RHYTHMIC COLLECTIVE BEHAVIOR IN MEAN-FIELD SYSTEMS

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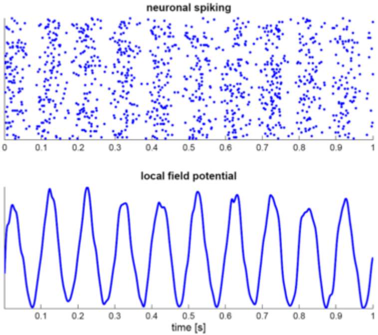
Life Sciences
Institut Henri Poincaré, Paris
May 16–18, 2017

Motivations

The emergence of self-sustained periodic behavior has been widely observed and (numerically) studied in neuroscience: neural networks (Pham, Pakdaman, Champagnat '98; Pakdaman, Perthame, Salort '10, Pakdaman, Perthame, Salort '10), nerve membranes (Fitzhugh '61), nerve axons (Nagumo, Arimoto, Yoshizawa '62),. . .



Problem

Emergence of periodicity in mean-field models

Analyze interacting systems that may exhibit a collective periodic behavior even though single units have no natural tendency of behaving periodically

MINIMAL CHARACTERISTICS

- Interaction
- Noise
- Reversibility breaking mechanism

Reversibility breaking mechanisms

DISSIPATION

- Collet, Dai Pra and Formentin. Collective periodicity in mean-field models of cooperative behavior. *NoDEA*, 22(5):1461–1482, 2015
- Dai Pra, Fischer and Regoli. A Curie-Weiss model with dissipation. *J. Stat. Phys.*, 152:37–53, 2013
- Dai Pra, Giacomin and Regoli. Noise-induced periodicity: some stochastic models for complex biological systems, in *Mathematical Models and Methods for Planet Earth*, Springer-Berlin, pp. 25-35, 2014
- Andreis and Tovazzi. Coexistence of stable limit cycles in a generalized Curie-Weiss model with dissipation. *IHP ground floor*, May 16–18, 2017

Reversibility breaking mechanisms (cont'd)

DELAY & INTERACTION NETWORK

Ditlevsen and Löcherbach. Multi-class oscillating systems of interacting neurons. *Stoch. Proc. Appl.*, 127(6): 1840–1869, 2017

Touboul. The hipster effect: When anticonformists all look the same. *arXiv preprint arXiv:1410.8001*, 2014

Reversibility breaking mechanisms (cont'd)

INTERACTION NETWORK TOPOLOGY

AIM

Understand the role of interaction network **topology** in enhancing the creation of rhythms in a spin system

Collet, Formentin and Tovazzi. Rhythmic behavior in a two-population mean field Ising model. *Phys. Rev. E*, 94(4): 042139, 2016

Two-population Curie-Weiss model

From micro...

- N particles ($i = 1, \dots, N$) with **all-to-all** coupling
- State of particle i:

$$x_i \in \{-1, +1\}$$

- **Two families** of particles ($N_1 + N_2 = N$):

$$\begin{array}{c|c} \text{Population } F_1 & \text{Population } F_2 \\ (x_1, x_2, \dots, x_{N_1} & x_{N_1+1}, \dots, x_N) \end{array}$$

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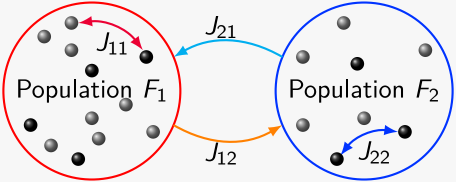
INTERACTION NETWORK



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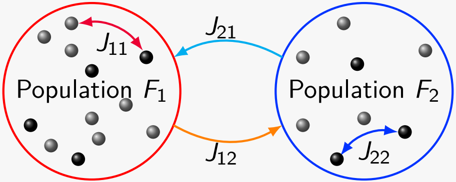
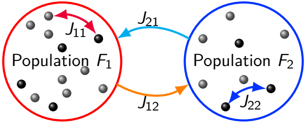


(!) **Ferromagnetic** and **anti-ferromagnetic** interactions are allowed

From micro. . . (cont'd)

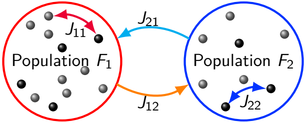


From micro... (cont'd)

Markovian evolution

$$x_i(t) \longrightarrow -x_i(t)$$

at rate



From micro... (cont'd)

