University of Nice

Diffusion versus Superdiffusion in noisy Hamiltonian systems

C. BERNARDIN

- *One dimensional conservative asymmetric* interacting particle systems display anomalous diffusion.
- It means that if the system at equilibrium is locally disturbed by adding some extra energy, the perturbation will not diffuse like a Brownian Motion but like some *superdiffusive process.*
- What is the superdiffusive process? How *universal* is it?

- We are interested in one dimensional interacting particle systems which conserve some quantities (energy, density, momentum ...).
- In a suitable *space-time scale*, the empirical conserved quantities (macroscopic, coarse-grained) will evolve according to some hyperbolic system of conservation laws (e.g. Euler equations). These are the *hydrodynamic limits* of the system.
- Starting from these macroscopic equations (ignoring the details of the microscopic dynamics), the *Nonlinear Fluctuating Hydrodynamics Theory* (Spohn) predicts very precisely the form of the fluctuations of the conserved quantities.

Microscopic models

- We consider a lattice field model $\{\eta_x(t) \in \mathbb{R}\,;\ x \in \mathbb{Z}\}$ whose dynamics is composed of a deterministic part and of a stochastic part [B. Stoltz'11], [Basile, B., Olla'06] .
- The stochastic part is introduced to provide a better control of the chaotic motion due to the nonlinearities of the interactions.

- Deterministic part: It is given by Hamilton equations:

$$d\eta_x = (V'(\eta_{x+1}) - V'(\eta_{x-1}))\,dt, \quad x \in \mathbb{Z},$$

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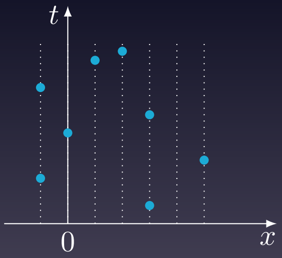
- Stochastic part :

Independent Poisson processes (clock) on each bond $\{x, x+1\}$. When the clock of $\{x, x+1\}$ rings, η_x is exchanged with η_{x+1}. The dynamics between two successive rings of the clocks is given by the Hamiltonian dynamics.



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2. The volume $\sum_x \eta_x$.

- The energy $\sum_x V(\eta_x)$ and the volume $\sum_x \eta_x$ are the **only** conserved quantities of the stochastic model (in a suitable sense) [B. Stoltz'11], [Fritz-Funaki-Lebowitz'93].
- The Gibbs equilibrium measures $\langle \cdot \rangle_{\tau,\beta}$ are parameterized by two parameters $(\tau, \beta) \in \mathbb{R} \times [0, \infty)$ and are product

$$\langle \cdot \rangle_{\tau,\beta} \sim \exp\{-\beta \sum_x (V(\eta_x) + \tau \eta_x)\} d\eta$$

are invariant for the dynamics.

Hydrodynamics: Euler equations

Theorem (B., Stoltz'11)

For $t < T^*$ (first shock), the hydrodynamic equations of the stochastic dynamics are given by the compressible Euler equations:

$$\begin{cases} \partial_t \mathfrak{v} = 2\partial_q \mathcal{P}, \\ \partial_t \mathfrak{e} = \partial_q \mathcal{P}^2. \end{cases}$$

In the harmonic case $V(r) = r^2/2$, $\mathcal{P}(\mathfrak{v}, \mathfrak{e}) = \mathfrak{v}$ and $T^* = \infty$ (no shocks).

Proof based on [Olla, Varadhan, Yau'91], [Fritz-Funaki-Lebowitz'93].

Nonlinear fluctuating hydrodynamics predictions

The theory of *nonlinear fluctuating hydrodynamics* (Spohn) predicts the long time behavior of the equilibrium time-space correlation functions of the conserved fields $g(x,t) = (\eta_x(t), e_x(t))$

$$S_{\alpha\alpha'}(x,t) = \langle g_\alpha(x,t)\, g_{\alpha'}(0,0)\rangle_{\tau,\beta} - \langle g_\alpha\rangle_{\tau,\beta}\ \langle g_{\alpha'}\rangle_{\tau,\beta}$$

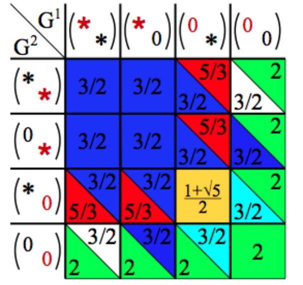
Gibbs measure: $$\langle\cdot\rangle_{\tau,\beta} \sim \exp\{-\beta \sum_x (e_x + \tau\eta_x)\}\, d\eta.$$

temperature: β^{-1} pressure: τ

- Spohn's theory: the long time behavior of the correlation functions of the conserved fields depends only on the function $(v, e) \to \mathcal{P}(v, e)$, and parameters τ, β, but **NOT** on the details of the microscopic dynamics.
- It is a *macroscopic* theory based on the validity of the hydrodynamics in the Euler time scale.

