Nonequilibirum stationary states

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Stochastic dynamics out of equilibrium
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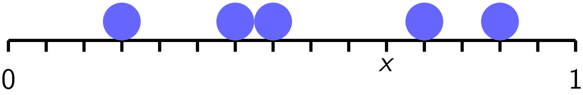
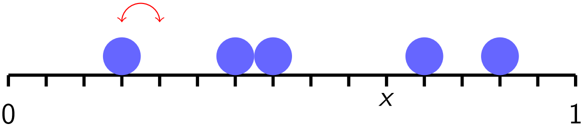
1. A reaction-diffusion model

Reaction-diffusion model

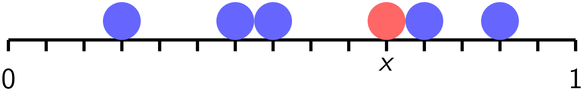
- $\mathbb{T}_N = \left\{0, 1/N, \ldots, 1 - 1/N\right\}$ $\{0,1\}^{\mathbb{T}_N}$ η $\eta(x) \in \{0,1\}$



Kawasaki dynamics speeded up by N^2:



Glauber dynamics: $c(\eta)$ local function $c(\tau_x \eta)$ $\eta(x) \to 1 - \eta(x)$



Problem

- μ_N stationary state
- $\rho_N = \rho_N(\eta)$ empirical density
- $\mu_N(\rho_N \in B) \to 1$

- Hydrodynamical limit

De Masi, Ferrari, Lebowitz (85)

- $\rho_N(0) \to \gamma(\theta)$
- $\rho_N(t) \to \rho(t, \theta)$

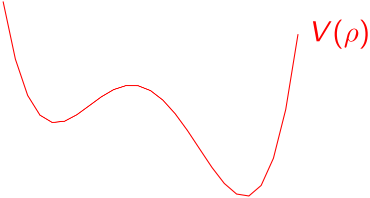
$$\begin{cases} \partial_t \rho = \Delta\rho + F(\rho) \\ \rho(0,\theta) = \gamma(\theta) \end{cases}$$

- $F(\rho) = B(\rho) - D(\rho)$
- $B(\rho) = E_{\nu_\rho}\left[\{1-\eta(0)\}c(\eta)\right] \qquad D(\rho) = E_{\nu_\rho}\left[\eta(0)c(\eta)\right]$

Hydrostatics

$$\begin{cases} \partial_t \rho \;=\; \Delta\rho \;+\; F(\rho) \\ \rho(0,\theta) = \gamma(\theta) \end{cases}$$

$$F(\rho) = -V'(\rho)$$

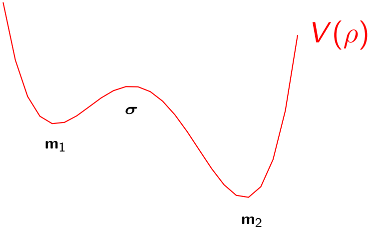


Chen, Matano (89)

- $\rho(t) \quad \lim_{t\to\infty} \rho(t) = \bar{\rho}$
- $0 = \Delta\bar{\rho} - V'(\bar{\rho})$
- S set of classical solutions
- $\mu_N(\rho_N \in \mathcal{N}_\epsilon(S)) \to 1$

