Invariant measures in coupled KPZ equations

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Plan of the talk

- Coupled KPZ (Kardar-Parisi-Zhang) equations
  – Motivation: nonlinear fluctuating hydrodynamics
- Quick overview of results with Hoshino (JFA **273**, 2017)
  – Two approximating equations
  – Trilinear condition (T) for coupling constants Γ
  – Invariant measure
  – Global-in-time existence
- Role of (T)
  – Invariant measure, renormalizations (for 4th order terms)
- Extensions of Ertaş-Kardar's example, not satisfying (T) but having Invariant measure

Multi-component coupled KPZ equation

- $\mathbb{R}^d$-valued KPZ eq for $h(t,x) = (h^\alpha(t,x))_{\alpha=1}^d$ on $\mathbb{T} = [0,1)$:

$$\partial_t h^\alpha = \frac{1}{2}\partial_x^2 h^\alpha + \frac{1}{2}\Gamma^\alpha_{\beta\gamma}\partial_x h^\beta \partial_x h^\gamma + \sigma^\alpha_\beta \xi^\beta \qquad (\sigma, \Gamma)_{KPZ}$$

- We use Einstein's convention.
- $\xi(t,x) = (\xi^\alpha(t,x))_{\alpha=1}^d \big(\equiv \dot{W}(t,x) \big)$ is an $\mathbb{R}^d$-valued space-time Gaussian white noise with covariance structure:

$$E[\xi^\alpha(t,x)\xi^\beta(s,y)] = \delta^{\alpha\beta}\delta(x-y)\delta(t-s).$$

- Coupled KPZ is ill-posed, since noise is irregular and doesn't match with nonlinear term. ($h \in C_{t,x}^{\frac{1}{4}-,\frac{1}{2}-}$ a.s. when $\Gamma = 0$)
- We need to introduce approximations with smooth noises and renormalization for $(\sigma, \Gamma)_{KPZ}$. Indeed, one can introduce two types of approximations: one is simple, the other is suitable to study invariant measures ($d = 1$: F-Quastel 2015).

- The constants $\Gamma^\alpha_{\beta\gamma}$ satisfy bilinear condition

$$\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta} \text{ for all } \alpha, \beta, \gamma,$$

and (sometimes) trilinear condition

$$\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta} = \Gamma^\gamma_{\beta\alpha} \text{ for all } \alpha, \beta, \gamma. \qquad \textbf{(T)}$$

(cf. Ferrari-Sasamoto-Spohn 2013, Kupiainen-Marcozz 2017)

- $\sigma = (\sigma^\alpha_\beta)$ is an invertible matrix.

- Since σ is invertible, $\hat{h} = \sigma^{-1} h$ transforms $(\sigma, \Gamma)_{KPZ}$ to $(I, \hat{\Gamma} = \sigma \circ \Gamma)_{KPZ}$, where

$$(\sigma \circ \Gamma)^{\alpha}_{\beta\gamma} := (\sigma^{-1})^{\alpha}_{\alpha'} \Gamma^{\alpha'}_{\beta'\gamma'} \sigma^{\beta'}_{\beta} \sigma^{\gamma'}_{\gamma}.$$

  Thus, the KPZ equation with $\sigma = I$ is considered as a canonical form.
- The operation (coordinate change) $\Gamma \mapsto \sigma \circ \Gamma$ keeps the bilinearity, but not the trilinearity.
- We should say (σ, Γ) satisfies trilinear condition, iff $\hat{\Gamma} := \sigma \circ \Gamma$ satisfies (T).
- In the following, we assume $\sigma = I$.

Two coupled KPZ approximating equations ($d=1$: FQ '15)

We replace the noise by smooth one: $\eta^\varepsilon = \frac{1}{\varepsilon}\eta(\frac{x}{\varepsilon}) \to \delta_0$ as usual.

- Approx. eq-1 (usual): $h^\alpha = h^{\varepsilon,\alpha}$

$$\partial_t h^\alpha = \frac{1}{2}\partial_x^2 h^\alpha + \frac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x h^\beta \partial_x h^\gamma - c^\varepsilon \delta^{\beta\gamma} - B^{\varepsilon,\beta\gamma}) + \xi^\alpha * \eta^\varepsilon, \quad (1)$$

where $c^\varepsilon = \frac{1}{\varepsilon}\|\eta\|^2_{L^2(\mathbb{R})} (= O(\frac{1}{\varepsilon}))$ and $B^{\varepsilon,\beta\gamma}$ $(= O(\log \frac{1}{\varepsilon})$ in general) is another renormalization factor.

