Diffusion versus Superdiffusion in noisy Hamiltonian systems

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One dimensional conservative asymmetric interacting particle systems display anomalous diffusion.

It means that if the system at equilibrium is locally disturbed by adding some extra energy, the perturbation will not diffuse like a Brownian Motion but like some superdiffusive process.

What is the superdiffusive process? How universal is it?
We are interested in one dimensional interacting particle systems which conserve some quantities (energy, density, momentum ...).

In a suitable space-time scale, the empirical conserved quantities (macroscopic, coarse-grained) will evolve according to some hyperbolic system of conservation laws (e.g. Euler equations). These are the hydrodynamic limits of the system.

Starting from these macroscopic equations (ignoring the details of the microscopic dynamics), the **Nonlinear Fluctuating Hydrodynamics Theory** (Spohn) predicts very precisely the form of the fluctuations of the conserved quantities.

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Microscopic models

• We consider a lattice field model \( \{ \eta_x(t) \in \mathbb{R} ; x \in \mathbb{Z} \} \) whose dynamics is composed of a deterministic part and of a stochastic part [B. Stoltz’11], [Basile, B., Olla’06].

• The stochastic part is introduced to provide a better control of the chaotic motion due to the nonlinearities of the interactions.
Nonlinear fluctuating hydrodynamics
Fractional Superdiffusion
Crossover between Diffusion and Superdiffusion

- Deterministic part: It is given by Hamilton equations:
  \[ \frac{d\eta_x}{dt} = (V'(\eta_x + 1) - V'(\eta_x - 1)), \quad x \in \mathbb{Z}, \]
  where \( V : \mathbb{R} \to \mathbb{R} \) is a well-behaved potential.

- Stochastic part: Independent Poisson processes (clock) on each bond \( \{x, x+1\} \).
  When the clock of \( \{x, x+1\} \) rings, \( \eta_x \) is exchanged with \( \eta_{x+1} \).
  The dynamics between two successive rings of the clocks is given by the Hamiltonian dynamics.
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• **Deterministic part:** It is given by Hamilton equations:

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• **Stochastic part:** Independent Poisson processes (clock) on each bond \( \{x, x + 1\} \). When the clock of \( \{x, x + 1\} \) rings, \( \eta_x \) is exchanged with \( \eta_{x+1} \). The dynamics between two successive rings of the clocks is given by the Hamiltonian dynamics.
The energy \[ \sum_x e^x = \sum_x V(\eta x) \].

The volume \[ \sum_x \eta x \].

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2. The volume \( \sum_x \eta_x \).
• The energy $\sum_x V(\eta_x)$ and the volume $\sum_x \eta_x$ are the only conserved quantities of the stochastic model (in a suitable sense) [B. Stoltz’11], [Fritz-Funaki-Lebowitz’93].

• The Gibbs equilibrium measures $\langle \cdot \rangle_{\tau,\beta}$ are parameterized by two parameters $(\tau, \beta) \in \mathbb{R} \times [0, \infty)$ and are product

$$\langle \cdot \rangle_{\tau,\beta} \sim \exp\left\{-\beta \sum_x (V(\eta_x) + \tau \eta_x)\right\} d\eta$$

are invariant for the dynamics.
Hydrodynamics: Euler equations

Theorem (B., Stoltz’11)

For $t < T^*$ (first shock), the hydrodynamic equations of the stochastic dynamics are given by the compressible Euler equations:

$$\begin{cases} 
\partial_t v = 2\partial_q \mathcal{P}, \\
\partial_t e = \partial_q \mathcal{P}^2.
\end{cases}$$

In the harmonic case $V(r) = r^2/2$, $\mathcal{P}(v, e) = v$ and $T^* = \infty$ (no shocks).

