The Polaron Measure
A second look

Paris, June 12, 2017

$$
\begin{aligned}
dQ_{\beta,\gamma,T} \\
&= \frac{1}{Z_{\beta,\gamma,T}} \exp\left[\frac{\beta\gamma}{2}\int_0^T\int_0^T \frac{e^{-\gamma|t-s|}}{|x(t)-x(s)|}dtds\right]dP \\
&= \frac{1}{Z_{\beta,\gamma,T}} \exp\left[\beta\gamma\int_0^T\int_{0\le s<t\le T} \frac{e^{-\gamma(t-s)}}{|x(t)-x(s)|}dtds\right]dP
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- $$Z_{\beta,\gamma,T} = E^P\left[\exp\left[\beta\gamma\int\int_{0\le s<t<T} \frac{e^{-\gamma(t-s)}}{|x(t)-x(s)|}dtds\right]\right]$$

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\begin{aligned}
g(\beta,\gamma) &= \lim_{T\to\infty}\frac{1}{T}\log Z_{\beta,\gamma,T} \\
&= \sup_{Q}\left[E^{Q}\left[\beta\gamma\int_{-\infty}^{0}\frac{e^{\gamma t}}{|x(t)-x(0)|}dt\right]-H(Q|P)\right]
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$$\frac{g(\beta,\gamma)}{\beta^2} = \sup_{Q} \left[E^{Q} [\int_{-\infty}^{0} \frac{\gamma e^{\gamma t}}{\beta |x(t)-x(0)|} dt] - \frac{1}{\beta^2} H(Q|P) \right]$$

$$= \sup_{Q} \left[E^{Q} [\int_{-\infty}^{0} \frac{\gamma e^{\gamma t}}{|x(\beta^2 t)-x(0)|} dt] - H(Q|P) \right]$$

$$= g(1, \frac{\gamma}{\beta^2})$$

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- $$\lim_{\gamma\to 0} g(1,\gamma) = \lim_{\beta\to\infty} \frac{g(\beta,1)}{\beta^2} = c$$
- $$c = \sup_{\phi:\|\phi\|_2=1} \left[\int\int \frac{\phi^2(x)\phi^2(y)}{|x-y|} dx dy - \frac{1}{2}\int |\nabla\phi|^2 dx \right]$$

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- If $\beta = 1$ and let $\gamma \to 0$, then the behavior of $Q_{1,\gamma,T}$ is closely related to the behavior as $T \to \infty$ of

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- It has been studied recently by Bolthausen, König and Mukherjee.

- It is shown that the distribution of the occupation measures $\frac{1}{t}\int_0^t \delta_{x(s)} ds$ under Q_T converges to the distribution of a random translate $[\phi_0^2 * \delta_z]dx$ of $\phi_0^2 dz$,with z having the distribution $\phi_0(z)dz$ suitably normalized.

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- Since ϕ is unique only up to translation the limit will be a convex combination of translations and they determine the precise limit.

- Our goal is to understand the measure

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\begin{aligned}
dQ_{\gamma,T} &= \frac{1}{Z(\gamma,T)} \exp[\frac{1}{2}\int_0^T \int_0^T \frac{\gamma e^{-\gamma|s-t|}}{|x(t)-x(s)|} dsdt] dP_T \\
&= \frac{1}{Z(\gamma,T)} \exp[\int_{0\le s<t\le T} \frac{\gamma e^{-\gamma(t-s)}}{|x(t)-x(s)|} dsdt] dP_T \\
&= \frac{\Psi(\gamma,T,\omega)}{Z(\gamma,T)} dP_T
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- Does the limit $Q_\gamma = \lim_{T\to\infty} Q_{\gamma,T}$ exist? What is it?
- How mixing is it?

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- What about the distribution of $\frac{x(T)-x(0)}{\sqrt{T}}$ under $Q_{\gamma,T}$ or Q_γ?
- Is there a CLT?
- What is the limiting variance $\sigma^2(\gamma)$
- How does the variance behave as $\gamma \to 0$?
- According to a heuristic argument of Spohn,

$$\sigma^2(\gamma) = c\gamma^2 + o(\gamma^2)$$

- It turns out that the distribution of $\frac{x(T)-x(0)}{\sqrt{T}}$ under $Q_{\gamma,T}$ is a convex combination of spherically symmetric Gaussians, i.e. $N(0, \theta I)$ with a random θ, $0 \leq \theta \leq 1$, having a distribution depending on γ and T.

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- By a law of large numbers, as $T \to \infty$, $\theta \to \sigma^2(\gamma)$ in probability
- It is a messy formula. We have not succeeded yet in unearthing its behavior as $\gamma \to 0$. We may as well take $\gamma = 1$ and proceed.

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- Expand the exponential.

$$\Psi(T,\omega) = \exp[\int_{0\le s<t\le T} \frac{e^{-(t-s)}}{|x(t) - x(s)|} ds dt]$$

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- $$\frac{1}{\|x\|}=\sqrt{\frac{2}{\pi}}\int e^{-\frac{\tau^2\|x\|^2}{2}}d\tau$$

- $$\int_{0\le s<t\le T}e^{-(t-s)}dtds=T-1+e^{-T}=q(T)$$

$$\Psi(T,\omega) = \sum_{n=0}^{\infty} \frac{q(T)^n}{n!}$$

$$\frac{1}{q(T)} \int_{-T \leq s_1 \leq t_1 \leq T} \cdots \frac{1}{q(T)} \int_{-T \leq s_n \leq t_n \leq T} e^{-\sum_{i=1}^{n} |s_i - t_i|}$$

$$\int_0^\infty \cdots \int_0^\infty \exp[-\frac{1}{2} \sum_{i=1}^{n} \tau_i^2 \|x(s_i) - x(t_i)\|^2]$$

$$\Pi ds_i \Pi dt_i \Pi \sqrt{\frac{2}{\pi}} d\tau_i$$

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$$k(s, t : n, \Delta, \theta\}) = \sum_{i=1}^{n} \tau_i^2 \chi_{[s_i, t_i]}(s) \chi_{[s_i, t_i]}(t)$$

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- $$k(s,t : n, \Delta, \theta\}) = \sum_{i=1}^{n} \tau_i^2 \chi_{[s_i,t_i]}(s)\chi_{[s_i,t_i]}(t)$$
- and the quadratic form

$$\int_0^T |f'(t)|^2 dt$$

$$+ \int_0^T \int_0^T \sum_{i=1}^{n} \tau_i^2 \chi_{[s_i,t_i]}(s)\chi_{[s_i,t_i]}(t) f'(s) f'(t) ds dt$$

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- We have a Poisson point process on the space of intervals $[s,t] \subset [0,T]$ with intensity $e^{-(t-s)} ds dt$ on $0 \leq s < t \leq T$.
- These intervals form clusters. Do not percolate.

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- But the measure has to be weighted with $Det|I + K|^{-\frac{3}{2}}$

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- In the busy period it is random has random length with $E[L(b)] = \ell$ and $E[V(b)] = v \leq \ell$
- CLT is valid with $\sigma^2 = \frac{v+1}{\ell+1}$

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- The lengths of periods of waiting for a new dynasty, as well as the length and structure of any dynasty, depends on the time left $(T - t)$
- The birth rate at time t is $(1 - e^{-(T-t)})dt$ and the death rate is a similar perturbation of 1 that is large as $t \to T$.
- The dependence disappears as $T \to \infty$

Last Slide

THE END