- Approx. eq-2 (suitable to study inv meas): $\tilde{h}^\alpha = \tilde{h}^{\varepsilon,\alpha}$

$$\partial_t \tilde{h}^\alpha = \frac{1}{2}\partial_x^2 \tilde{h}^\alpha + \frac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x \tilde{h}^\beta \partial_x \tilde{h}^\gamma - c^\varepsilon \delta^{\beta\gamma} - \tilde{B}^{\varepsilon,\beta\gamma}) * \eta_2^\varepsilon + \xi^\alpha * \eta^\varepsilon,$$

with a renormalization factor $\tilde{B}^{\varepsilon,\beta\gamma}$, where $\eta_2^\varepsilon = \eta^\varepsilon * \eta^\varepsilon$. (2)

- The idea behind (2) is the fluctuation-dissipation relation.
- Renorm-factor $c^\varepsilon \equiv c_\epsilon^{\vee} = O(\frac{1}{\varepsilon})$ is from 2nd order terms in the expansion, while R-factors $B^{\varepsilon,\beta\gamma}$ and $\tilde{B}^{\varepsilon,\beta\gamma} = O(log\frac{1}{\varepsilon})$ are from 4th order terms involving $C^\varepsilon = c_\epsilon^{\vee\vee}$, $D^\varepsilon = c_\epsilon^{\curlyvee}$.

Quick overview of results on coupled KPZ eq

(F-Hoshino, JFA 2017)

- Convergence of h^ε and $\tilde{h}^\varepsilon$ and Local well-posedness of coupled KPZ eq $(\sigma, \Gamma)_{KPZ}$ by applying paracontrolled calculus due to Gubinelli-Imkeller-Perkowski 2015
  (Cole-Hopf doesn't work for coupled eq. in general. In 1D, we used it and showed Boltzmann-Gibbs principle, FQ 2015)
- 2nd approx. fits to identify invariant measure under (T)
- Global solvability for a.s.-initial data under an invariant measure under (T) (similar to Da Prato-Debussche)
- Strong Feller property (due to Hairer-Mattingly 2016)
- Global well-posedness (existence, uniqueness) under (T) ergodicity and uniqueness of invariant measure
- A priori estimates for 1st approximation (1) under (T)

Convergence of h^ε and $\tilde{h}^\varepsilon$ and Local well-posedness of coupled KPZ eq $(\sigma, \Gamma)_{KPZ}$ (we take $\sigma = I$): $\mathcal{C}^\kappa = (\mathcal{B}^\kappa_{\infty,\infty}(\mathbb{T}))^d$, $\kappa \in \mathbb{R}$ denotes $\mathbb{R}^d$-valued Besov space on $\mathbb{T}$.

Theorem 1

(1) Assume $h_0 \in \cup_{\delta>0} \mathcal{C}^\delta$, then a unique solution h^ε of (1) exists up to some $T^\varepsilon \in (0, \infty]$ and $\bar{T} = \liminf_{\varepsilon \downarrow 0} T^\varepsilon > 0$ holds. With a proper choice of $B^{\varepsilon,\beta\gamma}$, h^ε converges in prob. to some h in $C([0,T], \mathcal{C}^{\frac{1}{2}-\delta})$ for every $\delta > 0$ and $0 < T \le \bar{T}$.

