Alignment processes on the sphere

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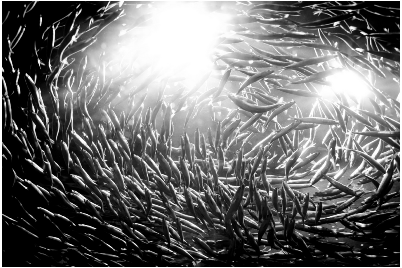
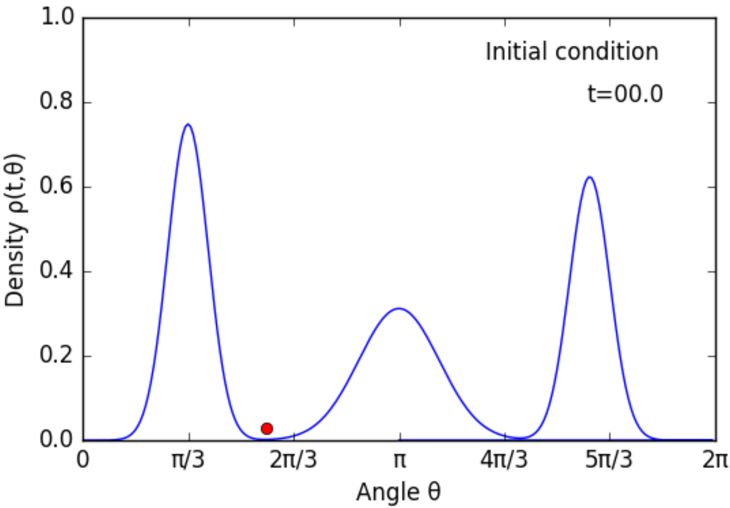
Stochastic Dynamics Out of Equilibrium
Institut Henri Poincaré, May 16th – 18th, 2017

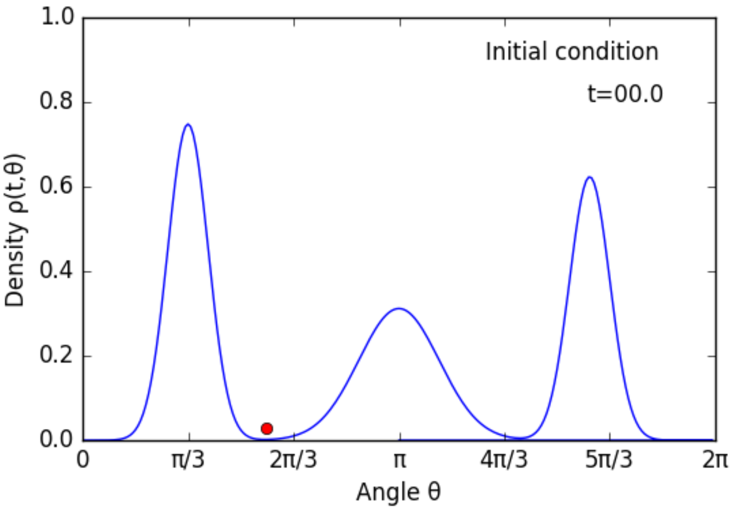
Context: alignment of self-propelled particles



- Unit speed, local interactions without leader
- Emergence of patterns

Purpose of this talk : alignment mechanisms of (kinetic) Vicsek and BDG models.
Focus on the alignment process only : no noise, no space !


Initial condition
t=00.0
Density ρ(t,θ)
1.0
0.8
0.6
0.4
0.2
0.0
0
π/3
2π/3
π
4π/3
5π/3
2π
Angle θ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Toy model in $\mathbb{R}^n$

Velocities $v_i(t) \in \mathbb{R}^n$, for $1 \leqslant i \leqslant N$. Each velocity is attracted by the others, with strengths m_j such that $\sum_{j=1}^N m_j = 1$.