- $n = 1$ case: 2 UC
  - Edwards-Wilkinson (Gaussian)
  - "KPZ fixed point" (non Gaussian) [a]
- $n \geq 2$ case:
  - richer (many UC),
  - different time scales involved

[a] scaling limit of the solution of the KPZ equation



Popkov et al., J. Stat. Phys. 160 (2015), $n = 2$

We consider now the harmonic case

$$V(r) = r^2/2 \;\Rightarrow\; e_x = \eta_x^2/2$$

Our first results confirm Spohn's predictions in this case.

We define the space-time correlation of the energy

$$E_t(x) = \Big\langle \big(e_0(0) - \tfrac{1}{\beta}\big)\big(e_t(x) - \tfrac{1}{\beta}\big) \Big\rangle_{\tau,\beta}$$

and the space-time correlation of the volume

$$V_t(x) = \big\langle \big(\eta_0(0) - \tau\big)\big(\eta_t(x) - \tau\big) \big\rangle_{\tau,\beta}$$

Theorem (B., Gonçalves, Jara'16)

- ***Volume:*** $\lim_{n\to\infty} V_{tn^2}\big([nq]\big) = \frac{2}{\beta}\,\mathcal{V}_t(q), \quad t>0, \quad q\in\mathbb{R},$

$$\partial_t \mathcal{V} = \Delta \mathcal{V}, \quad \textit{heat equation}$$

- ***Energy:*** $\lim_{n\to\infty} E_{tn^{3/2}}\big([nq]\big) = \frac{2}{\beta^2}\,\mathcal{E}_t(q), \quad t>0, \quad q\in\mathbb{R},$

$$\partial_t \mathcal{E} = -\frac{1}{\sqrt{2}}\left\{(-\Delta)^{3/4} - \nabla(-\Delta)^{1/4}\right\}\mathcal{E}, \quad \textit{skew fractional heat equation}$$

See also [Basile-B.-Olla'06,'09, Basile-Olla-Spohn'08, Mellet-Mischler-Mouhot'08, Jara-Komorowski-Olla'09, Delfini-Lepri-Livi-Mejia-Monasterio-Politi'10, Jara-Komorowski-Olla'15]

Proof:

We will relate the fractional Laplacian with solutions of the following extension problem. For a function $f : \mathbb{R}^n \to \mathbb{R}$, we consider the extension $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ that satisfies the equation

$$u(x, 0) = f(x) \tag{1.4}$$

$$\Delta_x u + \frac{a}{y} u_y + u_{yy} = 0 \tag{1.5}$$

The equation (1.5) can also be written as

$$\operatorname{div}(y^a \nabla u) = 0 \tag{1.6}$$

Which is clearly the Euler-Lagrange equation for the functional

$$J(u) = \int_{y>0} |\nabla u|^2 \; y^a \mathrm{d}X \tag{1.7}$$

We will show that

$$C(-\Delta)^s f = \lim_{y \to 0^+} -y^a u_y = \frac{1}{1-a} \lim_{y \to 0} \frac{u(x,y) - u(x,0)}{y^{1-a}}$$

Caffarelli &Silvestre, An extension problem related to the fractional Laplacian, CPDE '07

Weak anharmonicity limit: Universality persistence

Weak anharmonic potential : $V(r) = r^2 + \gamma_n\, r^4/4$.

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- If $\gamma_n = O(1)$ and $\tau = 0, \beta > 0,$ then NLFH [Spohn-Stoltz'15] indicates that the energy-volume fluctuation fields are described by the same processes.

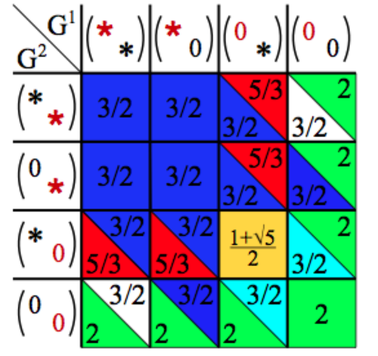
Theorem (B., Gonçalves, Jara, Simon ’17)

The harmonic scenario persists in tne nonlinear regime if γ_n *is sufficiently small. It is valid for the energy for* $\gamma_n \ll n^{-1/4}$ *and for the volume for* $\gamma_n \ll n^{-1/2}$*.*

Extension of this result outside of this regime requires new ideas.