(2) Similar result holds for the solution $\tilde{h}^\varepsilon$ of (2) with some limit $\tilde{h}$. Under proper choices of $B^{\varepsilon,\beta\gamma}$ and $\tilde{B}^{\varepsilon,\beta\gamma}$, we can actually make $h = \tilde{h}$.

$$\partial_t h^\alpha = \tfrac{1}{2}\partial_x^2 h^\alpha + \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x h^\beta \partial_x h^\gamma - c^\varepsilon \delta^{\beta\gamma} - B^{\varepsilon,\beta\gamma}) + \xi^\alpha * \eta^\varepsilon \qquad (1)$$

$$\partial_t \tilde{h}^\alpha = \tfrac{1}{2}\partial_x^2 \tilde{h}^\alpha + \tfrac{1}{2}\Gamma^\alpha_{\beta\gamma}(\partial_x \tilde{h}^\beta \partial_x \tilde{h}^\gamma - c^\varepsilon \delta^{\beta\gamma} - \tilde{B}^{\varepsilon,\beta\gamma}) * \eta_2^\varepsilon + \xi^\alpha * \eta^\varepsilon \qquad (2)$$

Results under (T): Cancellation in Log-Renormalizations, Invariant measure = Wiener measure, difference of two limits.

Theorem 2

*Assume the trilinear condition (**T**).*
(1) Then, $B^{\varepsilon,\beta\gamma}, \tilde{B}^{\varepsilon,\beta\gamma} = O(1)$ *so that the solutions of* (1) *with* $B = 0$ *and* (2) *with* $\tilde{B} = 0$ *converge. In the limit, we have*

$$\tilde{h}^\alpha(t,x) = h^\alpha(t,x) + c^\alpha t, \quad 1 \le \alpha \le d,$$

where

$$c^\alpha = \frac{1}{24} \sum_{\gamma,\gamma'} \Gamma^\alpha_{\alpha'\alpha''} \Gamma^{\alpha'}_{\gamma\gamma'} \Gamma^{\alpha''}_{\gamma\gamma'}.$$

(2) Moreover, the distribution of $\{\partial_x B\}_{x\in\mathbb{T}}$ *(*B *= periodic BM) is invariant under the tilt process* $u = \partial_x h$ *(or periodic Wiener measure on the quotient space* $\mathcal{C}^{\frac{1}{2}-\delta}/\sim$ *where* $h \sim h + c$*).*

- Remark (F-Quastel 2015, stationary case): When $d = 1$ (i.e., scalar-valued eq), (T) is automatic and solutions of two approx. eqs without log-renormalizations satisfy

$$\lim_{\varepsilon \downarrow 0} \tilde{h}^{\varepsilon} = \lim_{\varepsilon \downarrow 0} h^{\varepsilon} + \frac{t}{24} \ \left(= h_{CH} + \frac{t}{24} \right).$$

Global existence for a.s.-initial values under stationary measure

- We assume (T) and initial value $h(0)$ is given by $h(0,0)=0$ and $u(0) := \partial_x h(0) \underset{\text{law}}{=} (\partial_x B)_{x\in\mathbb{T}}$. Then, similarly to Da Prato-Debussche, $u = \partial_x h$ satisfies

Theorem 3

For every $T>0, p\geq 1, \kappa>0$, we have

$$E\left[\sup_{t\in[0,T]} \|u(t;u_0)\|^p_{-\frac{1}{2}-\kappa}\right] < \infty$$

In particular, $T_{survival}(u(0)) = \infty$ for a.a.-$u(0)$.

- Global existence for all given $u(0)$: In the scalar-valued case, this is immediate, since the limit is Cole-Hopf solution. Hairer-Mattingly 2016 proved this for coupled eq. by showing the strong Feller property on $\mathcal{C}^{\alpha-1}, \alpha\in(0,\frac{1}{2})$.

Cancellation of Log-Renorm's, $^\exists$Invariant measure without (T)

- Example (Ertaș and Kardar 1992: $d = 2$)

$$\begin{aligned} \partial_t h^1 &= \tfrac{1}{2}\partial_x^2 h^1 + \tfrac{1}{2}\{\lambda_1(\partial_x h^1)^2 + \lambda_2(\partial_x h^2)^2\} + \xi^1, \\ \partial_t h^2 &= \tfrac{1}{2}\partial_x^2 h^2 + \lambda_1 \partial_x h^1 \partial_x h^2 + \xi^2 \end{aligned} \qquad \text{(EK)}$$

  Γ satisfies (T) only when $\lambda_1 = \lambda_2$.