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = \sum_{j=1}^{N} m_j(v_j - v_i) = J - v_i \quad \text{where } J = \sum_{i=1}^{N} m_i v_i.$$

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Conservation of momentum : J is constant. We get

$$v_i(t) = J + e^{-t}\,a_i, \text{ with } a_i = v_i(0) - J \in \mathbb{R}^n \text{ and } \sum_{i=1}^N m_i a_i = 0.$$

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Gradient flow structure

If $\mathcal{E} = \frac{1}{2}\sum_{i,j} m_i m_j \|v_i - v_j\|^2$, then $\mathcal{E} = \sum_i m_i \|v_i - J\|^2$.
Then $\nabla_{v_i}\mathcal{E} = -2m_i \frac{\mathrm{d}v_i}{\mathrm{d}t}$, and so $\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = -2\sum_i m_i |\frac{\mathrm{d}v_i}{\mathrm{d}t}|^2 = -2\mathcal{E}$.

Coupled nonlinear ordinary differential equations

Velocities $v_i(t) \in \mathbb{S}$, the unit sphere of $\mathbb{R}^n$, for $1 \leqslant i \leqslant N$.
Each velocity is attracted by the others, with strengths m_j such that $\sum_{j=1}^N m_j = 1$, under the constraint that it stays on the sphere.
Notation : $P_{v^\perp} u = u - (v \cdot u) v$ (orth. proj. if $v \in \mathbb{S}$, $u \in \mathbb{R}^n$).

$$\frac{\mathrm{d}v_i}{\mathrm{d}t} = P_{v_i^\perp} \sum_{j=1}^N m_j (v_j - v_i) = P_{v_i^\perp} J \quad \text{where } J = \sum_{i=1}^N m_i v_i.$$

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Remark : $P_{v^\perp} u = \nabla_v (v \cdot u) = -\frac{1}{2}\nabla_v(\|u - v\|^2)$ for $v \in \mathbb{S}$.

No more conservation here !

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Special cases : $J = 0$ (nothing moves. . .), or $v_i(0) = v_j(0)$, for which $(m_i, m_j) \rightsquigarrow (m_i + m_j)$.

Same gradient flow structure

Define as in the toy model :

$$\mathcal{E} = \frac{1}{2}\sum_{i,j=1}^{N} m_i m_j \|v_i - v_j\|^2 = \sum_{i,j=1}^{N} m_i m_j (1 - v_i \cdot v_j) = 1 - |J|^2 \geqslant 0.$$

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Since $\nabla_v (v \cdot u) = P_{v^\perp} u$, (for $v \in \mathbb{S}$, $u \in \mathbb{R}^n$), we get

$$\nabla_{v_i} \mathcal{E} = -2 \sum_{j=1}^{N} m_i m_j P_{v_i^\perp} v_j = -2 m_i \frac{\mathrm{d} v_i}{\mathrm{d} t}.$$

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$$\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t}\left(= -\frac{\mathrm{d}|J|^2}{\mathrm{d}t}\right) = -2\sum_{i=1}^{N} m_i \left|\frac{\mathrm{d}v_i}{\mathrm{d}t}\right|^2 = -2\sum_{i=1}^{N} m_i (|J|^2 - (v_i \cdot J)^2) \leqslant 0.$$

Then $|J|$ is increasing, so if $J(0) \neq 0$, then $J(t) \neq 0$ for all t, and $\Omega(t) = \frac{J(t)}{|J(t)|}$ is well-defined.

Convergence, relatively to Ω

$$\frac{1}{2}\frac{\mathrm{d}|J|^2}{\mathrm{d}t} = |J|^2 \sum_{i=1}^{N} m_i(1-(v_i\cdot\Omega)^2) \geqslant 0.$$

Hence $|J|^2$ is increasing, bounded and with bounded second derivative (we compute it and get that everything is continuous on $\mathbb{S}$). Therefore its derivative must converge to 0 as $t \to +\infty$.