Proof based on [Olla, Varadhan, Yau’91], [Fritz-Funaki-Lebowitz’93].
Nonlinear fluctuating hydrodynamics predictions

The theory of *nonlinear fluctuating hydrodynamics* (Spohn) predicts the long time behavior of the equilibrium time-space correlation functions of the conserved fields \( g(x, t) = (\eta_x(t), e_x(t)) \)

\[
S_{\alpha \alpha'}(x, t) = \langle g_\alpha(x, t) g_{\alpha'}(0, 0) \rangle_{\tau, \beta} - \langle g_\alpha \rangle_{\tau, \beta} \langle g_{\alpha'} \rangle_{\tau, \beta}
\]

Gibbs measure: \( \langle \cdot \rangle_{\tau, \beta} \sim \exp\left\{ -\beta \sum_x (e_x + \tau \eta_x) \right\} d\eta. \)

**temperature:** \( \beta^{-1} \)

**pressure:** \( \tau \)
• Spohn’s theory: the long time behavior of the correlation functions of the conserved fields depends only on the function \((v, e) \rightarrow \mathcal{P}(v, e)\), and parameters \(\tau, \beta\), but **NOT** on the details of the microscopic dynamics.

• It is a *macroscopic* theory based on the validity of the hydrodynamics in the Euler time scale.
Nonlinear fluctuating hydrodynamics
Fractional Superdiffusion
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• $n = 1$ case: 2 UC
  • Edwards-Wilkinson (Gaussian)
  • "KPZ fixed point" (non Gaussian) $^a$

• $n \geq 2$ case:
  • richer (many UC),
  • different time scales involved

$^a$scaling limit of the solution of the KPZ equation

We consider now the harmonic case

\[ V(r) = \frac{r^2}{2} \Rightarrow e_x = \frac{\eta_x^2}{2} \]

Our first results confirm Spohn’s predictions in this case.
We define the space-time correlation of the energy

\[ E_t(x) = \langle (e_0(0) - 1/\beta) (e_t(x) - 1/\beta) \rangle_{\tau, \beta} \]

and the space-time correlation of the volume

\[ V_t(x) = \langle (\eta_0(0) - \tau) (\eta_t(x) - \tau) \rangle_{\tau, \beta} \]
Theorem (B., Gonçalves, Jara’16)

- **Volume:** \( \lim_{n \to \infty} V_{tn}^2(\lfloor nq \rfloor) = \frac{2}{\beta} \mathcal{V}_t(q), \quad t > 0, \quad q \in \mathbb{R}, \)
  \[ \partial_t \mathcal{V} = \Delta \mathcal{V}, \quad \text{heat equation} \]

- **Energy:** \( \lim_{n \to \infty} E_{tn}^{3/2}(\lfloor nq \rfloor) = \frac{2}{\beta^2} \mathcal{E}_t(q), \quad t > 0, \quad q \in \mathbb{R}, \)
  \[ \partial_t \mathcal{E} = -\frac{1}{\sqrt{2}} \left\{ (\Delta)^{3/4} - \nabla (\Delta)^{1/4} \right\} \mathcal{E}, \quad \text{skew fractional heat equation} \]

See also [Basile-B.-Olla’06,’09, Basile-Olla-Spohn’08, Mellet-Mischler-Mouhot’08, Jara-Komorowski-Olla’09, Delfini-Lepri-Livi-Mejia-Monasterio-Politi’10, Jara-Komorowski-Olla’15]
Proof:

We will relate the fractional Laplacian with solutions of the following extension problem. For a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, we consider the extension $u : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}$ that satisfies the equation

$$u(x, 0) = f(x) \quad (1.4)$$

$$\Delta_x u + \frac{a}{y} u_y + u_{yy} = 0 \quad (1.5)$$

The equation (1.5) can also be written as

$$\text{div} \left( y^a \nabla u \right) = 0 \quad (1.6)$$

Which is clearly the Euler-Lagrange equation for the functional

$$J(u) = \int_{y > 0} |\nabla u|^2 \ y^a \ dX \quad (1.7)$$

We will show that

$$C(-\Delta)^a f = \lim_{y \to 0^+} -y^a u_y = \frac{1}{1-a} \lim_{y \to 0} \frac{u(x, y) - u(x, 0)}{y^{1-a}}$$

Caffarelli & Silvestre, An extension problem related to the fractional Laplacian, CPDE ’07
Weak anharmonicity limit: Universality persistence

Weak anharmonic potential: $V(r) = r^2 + \gamma_n r^4 / 4$. 
Weak anharmonicity limit: Universality persistence

Weak anharmonic potential: $V(r) = r^2 + \gamma_n r^4/4$.

