?) eq. "balanced" on the "dissipation" observable $En(\vec{u}) = \int_{\mathcal{O}}(\partial\vec{u}(x))^2 dx$

$$\dot{\vec{u}} + (\vec{u}\cdot\boldsymbol{\partial})\vec{u} = -\boldsymbol{\partial} p + \vec{g} + \alpha(\vec{u})\Delta\vec{u}, \qquad \boldsymbol{\partial}\cdot\vec{u} = 0$$

$$\alpha(\vec{u}) \stackrel{def}{=} \frac{\sum_{\vec{k}} \vec{k}^2\, \vec{g}_{\vec{k}}\cdot\vec{u}_{-\vec{k}}}{\sum_{\vec{k}} \vec{k}^4 |\vec{u}_{\vec{k}}|^2}, \qquad D = 2$$

which yields an evolution with constant enstrophy $En(\vec{u}) = \sum_{\mathbf{k}} \mathbf{k}^2 |\mathbf{u}_{\mathbf{k}}|^2$

In Fourier transform, if $\mathbf{k} = (k_1, k_2) \in \mathcal{Z}^2$, it is (if $F = 1, L = 2\pi$)

$$\mathbf{u}(\mathbf{x}) = \sum_{\mathbf{k}} \mathbf{u}_{\mathbf{k}} \frac{i\mathbf{k}^{\perp}}{|\mathbf{k}|} e^{-2\pi i \mathbf{k}\cdot\mathbf{x}}, \qquad \mathbf{u}_{\mathbf{k}} = \int \mathbf{u}(\vec{x}) \cdot \frac{-i\mathbf{k}^{\perp}}{|\mathbf{k}|} e^{2i\pi\mathbf{k}\cdot\mathbf{x}} \frac{d\mathbf{x}}{(2\pi)^2}$$

and in mode space the equations become

$$\dot{u}_{\mathbf{k}} = - \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}} \frac{(\mathbf{k}_1^{\perp} \cdot \mathbf{k}_2)(\mathbf{k}_2^{\perp} \cdot \mathbf{k}^{\perp})}{|\mathbf{k}_1||\mathbf{k}_2||\mathbf{k}|} u_{\mathbf{k}_1} u_{\mathbf{k}_2} - \frac{1}{R}\mathbf{k}^2 u_{\mathbf{k}} + f_{\mathbf{k}}$$

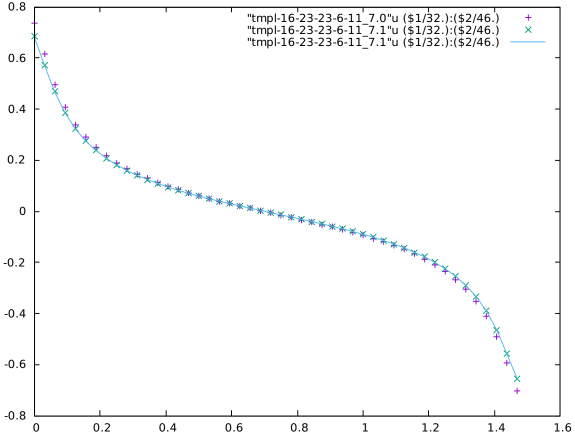
A groundbreaking work (SJ993),[2], presented evidence of equivalent reversible equations to NS 3D in the regime of developed turbulence provided the balance of the external work was imposed on a rather large number of observables (imposing the OK scaling on the energy). Several tests have been performed.

Early study in [15] up to $R = 10^3$ tests, in NS2D, the conjecture in a case with few modes (up to ~ 160); the possibility of balancing the external force by fixing instead of the enstrophy other observables, namely the energy or the "palinstrophy", seemed to follow the conjecture even for the Lyapunov spectrum, (surprisingly).

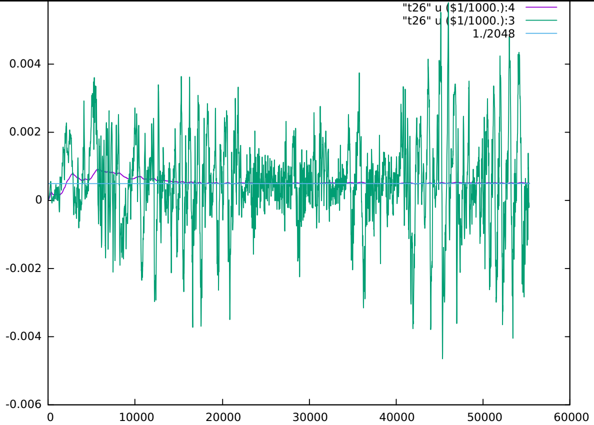
Later work, [16], shows that at higher number of modes equivalence might not extend to the entire Lyapunov spectrum: in particular at $R \sim 90$ and 960 modes ("31×31"). However the original conjecture, [3], refers to "local observales" (the Lyapunov exponents are not such).

A main computational difficulty seems to be the determination of the average value of the enstrophy En at fixed Reynolds R: this is time consuming as the average is reached on a long time scale.

It is interesting to present a few recent (preliminary) results (using the IHP cluster at the workshop).



Local Lyapunov spectrum in a 48 modes truncation (7×7) of NS2D: $(+)=$ viscous, $(\times)=$ reversible and $R = 128$.