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"Front" or "Back" particles

$$v_i(t) \cdot \Omega(t) \to \pm 1 \quad \text{as } t \to +\infty, \quad \text{for } 1 \leqslant i \leqslant N.$$

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Convergence of $\Omega(t)$? $\frac{\mathrm{d}\Omega}{\mathrm{d}t} = P_{\Omega^\perp} M\Omega$, with $M(t) = \sum_{i=1}^N m_i P_{v_i^\perp}$ (a $n\times n$ matrix). Easy to show that $|\frac{\mathrm{d}\Omega}{\mathrm{d}t}|$ is L^2 in time, but L^1 ?...

Only one can finish at the back

We denote $\lambda > 0$ the limit of $|J|$ (increasing). We have that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_i - v_j\|^2 = -|J|\,\Omega \cdot \frac{v_i + v_j}{2}\|v_i - v_j\|^2.$$

Repulsion of "back" particles

If $v_i(0) \neq v_j(0)$, we cannot have $v_i \cdot \Omega \to -1$ and $v_j \cdot \Omega \to -1$ (repulsion). Two possibilities :

- All particle are "front" : $v_i \cdot \Omega \to 1$.
- Up to renumbering, only v_N is "going to the back".

Second case : $|J| \to \sum_{i=1}^{N-1} m_i - m_N = 1 - 2m_N = \lambda$ (so $m_N < \frac{1}{2}$).

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Second case : $|J| \to \sum_{i=1}^{N-1} m_i - m_N = 1 - 2m_N = \lambda$ (so $m_N < \frac{1}{2}$).
Exponential estimate: $\|v_i - v_j\| = O(e^{-\lambda t})$ (for i, j "front").

Ω as a nearly conserved quantity

Theorem (if all particles are "front"):

There exists $\Omega_\infty \in \mathbb{S}$, and $a_i \in \{\Omega_\infty\}^\perp \subset \mathbb{R}^n$, for $1 \leqslant i \leqslant N$ such that $\sum_{i=1}^N m_i a_i = 0$ and that, as $t \to +\infty$,

$$v_i(t) = (1 - |a_i|^2 e^{-2t})\,\Omega_\infty + e^{-t}\,a_i + O(e^{-3t}) \quad \text{for } 1 \leqslant i \leqslant N,$$
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Theorem (if v_N is the only "back" particle — convexity argument):

There exists $\Omega_\infty \in \mathbb{S}$, and $a_i \in \{\Omega_\infty\}^\perp \subset \mathbb{R}^n$, for $1 \leqslant i < N$ such that $\sum_{i=1}^{N-1} m_i a_i = 0$ and that, as $t \to +\infty$,

$$v_i(t) = (1 - |a_i|^2 e^{-2\lambda t})\,\Omega_\infty + e^{-\lambda t}\,a_i + O(e^{-3\lambda t}) \quad \text{for } i \neq N,$$
$$v_N(t) = -\Omega_\infty + O(e^{-3\lambda t}), \quad \Omega(t) = \Omega_\infty + O(e^{-3\lambda t}).$$

Aggregation equation on the sphere

PDE for the empirical distribution

Define $f(t) = \sum_{i=1}^{N} m_i \delta_{v_i(t)} \in \mathcal{P}(\mathbb{S})$ (a probability measure on the sphere), then f is a weak solution of the following PDE:

$$\partial_t f + \nabla_v \cdot (f\, P_{v^\perp} J_f) = 0, \quad \text{where } J_f = \int_{\mathbb{S}} v \,\mathrm{d}f(v). \qquad (1)$$

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Theorem

Given a probability measure f_0, there exists a unique global (weak) solution to the aggregation equation given by (1).

Tools: optimal transport, or for this case harmonic analysis (Fourier for $n = 2$), which gives well-posedness in Sobolev spaces.