Elliptic equation

- $\Delta\rho - V'(\rho) = 0$
- $\partial_t^2 \rho - V'(\rho) = 0$

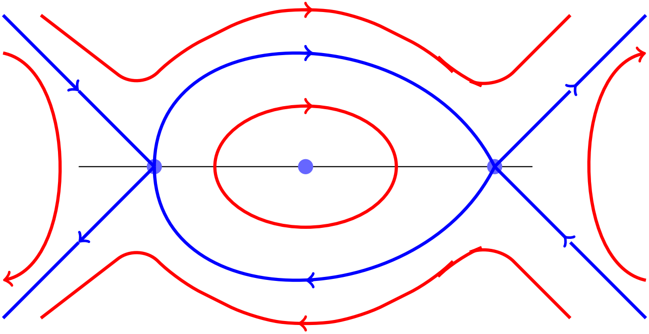


Hamiltonian dynamics

$$\begin{cases} \dot{u}(t) = v(t) \\ \dot{v}(t) = V'(u(t)) \end{cases}$$

Hamiltonian dynamics

- $\dot{u}(t) = v(t) \quad \dot{v}(t) = V'(u(t))$



- Time map $T : (m_1, m_2) \to \mathbb{R}_+$
- $u(0) = p \;\; v(0) = 0 \qquad T(p) = \inf\{t > 0 : v(t) = 0\}$

Time map

- $u(0) = p \quad v(0) = 0 \qquad T(p) = \inf\{t > 0 : v(t) = 0\}$

$\lim_{p\to\sigma} T(p)$

- $V''(\sigma) < 0 \Longrightarrow \lim_{p\to\sigma} T(p) = \pi/\sqrt{-V''(\sigma)}$
- $V''(\sigma) = 0 \Rightarrow \lim_{p\to\sigma} T(p) = +\infty$

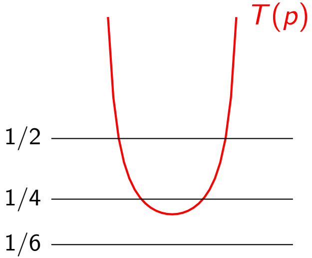
$\lim_{p\to m_1,m_2} T(p) = \infty$

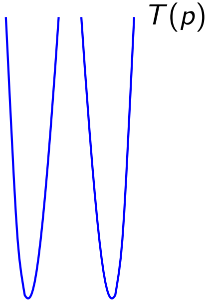
T is convex if [V is a polynomial]

- All zeroes of V' are real
- V has no critical points in (m_1, m_2) besides σ

Solutions of $\Delta u - V'(u) = 0$

- All zeroes of V' are real
- V has no critical points in (m_1, m_2) besides σ
- $2T(p) = 1/n$





Static large deviations (Farfán, L, Tsunoda)

- $B(\rho) = E_{\nu_\rho}\big[[1-\eta(0)]\, c(\eta)\big]$
- $D(\rho) = E_{\nu_\rho}\big[\eta(0)\, c(\eta)\big]$
- $-V'(\rho) = F(\rho) = B(\rho) - D(\rho)$

- $\Delta\rho = V'(\rho)$ has finite number of solutions
- $B \quad D$ concave functions

- $\mathcal{C}$ closed $\quad \mathcal{O}$ open

$$\limsup_{N\to\infty} \frac{1}{N} \log \mu_N(\rho_N \in \mathcal{C}) \;\le\; -\inf_{\vartheta\in\mathcal{C}} W(\vartheta)\;,$$
$$\liminf_{N\to\infty} \frac{1}{N} \log \mu_N(\rho_N \in \mathcal{O}) \;\ge\; -\inf_{\vartheta\in\mathcal{O}} W(\vartheta)\;.$$

W quasi-potential

- bounded, lower semicontinuous, compact level sets

Dynamical large deviations

Kipnis, Olla, Varadhan (89)   Donsker, Varadhan (89)

Jona-Lasinio, L, Vares (93)   Bodineau, Lagouge (10, 12)   L, Tsunoda (15)

- initial state η^N $\rho_N(\eta^N) \to \gamma(\theta)$

Hydrodynamical limit:

$$\mathbb{P}_{\eta^N}\big[\rho_N(t) \sim \rho_t \ , \ 0 \le t \le T\big] \ \to \ 1 \qquad \begin{cases} \partial_t \rho = \Delta\rho + F(\rho) \\ \rho(0,\theta) = \gamma(\theta) \end{cases}$$

Large deviations:

- $u_t(\theta) = u(t,\theta) \quad 0 \le t \le T$

$$\mathbb{P}_{\eta^N}\big[\rho_N(t) \sim u_t \ , \ 0 \le t \le T\big] \ \approx \ e^{-N\, I_{[0,T]}(u)}$$

Large Deviations rate function

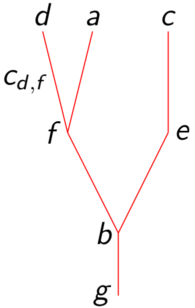
- $B(\rho) = E_{\nu_\rho}\big[\{1-\eta(0)\}\, c(\eta)\big] \quad D(\rho) = E_{\nu_\rho}\big[\eta(0)\, c(\eta)\big]$
- $F(\rho) = B(\rho) - D(\rho)$
- $\sigma(\rho) = \rho(1-\rho)$

$$\partial_t u = \Delta u + F(u) - \partial_\theta[\sigma(u)\partial_\theta H] + B(u)(e^H - 1) - D(u)(e^{-H} - 1)$$

$$\begin{aligned} I_T(u) \;=\; & \frac{1}{2}\int_0^T dt \int_{\mathbb{T}} d\theta\, \sigma(u)\,(\nabla H)^2 \\ & + \int_0^T dt \int_{\mathbb{T}} d\theta \Big\{ B(u)\,(1 - e^H + He^H) + D(u)\,(1 - e^{-H} - He^{-H}) \Big\} \end{aligned}$$

Quasi-potential

- $\bar{\rho}_1, \ldots, \bar{\rho}_p$ stationary solutions $\mathcal{M}_1, \ldots, \mathcal{M}_p$
- $V_i(\gamma) = \inf_{T>0} \inf\{I_{[0,T]}(u) : u_0 \in \mathcal{M}_i \;,\; u(T) = \gamma\}$
- $c_{i,j} = V_i(\bar{\rho}_j)$
- $T(i)$ trees i root
- $\mathfrak{g} \in T(i) \quad \kappa(\mathfrak{g}) = \sum_{(a,b)\in\mathfrak{g}} c_{a,b}$
- $w_i = \min_{\mathfrak{g}\in T(i)} \kappa(\mathfrak{g})$
- $v_i = w_i - \min_k w_k$
- $\mu_N(\mathcal{N}_\epsilon(\mathcal{M}_i)) \approx e^{-Nv_i}$



$\kappa(\mathfrak{g}) \quad \mathfrak{g} \in T(g)$

$$W(\gamma) = \min_i \{v_i + V_i(\gamma)\}$$

Heteroclinic orbits

- $\phi \to \psi \quad \lim_{t\to+\infty} \rho(t) = \psi \quad \lim_{t\to-\infty} \rho(t) = \phi$
- $\rho_j \to \rho_k \implies c_{j,k} = 0$

- $\{\bar\rho_1, \dots, \bar\rho_p\} \quad \mu_N(\rho_N \in \mathcal{N}_\epsilon(S)) \to 1$
- $\mu_N(\mathcal{N}_\epsilon(\mathcal{M}_i)) \approx e^{-Nv_i}$

- m loc. min. $V \quad \bar\rho_i = m \quad \inf_{\gamma\notin\mathcal{N}_\epsilon(\bar\rho_i)} V_i(\gamma) \geq c_\epsilon > 0 \quad \implies c_{i,\ell} > 0 \; \forall \, \ell$
- $c_{j,k} = 0 \quad c_{k,\ell} > 0 \; \forall \, \ell \implies v_j > 0$

Fiedler, Rocha, Wolfrum (04)

- $V''(\sigma_j) \neq 0 \quad (T'(p) \neq 0) \quad ((\phi \text{ hyperbolic}, \mathcal{L}_\phi))$
- ϕ non-constant solution $\quad m_i < \phi(\theta) < m_{i+1}$
- There are heteroclinic orbits $\phi \to \bar\rho_i$, $\phi \to \bar\rho_{i+1}$