- If $\gamma_n = 0$ then the energy fluctuation field is described in the time scale $tn^{3/2}$ by a $3/2$ stable asymmetric Lévy process and the volume fluctuation field is described in the time scale $tn^2$ by a Brownian motion.
Weak anharmonicity limit: Universality persistence

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- If \( \gamma_n = O(1) \) and \( \tau = 0, \beta > 0 \), then NLFH [Spohn-Stoltz’15] indicates that the energy-volume fluctuation fields are described by the same processes.
Theorem (B., Gonçalves, Jara, Simon ’17)

The harmonic scenario persists in the nonlinear regime if $\gamma_n$ is sufficiently small. It is valid for the energy for $\gamma_n \ll n^{-1/4}$ and for the volume for $\gamma_n \ll n^{-1/2}$.

Extension of this result outside of this regime requires new ideas.
Crossover between Diffusion and Superdiffusion

How can we cross different Universality Classes by tuning the parameters of the model?

Weakly harmonic chain

- Consider the harmonic chain where the potential $V(r)$ is now
  \[ V(r) = \frac{c}{n^b} r^2 \]
  with $c, b > 0$ two positive constants.
- We look at the system in the time scale $tn^a$, $a > 0$, such that the energy field has a non-trivial limit.
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Diffusion versus Superdiffusion
For $b = 1/3$, $a = 2$ the energy limiting field is described by a Levy process interpolating between the asymmetric stable Levy process and the Brownian motion.
The generator of the interpolating Levy process is

$$\Delta - \frac{c^{3/2}}{\sqrt{2}} \left\{ (-\Delta)^{3/4} - \nabla (-\Delta)^{1/4} \right\}.$$  

As $c \to \infty$, scaled with $c$, it goes to the skew $3/4$-fractional Laplacian.

As $c \to 0$, it goes to the Laplacian.
Consider the initial process (harmonic chain + exchange noise) and add a second stochastic perturbation with intensity $\gamma_n = c_n - b$, $c, b > 0$, which consists to flip independently on each site at exponential times the variable $\eta_x$ into $-\eta_x$.

The energy is conserved but the volume $\sum_x \eta_x$ is not (stricto sensu, only if $b = \infty$).

We look at the system in the time scale $t_n^a$, $a > 0$, such that the energy field has a non-trivial limit.

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**Evanescent flip noise limit**
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- Consider the initial process (harmonic chain + exchange noise) and add a second stochastic perturbation with intensity \( \gamma_n = cn^{-b}, \) \( c, b > 0, \) which consists to flip independently on each site at exponential times the variable \( \eta_x \) into \( -\eta_x. \)
- The energy is conserved but the volume \( \sum_x \eta_x \) is not (stricto sensu, only if \( b = \infty \)).
Evanescent flip noise limit

- Consider the initial process (harmonic chain + exchange noise) and add a second stochastic perturbation with intensity \( \gamma_n = cn^{-b} \), \( c, b > 0 \), which consists to flip independently on each site at exponential times the variable \( \eta_x \) into \(-\eta_x\).
- The energy is conserved but the volume \( \sum_x \eta_x \) is not (stricto sensu, only if \( b = \infty \)).
- We look at the system in the time scale \( tn^a \), \( a > 0 \), such that the energy field has a non-trivial limit.
For $b = 1$, $a = 3/2$ the energy limiting field is described by a Levy process interpolating between the Brownian motion and the asymmetric stable Levy process.
• The Fourier symbol of the generator of the interpolating Levy process is

\[ \frac{1}{2\sqrt{3}} \frac{(2\pi k)^2}{\sqrt{c + i\pi k}}, \quad k \in \mathbb{R}. \]

• As \( c \to 0 \) it goes to the Fourier symbol of the skew 3/4-fractional Laplacian.

• As \( c \to \infty \), scaled with \( c \), it goes to the Fourier symbol of the Laplacian.
Nonlinear fluctuating hydrodynamics
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Space of UC

KPZ / KPZ

??

2 / 2

Coupled KPZ eq. ??

Evanescent flip noise

Weak harmonicity

\( \frac{3}{2} / 2 \)

\( \frac{5}{3} / \text{KPZ} \)

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