Properties of the model

Characteristics, if the function $J(t)$ is given

Define Φ_t as the flow of the one-particle ODE ($\frac{\mathrm{d}v}{\mathrm{d}t} = P_{v^\perp} J$) :

$$\frac{\mathrm{d}\Phi_t(v)}{\mathrm{d}t} = P_{\Phi_t(v)^\perp} J(t) \quad \text{with } \Phi_0(v) = v.$$

Then the solution to the (linear) equation $\partial_t f + \nabla_v \cdot (f\, P_{v^\perp} J) = 0$ is given by $f(t) = \Phi_t \# f_0$ (push-forward):

$$\int_{\mathbb{S}} \psi(v) \mathrm{d}(\Phi_t \# f_0)(v) = \int_{\mathbb{S}} \psi(\Phi_t(v)) \mathrm{d}f_0(v) \text{ for } \psi \in C^0(\mathbb{S}).$$

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We can perform computations exactly as for the particles:

$$\frac{\mathrm{d}}{\mathrm{d}t} J_f = \int_{\mathbb{S}} P_{v^\perp} J_f \mathrm{d}f(v) = \langle P_{v^\perp} \rangle_f J_f = M_f J_f.$$

Increase of $|J_f(t)|$, integrability of $J_f \cdot M_f J_f$. All moments are C^∞.

Convergence of $\Omega(t)$

We have $\dot\Omega = \frac{\mathrm{d}\Omega}{\mathrm{d}t} = P_{\Omega^\perp} M_f \Omega$, which is L^2 in time, but L^1 ?
After a few computations, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|\dot\Omega| = |\dot\Omega|\big(1 - \Omega\cdot M_f\Omega - \langle (u\cdot P_{\Omega^\perp} v)^2\rangle_f\big) \\ + 2|J|\langle (1-(v\cdot\Omega)^2) u\cdot P_{\Omega^\perp} v\rangle_f ,$$

where $u = \frac{\dot\Omega}{|\dot\Omega|}$ (when it is well-defined, 0 otherwise).

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where $u = \frac{\dot{\Omega}}{|\dot{\Omega}|}$ (when it is well-defined, 0 otherwise).

Lemma: Tail of a perturbed ODE (integrability)

If $\frac{\mathrm{d}x}{\mathrm{d}t} = x + g$ where x is bounded, $g \in L^1(\mathbb{R}_+)$,
then $x(t) \in L^1(\mathbb{R}_+)$.

Therefore we get that $|\dot{\Omega}|$ is integrable, and then $\Omega(t) \to \Omega_\infty \in \mathbb{S}$.

Convergence of f

Proposition: unique back (same idea of "repulsion")

Suppose that $J(t)$ is given (continuous, $|J| \geqslant c > 0$), with $\Omega(t) = \frac{J(t)}{|J(t)|}$ converging to $\Omega_\infty \in \mathbb{S}$. Then there exists a unique $v_{\text{back}} \in \mathbb{S}$ such that the solution $v(t)$ of $\frac{\mathrm{d}v}{\mathrm{d}t} = P_{v^\perp} J(t)$ with $v(0) = v_{\text{back}}$ satisfies $v(t) \to -\Omega_\infty$ as $t \to +\infty$.

Conversely, if $v(0) \neq v_{\text{back}}$, then $v(t) \to \Omega_\infty$ as $t \to \infty$.

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Theorem

Convergence in Wasserstein distance to $m\delta_{-\Omega_\infty} + (1-m)\delta_{\Omega_\infty}$, where m is the mass of $\{v_b\}$ with respect to the measure f_0. In particular, if f_0 has no atoms, then $f \to \delta_{\Omega_\infty}$.

No rate …

Toy model in $\mathbb{R}^n$: midpoint collision (sticky particles)

Speeds $v_i \in \mathbb{R}^n$ $(1 \leqslant i \leqslant N)$, Poisson clocks: $v_i, v_j \rightsquigarrow \frac{v_i+v_j}{2}$.