- $\mu_N(\rho_N \in \mathcal{N}_\epsilon(\mathrm{loc.\ min.})) \to 1$

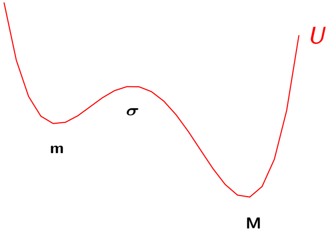
2. Potential theory for non-reversible diffusions

Problem

- $b : \mathbb{T}^d \to \mathbb{R}^d$ smooth vector field
- $b = \nabla U \quad b = \nabla U + \ell \quad \nabla U \cdot \ell = 0$
- $(\mathcal{L}_\epsilon f)(x) \ = \ \epsilon\,(\Delta f)(x) + b(x) \cdot (\nabla f)(x)$
- $\mu_\epsilon(dx) = e^{-\epsilon^{-1} U_\epsilon(x)}\, dx$
- FW: $U_\epsilon(x) = W(x) + o_\epsilon(1)$
- $\mu_\epsilon(dx) = e^{-\epsilon^{-1}[1+o_\epsilon(1)] W(x)}\, dx$
- $U_\epsilon(x) = W(x) + \epsilon W_1(x) + o(\epsilon)$
- $\mu_\epsilon(dx) = [1 + o_\epsilon(1)]\, m(x)\, e^{-\epsilon^{-1} W(x)} dx$

Small perturbations of dynamical systems

- $U : \mathbb{R}^d \to \mathbb{R}$
- $\mathbb{M}(x)$ uniformly positive-definite matrix



$$(\mathcal{L}_\epsilon f)(\mathbf{x}) \;=\; \epsilon\, e^{U(\mathbf{x})/\epsilon}\, \nabla \cdot \Big[e^{-U(\mathbf{x})/\epsilon}\, \mathbb{M}(x) (\nabla f)(\mathbf{x}) \Big]$$

$$dX_t^\epsilon \;=\; -\mathbb{M}(X_t^\epsilon)\, (\nabla U)(X_t^\epsilon)\, dt \,+\, \sqrt{2\epsilon}\, \mathbb{K}(X_t^\epsilon)\, dW_t$$

- $\mathbb{S} = (1/2)(\mathbb{M} + \mathbb{M}^\dagger) \qquad \mathbb{S} = \mathbb{K}\mathbb{K}$
- $\mu_\epsilon(dx) = Z_\epsilon^{-1}\, e^{-U(x)/\epsilon}\, dx$

Transition time

- Arrhenius' law (1889)

$$\mathbb{E}_{\mathbf{m}}\left[H_{B(\mathbf{M})}\right] \sim e^{[U(\boldsymbol{\sigma})-U(\mathbf{m})]/\epsilon}$$

- Eyring-Kramers formula (1935) [reversible dynamics]

$$\mathbb{E}_{\mathbf{m}}\left[H_{B(\mathbf{M})}\right] = \left[1+o_\epsilon(1)\right] e^{[U(\boldsymbol{\sigma})-U(\mathbf{m})]/\epsilon} \frac{2\pi}{|\lambda_{\boldsymbol{\sigma}}|}\sqrt{\frac{-\det(\text{Hess } U)(\boldsymbol{\sigma})}{\det(\text{Hess } U)(\mathbf{m})}}$$

- $\lambda_{\boldsymbol{\sigma}}$ negative e. v. of (Hess U)($\boldsymbol{\sigma}$)
- Bovier, Eckhoff, Gayrard, Klein (04)

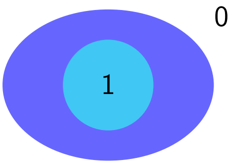
Potential theory

- $\Omega \subset \mathbb{R}^d$, $d \geq 2$ smooth, bounded domain
- $f : \mathbb{R}^d \to \mathbb{R}$
- $v : \overline{\Omega} \to \mathbb{R}$; $v = f$ on $\partial\Omega$,

$$\mathcal{E}(u) = \int_\Omega \|\nabla u(x)\|^2 \, dx$$

- $\Delta u = 0$ on Ω and $u = f$ on $\partial\Omega$.