Markovian evolution

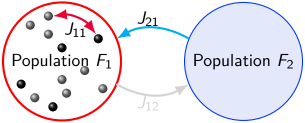
$$x_i(t) \longrightarrow -x_i(t)$$

at rate

$$e^{-x_i\left[\alpha J_{11} m_{N_1}(t)+(1-\alpha) J_{21} m_{N_2}(t)\right]}$$

if $i \in F_1$

with $\alpha = \frac{N_1}{N}$ and $m_{N_j}(t) = \frac{1}{N_j}\sum_{i\in F_j} x_i(t)$ for $j = 1, 2$



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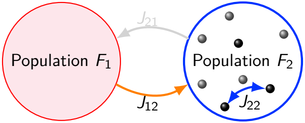

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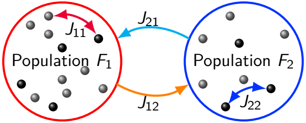

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if $i \in F_1$ if $i \in F_2$

with $\alpha = \frac{N_1}{N}$ and $m_{N_j}(t) = \frac{1}{N_j} \sum_{i \in F_j} x_i(t)$ for $j = 1, 2$

ORDER PARAMETER

$$(m_{N_1}(t), m_{N_2}(t))$$

. . . to macro

Infinite volume dynamics:

$$(\dot{m}_1(t), \dot{m}_2(t)) = V_{\alpha,J}(m_1(t), m_2(t))$$

where

$$V_{\alpha,J}(x,y) = \begin{pmatrix} 2\sinh\left[\alpha J_{11}x + (1-\alpha)J_{21}y\right] - 2x\cosh\left[\alpha J_{11}x + (1-\alpha)J_{21}y\right] \\ 2\sinh\left[\alpha J_{12}x + (1-\alpha)J_{22}y\right] - 2y\cosh\left[\alpha J_{12}x + (1-\alpha)J_{22}y\right] \end{pmatrix}$$

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Phase Transition

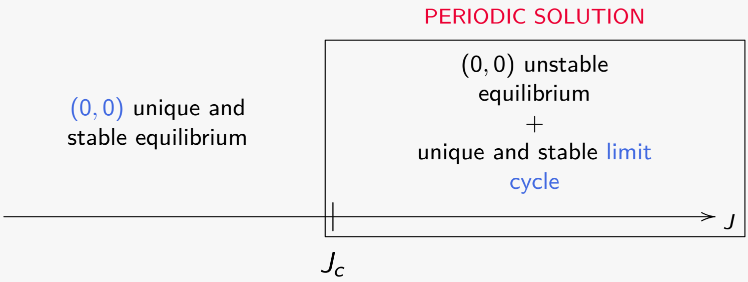
$$\begin{aligned} &\alpha J_{11} = 2 - (1-\alpha)J_{22} \\ &\left[(1-\alpha)J_{22} - 1\right]^2 + \alpha(1-\alpha)J_{12}J_{21} < 0 \end{aligned} \iff$$

HOPF BIFURCATION AT $(0,0)$

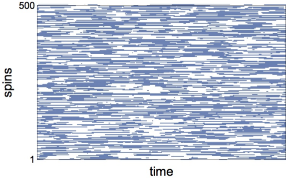
. . . to macro (cont'd)

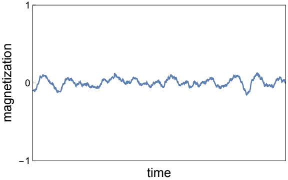
We fix α, J_{12}, J_{21} and we assume $J_{11} = J_{22} = J$.
Then,

Phase Diagram



Self-organization

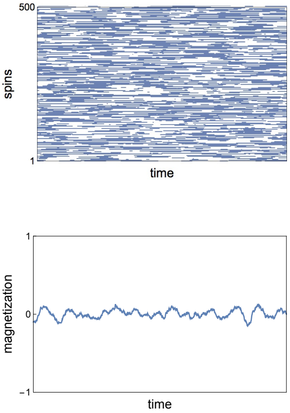




$J_{11} = J_{22} = J = 0.5$

Time evolution of population F_1. Simulations have been run with $N = 1000$, $\alpha = 0.5$, $J_{12} = -6$, $J_{21} = 5$. Critical threshold: $J_c = 2$.