Kinetic version (large N) : evolution of $f_t(v) \in \mathcal{P}_2(\mathbb{R}^n)$

$$\partial_t f_t(v) = \int_{\mathbb{R}^n} f_t(v+w) f_t(v-w) \mathrm{d}w - f_t(v).$$

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$$\partial_t f_t(v) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \delta_{\frac{v_*+v'_*}{2}}(v)\, \mathrm{d}f_t(v_*)\mathrm{d}f_t(v'_*) - f(t,v).$$

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- Conservation of center of mass $\bar{v} = \int_{\mathbb{R}^n} v\, \mathrm{d}f(v)$.
- Second moment $m_2 = \int_{\mathbb{R}^n} |v-\bar{v}|^2\, \mathrm{d}f(v)$: $\frac{\mathrm{d}}{\mathrm{d}t} m_2 = -\frac{m_2}{2}$.

Exponential convergence towards a Dirac mass

$$W_2(f_t, \delta_{\bar{v}}) = W_2(f_0, \delta_{\bar{v}}) e^{-\frac{t}{4}}.$$

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Exponential convergence towards a Dirac mass

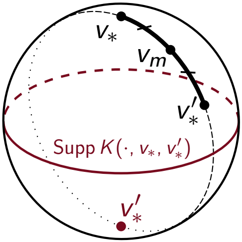
$$W_2(f_t, \delta_{\bar{v}}) = W_2(f_0, \delta_{\bar{v}})e^{-\frac{t}{4}}.$$

- Decreasing energy: $E(f) = \int_{\mathbb{R}^n\times\mathbb{R}^n} |v-u|^2\,\mathrm{d}f(v)\,\mathrm{d}f(u)$ (equal to $2m_2$), no need for $\bar{v}$ in this definition.

Midpoint model on the sphere

Kernel $K(v, v_*, v_*')$: probability density that a particle at position v_* interacting with another one at v_*' is found at v after collision.

$$\partial_t f_t(v) = \int_{\mathbb{S}\times\mathbb{S}} K(v, v_*, v_*')\, \mathrm{d}f_t(v_*)\mathrm{d}f_t(v_*') - f_t(v).$$



When $v_* \neq v_*'$, $K(\cdot, v_*, v_*') = \delta_{v_m}$, where $v_m = \frac{v_*+v_*'}{\|v_*+v_*'\|}$.

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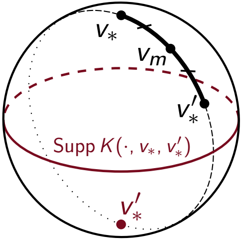

When $v_* \neq v'_*$, $K(\cdot, v_*, v'_*) = \delta_{v_m}$,
where $v_m = \frac{v_*+v'_*}{\|v_*+v'_*\|}$.

Energy

$$E(f) = \int_{\mathbb{S}\times\mathbb{S}} d(v,u)^2\, \mathrm{d}f(v)\, \mathrm{d}f(u).$$

Link "Energy – Wasserstein"

Useful Lemma – Markov inequalities

For $f \in \mathcal{P}(\mathbb{S})$, there exists $\bar{v} \in \mathbb{S}$ such that for all $v \in \mathbb{S}$:

$$W_2(f, \delta_{\bar{v}})^2 \leqslant E(f) \leqslant 4\, W_2(f, \delta_v)^2,$$

For such a $\bar{v}$ and for all $\kappa > 0$, we have

$$\int_{\{v \in \mathbb{S};\, d(v,\bar{v}) \geqslant \kappa\}} \mathrm{d}f(v) \leqslant \frac{1}{\kappa^2} E(f),$$

$$\int_{\{v \in \mathbb{S};\, d(v,\bar{v}) \geqslant \kappa\}} d(v, \bar{v})\, \mathrm{d}f(v) \leqslant \frac{1}{\kappa} E(f).$$

Evolution of the energy

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E(f) = \int_{\mathbb{S}\times\mathbb{S}\times\mathbb{S}} \alpha(v_*, v_*', u)\,\mathrm{d}f(v_*)\,\mathrm{d}f(v_*')\,\mathrm{d}f(u).$$