At 960 modes and $R = 2048$: the evolution of the observable "reversible viscosity": $\alpha(\mathbf{u}) = \dfrac{\sum |\mathbf{k}|^2 F_{\mathbf{k}} \overline{\mathbf{u}}_{\mathbf{k}}}{\sum |\mathbf{k}|^4 |\mathbf{u}_{\mathbf{k}}|^2}$

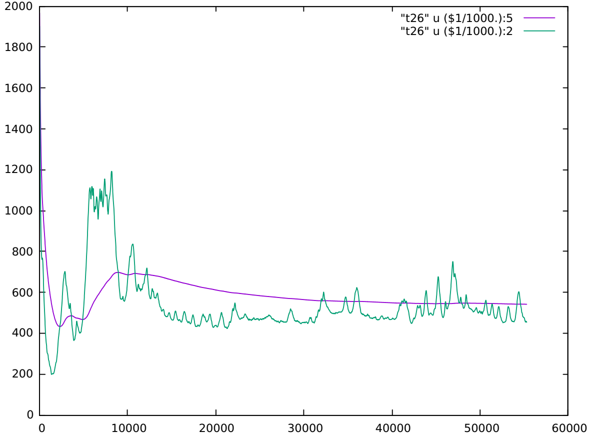
According to the equivalence the time average of α should be $\mathbf{\frac{1}{R}}$. Represents the fluctuating values of α at intervals of 10^4 steps (see below); the middle line is the running average of α (at intervals of 100 steps) and it converges to $\mathbf{\frac{1}{R}}$ (horiz. line).

The graph gives the values only every 1000 interaction steps (otherwise it would be just a black stain).

We see that once the attractor might be considered reached, for instance if the running average of the reversible viscosity is close to $\frac{1}{R}$ there are still wild fluctuations and the statisticscan be sampled.

For comparing the reversible and irreversible Lyapunov spectra it should be necessary to compute them over a time scale of 10^6 time steps. This is at the moment being attempted.

A further consequence of the equivalence could be that in the irreversible system the observable $\alpha(x)$ has the same statistics it has in the reversible system: hence it should satisfy the fluctuation relation. Thanks to the equivalence the fluctuation relation may predict the same symmetry of the large deviations.



The figure shows the slow approach of the running average to the average is slow (again 960 modes).
Further relevant references in [17, 18, 19, 20, 21].

Quoted references and related ones

[1] D. J. Evans and G. P. Morriss.
Statistical Mechanics of Nonequilibrium Fluids.
Academic Press, New-York, 1990.

[2] Z.S. She and E. Jackson.
Constrained Euler system for Navier-Stokes turbulence.
Physical Review Letters, 70:1255–1258, 1993.

[3] G. Gallavotti.
Equivalence of dynamical ensembles and Navier Stokes equations.
Physics Letters A, 223:91–95, 1996.

[4] G. Gallavotti.
Dynamical ensembles equivalence in fluid mechanics.
Physica D, 105:163–184, 1997.

[5] D. Ruelle.
A remark on the equivalence of isokinetic and isoenergetic thermostats in the thermodynamic limit.
Journal of Statistical Physics, 100:757–763, 2000.

[6] G. Gallavotti.
Breakdown and regeneration of time reversal symmetry in nonequilibrium statistical mechanics.
Physica D, 112:250–257, 1998.

[7] G. Gallavotti and V. Lucarini.
Equivalence of Non-Equilibrium Ensembles and Representation of Friction in Turbulent Flows: The Lorenz 96 Model.
Journal of Statistical Physics, ***156****, 1027-1065, 2014*, 156:1027–10653, 2014.

[8] D. Ruelle.
Measures describing a turbulent flow.
Annals of the New York Academy of Sciences, 357:1–9, 1980.

[9] G. Gallavotti and D. Cohen.
Dynamical ensembles in stationary states.
Journal of Statistical Physics, 80:931–970, 1995.

[10] D. Ruelle.
Turbulence, strange attractors and chaos.
World Scientific, New-York, 1995.

[11] G. Gallavotti.
Ergodicity: a historical perspective. equilibrium and nonequilibrium.
European Physical Journal H, 41:181–259, 2016.

[12] G. Gallavotti.
Nonequilibrium and irreversibility.
Theoretical and Mathematical Physics. Springer-Verlag and http://ipparco.roma1.infn.it & arXiv 1311.6448, Heidelberg, 2014.

[13] D. Ruelle and F. Takens.
On the nature of turbulence.
Communications in Mathematical Physics, 20:167–192, 1971.

[14] R. Livi, A. Politi, and S. Ruffo.
Distribution of characteristic exponents in the thermodynamic limit.
Journal of Physics A, 19:2033–2040, 1986.

[15] G. Gallavotti, L. Rondoni, and E. Segre.
Lyapunov spectra and nonequilibrium ensembles equivalence in 2d fluid.
Physica D, 187:358–369, 2004.

[16] C. Mejia-Monasterio and L. Rondoni.
On the fluctuation relation for Nosé-Hoover locally thermostated systems.
Journal of Statistical Physics, 133:617–637, 2008.

[17] G. Gallavotti.
Non equilibrium in statistical and fluid mechanics. ensembles and their equivalence. entropy driven intermittency.
Journal of Mathematical Physics, 41:4061–4081, 2000.

[18] G. Gallavotti.
Fluctuations and entropy driven space–time intermittency in Navier–Stokes fluids, in Mathematical Physics 2000, Ed. E. Fokas, A. Grigoryan, T. Kibble, B. Zegarlinski.
World Scientific, London, 2000.

[19] G. Gallavotti.
Microscopic chaos and macroscopic entropy in fluids.
Journal of Statistical Mechanics (JSTAT), 2006:P10011 (+9), 2006.

[20] G. Gallavotti.
On thermostats: Isokinetic or Hamiltonian? finite or infinite?
Chaos, 19:013101 (+7), 2008.

[21] G. Gallavotti and E. Presutti.
Fritionless thermostats and intensive constants of motion.
Journal of Statistical Physics, 139:618–629, 2010.

Also: http://arxiv.org & http://ipparco.roma1.infn.it