Self-organization



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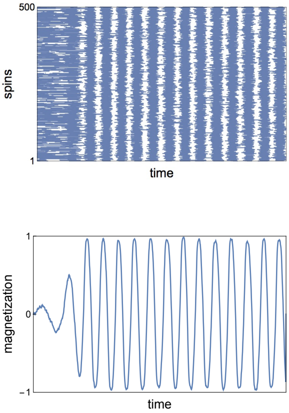

$J_{11} = J_{22} = J = 3$

Time evolution of population F_1. Simulations have been run with $N = 1000$, $\alpha = 0.5$, $J_{12} = -6$, $J_{21} = 5$. Critical threshold: $J_c = 2$.

(Partial) summary of the results

HOPF BIFURCATION THRESHOLD

$$\alpha J_{11} = 2 - (1-\alpha) J_{22} \quad \& \quad \big[(1-\alpha) J_{22} - 1\big]^2 + \alpha(1-\alpha) J_{12} J_{21} < 0$$

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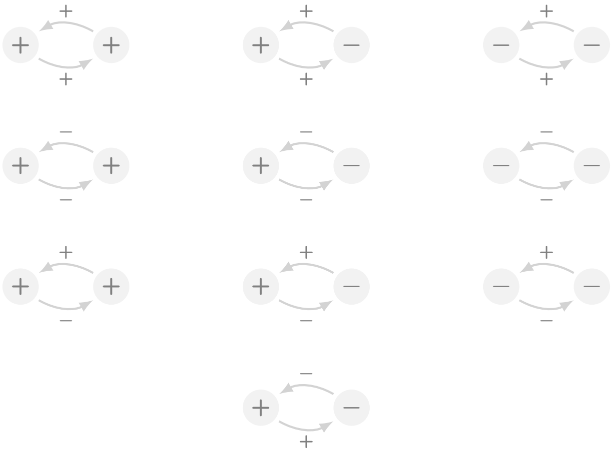

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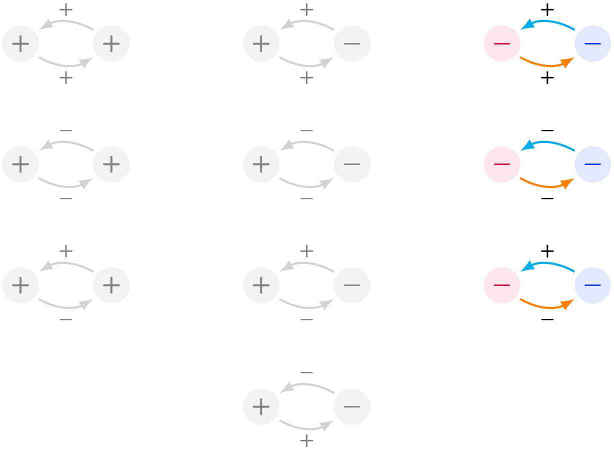

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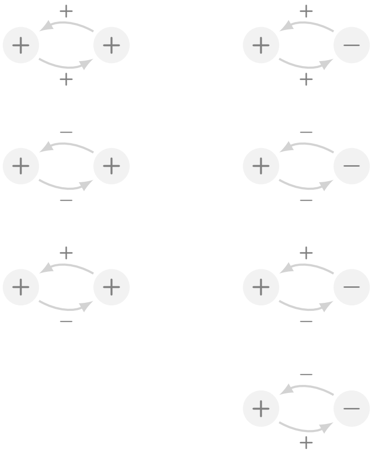

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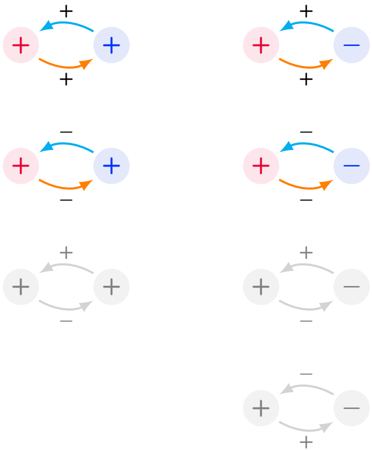

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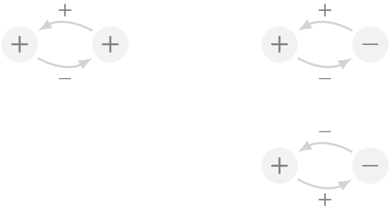

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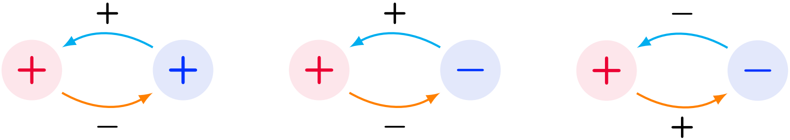
(Partial) summary of the results

FRUSTRATED NETWORKS



SELF-SUSTAINED RHYTHMIC BEHAVIOR

Adding delay

From micro...