- $A \subset D$ $\Omega = D \setminus \overline{A}$
- $f = \chi_A$ $A \cup D^c$
- $h_{A,D}$ equilibrium potential



$$\mathrm{cap}(A, D) = \inf_u \mathcal{E}(u) = \mathcal{E}(h) = -\int_{\partial A} \nabla h \cdot \mathbf{n}_\Omega \, d\sigma$$

Potential theory for nonreversible dynamics

$$(\mathcal{L}_\epsilon f)(\mathbf{x}) \;=\; \epsilon\, e^{U(\mathbf{x})/\epsilon}\, \nabla \,\cdot\, \left[e^{-U(\mathbf{x})/\epsilon}\, \mathbb{M}(x)(\nabla f)(\mathbf{x})\right]$$

$$\mu_\epsilon(dx) = Z_\epsilon^{-1}\, e^{-U(\mathbf{x})/\epsilon}\, dx$$

- $A \cap B = \varnothing$

$$\begin{cases} (\mathcal{L}_\epsilon h)(x) \;=\; 0\,, \quad x \in (A \cup B)^c\,, \\ h(x) \;=\; 1\,, \quad x \in A \quad h(x) \;=\; 0\,, \quad x \in B \end{cases}$$

$$\mathrm{cap}(A, B) \;=\; -\frac{1}{Z_\epsilon} \int_{\partial A} \left[\mathbb{M}(x)\, \nabla h_{A,B}(x)\right] \cdot \mathbf{n}_\Omega(x)\, e^{-U(x)/\epsilon} \sigma(dx)$$

- $h(x) = \mathbb{P}_x[H_A < H_B]$

$$\mathrm{cap}(A, B) \;=\; \int_{\mathbb{R}^d} \nabla h(x) \cdot \mathbb{S}(x)\, \nabla h(x)\, \mu_\epsilon(dx) \;=\; \mathcal{E}(h)$$

Variational formulae for the capacity

Pinsky (88), Doyle (94), Gaudillière, L (14), Slowik (12)

- Countable state Markov chain E
- $A \cap B = \varnothing$

$$\mathrm{cap}(A, B) \;=\; \sum_{x \in A} \mu(x)\, \lambda(x)\, \mathbb{P}_x\big[H_B < H_A^+\big]$$

$$\mathrm{cap}(A, B) \;=\; \inf_F \sup_H \Big\{ 2\langle L^* F \,,\, H\rangle_\mu \;-\; \langle H, (-S) H\rangle_\mu \Big\}$$

- $H|_A = \text{cte} \qquad H|_B = \text{cte}$
- $F|_A = 1 \qquad F|_B = 0$
- $F_{\mathrm{opt}} = (1/2)\{h_{A,B} + h^*_{A,B}\}$

Variational formulae for the capacity

Gaudillière, L. (14), Slowik (12), L., Mariani, Seo (16)

- $A \cap B = \varnothing \qquad \Omega = \mathbb{R}^d \setminus (\overline{A} \cup \overline{B})$
- $\varphi : \Omega \to \mathbb{R}^d$
- $\langle \varphi \,,\, \psi \rangle \;=\; \int_\Omega \varphi(x) \cdot \mathbb{S}(x)^{-1} \psi(x)\, e^{U(x)/\epsilon}\, dx$
- $c \in \mathbb{R} \qquad \mathcal{F}^{(c)}$
- $(\nabla \cdot \varphi) \;=\; 0 \text{ on } \Omega \qquad - \int_{\partial A} \varphi(x) \cdot \mathbf{n}(x)\, \sigma(dx) \;=\; c$
- $\mathcal{C}^{a,b}_{A,B} \quad f : \mathbb{R}^d \to \mathbb{R} \quad f|_A = a \quad f|_B = b$
- $\Phi_g \;=\; e^{-U/\epsilon}\, \mathbb{M}^\dagger \nabla g$