- However, under the transform $\hat h = sh$ with $s = \begin{pmatrix} \lambda_1 & (\lambda_1\lambda_2)^{1/2} \\ \lambda_1 & -(\lambda_1\lambda_2)^{1/2} \end{pmatrix}$, (EK) is transformed into

$$\partial_t \hat h^\alpha = \tfrac{1}{2}\partial_x^2 \hat h^\alpha + \tfrac{1}{2}(\partial_x \hat h^\alpha)^2 + s^\alpha_\beta \xi^\beta. \qquad (\mathrm{EK}_T)$$

- $\hat\Gamma = s \circ \Gamma$ in (EK_T) is given by $\hat\Gamma^\alpha_{\alpha\alpha} = 1, = 0$ otherwise, so that $\hat\Gamma$ satisfies (T). But, (EK) is the canonical form (with $\sigma = I$) and not (EK_T).

- (EK) doesn't satisfy (T).
- However, since nonlinear term is decoupled in (EK_T), the Cole-Hopf transform $Z^\alpha = \exp \hat{h}^\alpha$ works for each component so that global well-posedness follows.
- Log-renormalization factors are unnecessary.
- Invariant measure exists whose marginals are Wiener measures, but the joint distribution of such invariant measure is unclear (presumably non-Gaussian).
- Indeed, with the help of Rellich type theorem, one can easily check the tightness on the space $\mathcal{C}_0^{\delta-1}/\sim$ of the Cesàro mean $\mu_T = \frac{1}{T}\int_0^T \mu(t)dt$ of the distributions $\mu(t)$ of $\partial_x \hat{h}(t)$ having an initial distribution $\otimes_\alpha \mu_\alpha$, so that the limit of μ_T as $T \to \infty$ is an invariant measure.
- Invariance of marginals means that of $E[\Phi(h(t))]$ in t only for a subclass of Φ s.t. $\Phi = \Phi(h^\alpha)$ for $\alpha = 1$ or 2.

Reason of cancellation of log-renormalization factors

- Formulas of Renormalization factors $B^{\varepsilon,\beta\gamma}, \tilde{B}^{\varepsilon,\beta\gamma}$

$$B^{\varepsilon,\beta\gamma} = F^{\beta\gamma} C^{\varepsilon} + 2G^{\beta\gamma} D^{\varepsilon}, \; \tilde{B}^{\varepsilon,\beta\gamma} = F^{\beta\gamma} \tilde{C}^{\varepsilon} + 2G^{\beta\gamma} \tilde{D}^{\varepsilon},$$

where

$$F^{\beta\gamma} = \Gamma^{\beta}_{\gamma_1\gamma_2} \Gamma^{\gamma}_{\gamma_1\gamma_2}, \; G^{\beta\gamma} = \Gamma^{\beta}_{\gamma_1\gamma_2} \Gamma^{\gamma_1}_{\gamma\gamma_2},$$

$$C^{\varepsilon} + 2D^{\varepsilon} = -\tfrac{1}{12} + O(\varepsilon), \quad \tilde{C}^{\varepsilon} + 2\tilde{D}^{\varepsilon} = 0,$$

$(c^{\varepsilon} =$ , $C^{\varepsilon} =$ , $D^{\varepsilon} =$ $)$

- Trilinear condition (T) $\iff$ "$F = G$" $\iff$ $B, \tilde{B} = O(1)$
- But, for cancellation of log-renormalization factors, what we really need is: "$\Gamma B, \Gamma \tilde{B} = O(1)$". This holds if $\Gamma F = \Gamma G$.

$$\partial_t h^{\alpha} = \tfrac{1}{2}\partial_x^2 h^{\alpha} + \tfrac{1}{2}\Gamma^{\alpha}_{\beta\gamma}(\partial_x h^{\beta}\partial_x h^{\gamma} - c^{\varepsilon}\delta^{\beta\gamma} - B^{\varepsilon,\beta\gamma}) + \xi^{\alpha} * \eta^{\varepsilon} \quad (1)$$

$$\partial_t \tilde{h}^{\alpha} = \tfrac{1}{2}\partial_x^2 \tilde{h}^{\alpha} + \tfrac{1}{2}\Gamma^{\alpha}_{\beta\gamma}(\partial_x \tilde{h}^{\beta}\partial_x \tilde{h}^{\gamma} - c^{\varepsilon}\delta^{\beta\gamma} - \tilde{B}^{\varepsilon,\beta\gamma}) * \eta_2^{\varepsilon} + \xi^{\alpha} * \eta^{\varepsilon} \quad (2)$$