- N particles ($i = 1, \ldots, N$) with **all-to-all** coupling
- State of particle i:

$$x_i \in \{-1, +1\}$$

- **Two families** of particles ($N_1 + N_2 = N$):

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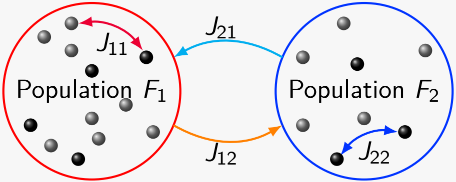
INTERACTION NETWORK



(**!**) **Ferromagnetic** and **anti-ferromagnetic** interactions are allowed

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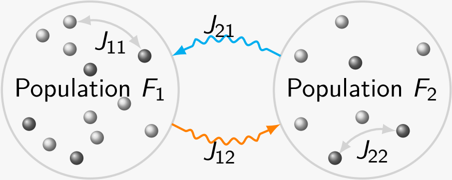


(**!**) **Ferromagnetic** and **anti-ferromagnetic** interactions are allowed

From micro... (cont'd)

Markovian evolution

$$x_i(t) \longrightarrow -x_i(t)$$

at rate

$$e^{-x_i\left[\alpha J_{11} m_{N_1}(t)+(1-\alpha)J_{21}\gamma^{(n)}_{N_2}(t)\right]} \qquad e^{-x_i\left[\alpha J_{12}\gamma^{(n)}_{N_1}(t)+(1-\alpha)J_{22} m_{N_2}(t)\right]}$$

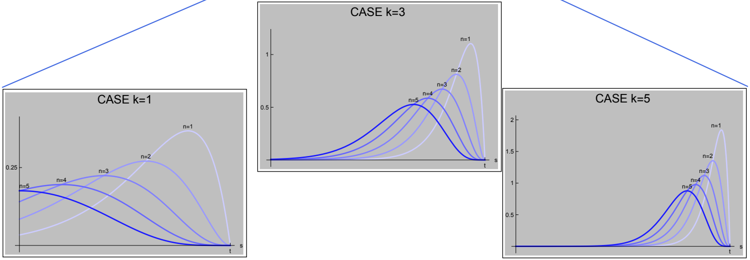
if $i \in F_1$ if $i \in F_2$

where, for $n \in \mathbb{N}$ and $k \in \mathbb{N} \setminus \{0\}$, we define

$$\gamma^{(n)}_{N_j}(t) = \int_0^t \frac{(t-s)^n}{n!} k^{n+1} e^{-k(t-s)}\, m_{N_j}(s)\, ds$$

Delay kernel features

$$\gamma_{N_j}^{(n)}(t) = \int_0^t \boxed{\frac{(t-s)^n}{n!} k^{n+1} e^{-k(t-s)}} \, m_{N_j}(s)\, ds$$



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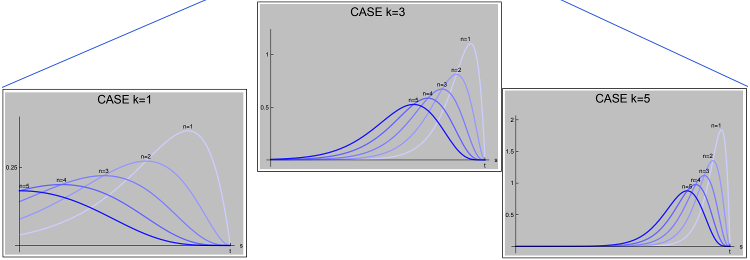


FINITE-DIMENSIONAL ORDER PARAMETER

$$\left(m_{N_1}(t), m_{N_2}(t), \gamma_{N_1}^{(0)}(t), \ldots, \gamma_{N_1}^{(n)}(t), \gamma_{N_2}^{(0)}(t), \ldots, \gamma_{N_2}^{(n)}(t) \right)$$