Local contribution to the variation of energy

$$\alpha(v_*, v_*', u) = \int_{\mathbb{S}} d(v,u)^2 K(v, v_*, v_*')\,\mathrm{d}v - \frac{d(v_*,u)^2 + d(v_*',u)^2}{2}.$$

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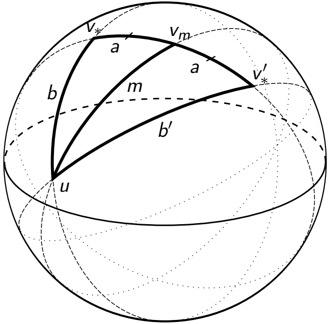

Configuration of Apollonius:

$$\alpha(v_*, v_*', u) = m^2 - \frac{b^2 + b'^2}{2}.$$

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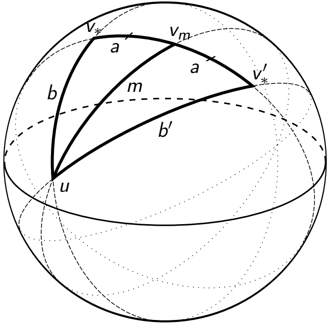

Configuration of Apollonius:

$$\alpha(v_*, v_*', u) = m^2 - \frac{b^2 + b'^2}{2}.$$

Flat case (we get $-a^2$):

$$\alpha(v_*, v_*', u) = -\frac{1}{4}d(v_*, v_*')^2.$$

Error estimates in Apollonius' formula

Lemma: global estimate (only triangular inequalities)

For all $v_*,\ v_*',\ u \in \mathbb{S}$, we have

$$\alpha(v_*, v_*', u) \leqslant -\frac{1}{4} d(v_*, v_*')^2 + 2\, d(v_*, v_*') \min\big(d(v_*, u),\ d(v_*', u)\big).$$

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Lemma: local estimate, more precise

For all $\kappa_1 < \frac{2\pi}{3}$, there exists $C_1 > 0$ such that for all $\kappa \leqslant \kappa_1$, and all v_*, v_*', $u \in \mathbb{S}$ with $\max(d(v_*, u), d(v_*', u), d(v_*, v_*')) \leqslant \kappa$, we have

$$\alpha(v_*, v_*', u) \leqslant -\frac{1}{4} d(v_*, v_*')^2 + C_1\, \kappa^2\, d(v_*, v_*')^2.$$

Spherical Apollonius: $\frac{1}{2}(\cos b + \cos b') = \cos a \cos m$.

Decreasing energy – Control on displacement

We set $\bar{\omega} := \{v \in \mathbb{S};\ d(v, \bar{v}) \leqslant \frac{1}{2}\kappa\}$, and we cut the triple integral in four parts following if v_*, v'_*, u is in $\bar{\omega}$ or not.

$$\frac{1}{2}\frac{d}{dt}E(f) + \frac{1}{4}E(f) \leqslant \underbrace{C\,\kappa^2\,E(f)}_{\text{Local lemma}} + \underbrace{12\frac{E(f)^{\frac{3}{2}}}{\kappa} + 24\frac{E(f)^2}{\kappa^2}}_{\substack{\text{Global lemma + Markov} \\ \text{(and Cauchy-Schwarz)}}}.$$

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Theorem: local stability of Dirac masses

There exists $C_1 > 0$ and $\eta > 0$ such that for all solution $f \in C(\mathbb{R}_+, \mathcal{P}(\mathbb{S}))$ with initial condition f_0 satisfying $W_2(f_0, \delta_{v_0}) < \eta$ for a $v_0 \in \mathbb{S}$, there exists $v_\infty \in \mathbb{S}$ such that

$$W_2(f_t, \delta_{v_\infty}) \leqslant C_1 W_2(f_0, \delta_{v_0})\, e^{-\frac{1}{4}t}.$$

Thanks!