- "$\Gamma F = \Gamma G$" holds iff Γ satisfies the condition

$$\Gamma^{\alpha}_{\beta\gamma}\Gamma^{\beta}_{\gamma_1\gamma_2}\Gamma^{\gamma}_{\gamma_1\gamma_2} = \Gamma^{\alpha}_{\beta\gamma}\Gamma^{\beta}_{\gamma_1\gamma_2}\Gamma^{\gamma_1}_{\gamma\gamma_2}, \quad {}^{\forall}\alpha$$

- This holds under (T) and also for Ertaş-Kardar's example.
- We can summarize as

$$\begin{aligned}(T) &\iff \text{"}F = G\text{"} \\ &\Longrightarrow \text{"}\Gamma F = \Gamma G\text{"} \\ &\iff \text{Cancellation of log-renormalization factors}\end{aligned}$$

Infinitesimal invariance (to explain the role of (T))

- $\mathcal{L} = \mathcal{L}_0 + \mathcal{A}$: genetaror of KPZ eq ($\sigma = I$).
- $\mathcal{L}_0$ is the generator of OU-part, while $\mathcal{A}$ is that of nonlinear part (we ignore renormalization factors):

$$\mathcal{L}_0\Phi = \frac{1}{2}\sum_{\alpha}\left\{\int_{\mathbb{T}} D^2_{h^\alpha(x)}\Phi\,dx + \int_{\mathbb{T}} \ddot{h}^\alpha(x) D_{h^\alpha(x)}\Phi\,dx\right\}$$

$$\mathcal{A}\Phi = \frac{1}{2}\sum_{\alpha,\beta,\gamma}\Gamma^\alpha_{\beta\gamma}\int_{\mathbb{T}} \dot{h}^\beta(x)\dot{h}^\gamma(x) D_{h^\alpha(x)}\Phi\,dx,$$

and $\dot{h}^\beta(x) := \partial_x h^\beta(x)$, $\ddot{h}^\alpha(x) := \partial_x^2 h^\alpha(x)$.

- The infinitesimal invariance $(ST)_{\mathcal{L}}$ for ν $\underset{\text{def}}{\Longleftrightarrow}$ "$\int \mathcal{L}\Phi d\nu = 0,^\forall \Phi$"

- If the invariant measure ν is Gaussian, $(ST)_{\mathcal{L}_0}$ is the condition for 2nd order Wiener chaos of Φ, while $(ST)_{\mathcal{A}}$ is that for 3rd order Wiener chaos of Φ. Therefore, the condition $(ST)_{\mathcal{L}}$ is separated into two conditions:

$$(ST)_{\mathcal{L}} \Longleftrightarrow (ST)_{\mathcal{L}_0} + (ST)_{\mathcal{A}}$$

- $\mathcal{L}_0$ is OU-op and $(ST)_{\mathcal{L}_0}$ determines $\nu =$ Wiener meas.

Trilinear condition (T) $\iff \nu$ satisfies $(ST)_{\mathcal{A}}$

- We have the integration-by-parts formula for $\nu =$ Wiener measure (actually we need to discuss at ε-level):

$$\int \mathcal{A}\Phi d\nu = -\frac{1}{2}\Gamma^{\alpha}_{\beta\gamma} c^{\beta\gamma}_{\alpha},$$

where

$$c^{\beta\gamma}_{\alpha} \equiv c^{\beta\gamma}_{\alpha}(\Phi) := E^{\nu}\left[\Phi \int_{\mathbb{T}} \dot{h}^{\beta}(x)\dot{h}^{\gamma}(x)\ddot{h}^{\alpha}(x)dx\right].$$

- (1) (bilinearity) $c^{\beta\gamma}_{\alpha} = c^{\gamma\beta}_{\alpha}$
(2) (integration by parts on $\mathbb{T}$) $c^{\beta\gamma}_{\alpha} + c^{\gamma\alpha}_{\beta} + c^{\alpha\beta}_{\gamma} = 0$
- In particular, $c^{\alpha\alpha}_{\alpha} = 0, {}^{\forall}\alpha$. When $d = 1$, this implies $(ST)_{\mathcal{A}}$: $\int \mathcal{A}\Phi d\nu = 0$ for ${}^{\forall}\Phi$.