... to macro

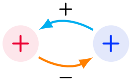
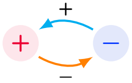
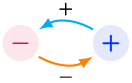
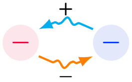
Infinite volume dynamics:

$$
\begin{aligned}
\dot{m}_1(t) &= 2\sinh[\alpha J_{11} m_1(t) + (1-\alpha) J_{21} \gamma_2^{(n)}(t)] \\
&\quad -2 m_1(t) \cosh[\alpha J_{11} m_1(t) + (1-\alpha) J_{21} \gamma_2^{(n)}(t)] \\
\dot{m}_2(t) &= 2\sinh[\alpha J_{12} \gamma_1^{(n)}(t) + (1-\alpha) J_{22} m_2(t)] \\
&\quad -2 m_2(t) \cosh[\alpha J_{12} \gamma_1^{(n)}(t) + (1-\alpha) J_{22} m_2(t)] \\
\dot{\gamma}_1^{(0)}(t) &= k[-\gamma_1^{(0)}(t) + m_1(t)] \\
\dot{\gamma}_1^{(n)}(t) &= k[-\gamma_1^{(n)}(t) + \gamma_1^{(n-1)}(t)] \quad (\text{for } n > 0) \\
\dot{\gamma}_2^{(0)}(t) &= k[-\gamma_2^{(0)}(t) + m_2(t)] \\
\dot{\gamma}_2^{(n)}(t) &= k[-\gamma_2^{(n)}(t) + \gamma_2^{(n-1)}(t)] \quad (\text{for } n > 0)
\end{aligned}
$$

GOAL

THERE EXISTS A SUBSPACE OF THE PARAMETER SPACE WHERE A HOPF BIFURCATION OCCURS

Summary of the results

DYNAMICS / INTERACTION NETWORK	**without** delay	**with** delay
	Rhythmic behavior	Rhythmic behavior
	Rhythmic behavior	Rhythmic behavior
	Rhythmic behavior	Rhythmic behavior
	~~Rhythmic behavior~~	Rhythmic behavior

Conclusions

- Network topology. A robust choice of the coupling constants and of the population sizes is sufficient for a limit cycle to arise.
- Delay. In the case when the choice of the parameters does not suffice to favor the transition to a rhythm, delay may help in this respect.
- Beyond mean-field. Emergence of a periodic behavior through frustrated dynamics is very much related to the mean-field setting.
- More general networks. . . ?!

Currently

with Luisa Andreis, Marco Formentin and Daniele Tovazzi

Understand if frustration may work also for diffusions

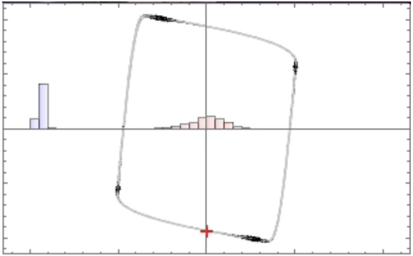
- N particles ($i = 1, \ldots, N$) with **all-to-all** coupling
- State of particle i: $x_i \in \mathbb{R}$
- **Two families**: $|F_1| = N_1$, $|F_2| = N_2$ and $N_1 + N_2 = N$
- Dynamics:

$$\text{If } i \in F_1: \; dx_i(t) = \left[-x_i^3(t) + x_i(t)\right] dt - \alpha \boldsymbol{J_{11}} \left[x_i(t) - m_{N_1}(t)\right] dt \\ - (1-\alpha) \boldsymbol{J_{21}} \left[x_i(t) - m_{N_2}(t)\right] dt + \sigma \, dw_i(t)$$

$$\text{If } i \in F_2: \; dx_i(t) = \left[-x_i^3(t) + x_i(t)\right] dt - \alpha \boldsymbol{J_{12}} \left[x_i(t) - m_{N_1}(t)\right] dt \\ - (1-\alpha) \boldsymbol{J_{22}} \left[x_i(t) - m_{N_2}(t)\right] dt + \sigma \, dw_i(t)$$

(**!**) **Positive** and **negative** interactions are allowed

Noise-induced periodicity



Time evolution of populations F_1 and F_2. Simulations have been run with $N = 1000$, $\alpha = 0.7$, $J_{11} = J_{22} = 1$, $J_{12} = 0.5$ and $J_{21} = -3.5$. Sufficiently large σ.

Thank you very much
for your attention!

References

- Collet, Dai Pra and Formentin. Collective periodicity in mean-field models of cooperative behavior. *NoDEA*, 22(5):1461–1482, 2015

- Collet, Formentin and Tovazzi. Rhythmic behavior in a two-population mean field Ising model. *Phys. Rev. E*, 94(4): 042139, 2016

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