$$\mathrm{cap}(A,B) \;=\; \inf_{f \in \mathcal{C}^{1,0}_{A,B}} \; \inf_{\varphi \in \mathcal{F}^{(0)}} \langle \Phi_f - \varphi \,,\, \Phi_f - \varphi \rangle$$

$$\frac{1}{\mathrm{cap}(A,B)} \;=\; \inf_{\varphi \in \mathcal{F}^{(1)}} \; \inf_{f \in \mathcal{C}^{0,0}_{A,B}} \langle \varphi - \Phi_f \,,\, \varphi - \Phi_f \rangle$$

Optimal flows

- $\Phi_g \;=\; e^{-U/\epsilon}\,\mathbb{M}^\dagger \nabla g$
- $\Psi_f \;=\; e^{-U/\epsilon}\mathbb{S}\nabla f$

$$\mathrm{cap}(A,B) \;=\; \inf_{f\in\mathcal{C}^{1,0}_{A,B}} \; \inf_{\varphi\in\mathcal{F}^{(0)}} \langle \Phi_f - \varphi \,,\, \Phi_f - \varphi \rangle$$

- $f = (1/2)(h_{A,B} + h^*_{A,B})$
- $\varphi = \Phi_f - \Psi_{h_{A,B}}$

$$\frac{1}{\mathrm{cap}(A,B)} \;=\; \inf_{\varphi\in\mathcal{F}^{(1)}} \; \inf_{f\in\mathcal{C}^{0,0}_{A,B}} \langle \varphi - \Phi_f \,,\, \varphi - \Phi_f \rangle$$

- $f = (h_{A,B} - h^*_{A,B})/2\,\mathrm{cap}(A,B)$
- $\varphi = \Phi_f - \Psi_{g_{A,B}} \qquad g_{A,B} = h_{A,B}/\mathrm{cap}(A,B)$

Harmonic measure

- Harmonic measure $\nu_{A,B}$ on ∂A

$$\nu_{A,B}(dx) \;=\; \frac{1}{\mathrm{cap}(A,B)}\, \mathbb{M}^{\dagger}(x)\,(\nabla h^{*}_{A,B})(x)\cdot \mathbf{n}(x)\, \frac{1}{Z_\epsilon}\, e^{-U(x)/\epsilon}\, \sigma(dx)$$

$$\mathbb{E}_{\nu_{A,B}}\Big[\int_0^{H_B} f(X_s)\, ds\Big] \;=\; \frac{1}{\mathrm{cap}(A,B)} \int_{\mathbb{R}^d} h^{*}_{A,B}(x)\, f(x)\, \mu_\epsilon(dx)$$

Eyring-Kramers formula for non-reversible dynamics

Bouchet, Reygnier (15) L, Mariani, Seo (15)

$$\mathbb{E}_{\mathbf{m}}\left[H_{B(\mathbf{M})}\right] = \left[1 + o_\epsilon(1)\right] e^{[U(\boldsymbol{\sigma}) - U(\mathbf{m})]/\epsilon} \frac{2\pi}{|\mu_{\boldsymbol{\sigma}}|} \sqrt{\frac{-\det(\text{Hess } U)(\boldsymbol{\sigma})}{\det(\text{Hess } U)(\mathbf{m})}}$$

- $dX_t = -\mathbb{M}(X_t^\epsilon)\,(\nabla U)(X_t^\epsilon)\,dt + \sqrt{2\epsilon}\,\mathbb{K}(X_t) dW_t$
- $b(x) = \mathbb{M}(x)\,(\nabla U)(x)$
- $\mu_{\boldsymbol{\sigma}}$ negative e. v. of Jacobian $(Db)(\boldsymbol{\sigma})$

- Metastability