- (F: LNM **2137**, 2015) If Γ satisfies (T), by (2) for $c_\alpha^{\beta\gamma}$

$$\Gamma_{\beta\gamma}^\alpha c_\alpha^{\beta\gamma} = \frac{1}{3}\Gamma_{\beta\gamma}^\alpha (c_\alpha^{\beta\gamma} + c_\beta^{\gamma\alpha} + c_\gamma^{\alpha\beta}) = 0$$

Therefore, (T) implies $(ST)_{\mathcal{A}}$.

- Conversely, $(ST)_{\mathcal{A}}$ implies (T). In fact, by (2) for $c_\alpha^{\beta\gamma}$

$$\begin{aligned}-2\int \mathcal{A}\Phi d\nu =& \sum_{\alpha\neq\beta}(\Gamma_{\beta\beta}^\alpha - \Gamma_{\alpha\beta}^\beta)c_\alpha^{\beta\beta} \\ &+ 2\sum_{\alpha>\beta>\gamma}(\Gamma_{\beta\gamma}^\alpha - \Gamma_{\alpha\beta}^\gamma)c_\alpha^{\beta\gamma} + 2\sum_{\beta>\alpha>\gamma}(\Gamma_{\beta\gamma}^\alpha - \Gamma_{\alpha\beta}^\gamma)c_\alpha^{\beta\gamma}\end{aligned}$$

and $c_\alpha^{\beta\beta}, c_\alpha^{\beta\gamma} (\alpha > \beta > \gamma, \beta > \alpha > \gamma)$ move freely.

- Ertaş-Kardar's example does not satisfy (T), but has an invariant measure. This should be "non-separating class" and the invariant measure is presumably non-Gaussian (but has Gaussian marginal).

Extensions of Ertaș-Kardar's example

- Consider KPZ ($\sigma = I, \Gamma$).
- This has an invariant measure if ${}^{\exists}s \in GL(d)$, ${}^{\exists}$decomposition $\Delta = \cup_{i=1}^{k} I_i$ (disjoint) of $\{1, \dots, d\}$ such that
    - $s \circ \Gamma$ is decoupled under Δ, i.e., $(s \circ \Gamma)^{\alpha}_{\beta\gamma} = 0$ if $\{\alpha, \beta, \gamma\} \not\subset I_i$ for ${}^{\forall}i$
    - $(\sigma_i, s \circ \Gamma|_{I_i})$ are trilinear i.e., $\sigma_i \in GL(|I_i|)$ and $\sigma_i \circ (s \circ \Gamma|_{I_i})$ satisfy (T),

  where $\sigma_i = \sqrt{(\sum_{\gamma=1}^{d} s_{\gamma}^{\alpha} s_{\gamma}^{\beta})_{\alpha,\beta \in I_i}}$ and $\Gamma|_{I_i} = (\Gamma^{\alpha}_{\beta\gamma})|_{\alpha,\beta,\gamma \in I_i}$.
- Γ does not satisfy (T) in general.

One can prove infinitesimal invariance for subclasses of Φ.
(e.g., reflection-inv or shift-inv for each component)

Conjecture: For every Γ, invariant measure exists.

Summary of the talk.

1. Coupled KPZ equation (with $\sigma = I$):

$$\partial_t h^\alpha = \frac{1}{2}\partial_x^2 h^\alpha + \frac{1}{2}\Gamma^\alpha_{\beta\gamma}\partial_x h^\beta \partial_x h^\gamma + \xi^\alpha, \quad x \in \mathbb{T}.$$

2. For $^\forall\Gamma$, convergence of two approximating solutions $h^\varepsilon, \tilde{h}^\varepsilon$ and local well-posedness of coupled KPZ eq (σ, Γ).
3. For Γ satisfying (T), Wiener measure is invariant and global well-posedness of KPZ holds.
4. $(T) \iff$ "$F = G$" $\iff (ST)_{\mathcal{A}}$ for Wiener meas. ν
$\implies$ "$\Gamma F = \Gamma G$" $\iff$ Cancellation of log-renormalization factors
5. Extensions of Ertaş-Kardar's example

Thank you for your attention!