Crossover between Diffusion and Superdiffusion

How can we cross different Universality Classes by tuning the parameters of the model ?

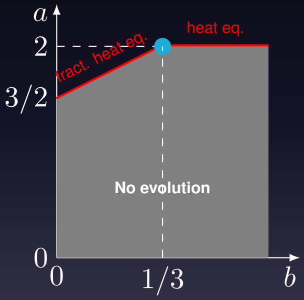


Popkov et al., J. Stat. Phys. 160 (2015)

Weakly harmonic chain

- Consider the harmonic chain where the potential $V(r)$ is now
$$V(r) = \frac{c}{n^b}\, r^2$$
with $c, b > 0$ two positive constants.
- We look at the system in the time scale tn^a, $a > 0$, such that the energy field has a non-trivial limit.

[B., Gonçalves, Jara'16]



For $b = 1/3$, $a = 2$ the energy limiting field is described by a Levy process interpolating between the asymmetric stable Levy process and the Brownian motion.

- The generator of the interpolating Levy process is

$$\Delta - \frac{c^{3/2}}{\sqrt{2}}\{(-\Delta)^{3/4} - \nabla(-\Delta)^{1/4}\}.$$

- As $c \to \infty$, scaled with c, it goes to the skew $3/4$-fractional Laplacian.
- As $c \to 0$, it goes to the Laplacian.

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- Consider the initial process (harmonic chain + exchange noise) and add a second stochastic perturbation with intensity $\gamma_n = cn^{-b}$, $c, b > 0$, which consists to flip independently on each site at exponential times the variable η_x into $-\eta_x$.

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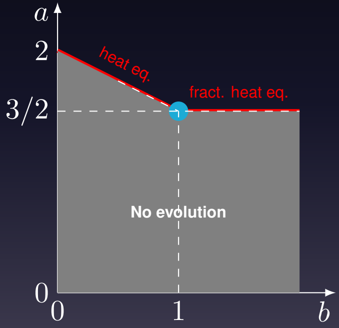
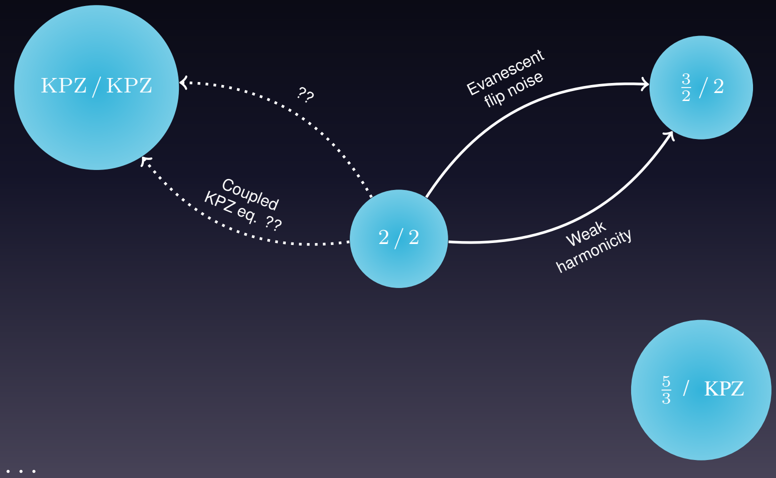
[B., Gonçalves, Jara, Sasada, Simon '15], [B., Gonçalves, Jara, Simon '16]



For $b = 1$, $a = 3/2$ the energy limiting field is described by a Levy process interpolating between the Brownian motion and the asymmetric stable Levy process.

- The Fourier symbol of the generator of the interpolating Levy process is
$$\frac{1}{2\sqrt{3}}\frac{(2i\pi k)^2}{\sqrt{c+i\pi k}}, \qquad k \in \mathbb{R}.$$
- As $c \to 0$ it goes to the Fourier symbol of the skew $3/4$-fractional Laplacian.
- As $c \to \infty$, scaled with c, it goes to the Fourier symbol of the Laplacian.

Space of UC


KPZ / KPZ
??
Coupled KPZ eq. ??
2 / 2
Evanescent flip noise
$\frac{3}{2}$ / 2
Weak harmonicity
$\frac{5}{3}$ / KPZ
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