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Entropy method for hypocoercive & non-symmetric Fokker-Planck equations with linear drift

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degenerate Fokker-Planck equations with linear drift

evolution of probability density $f(x,t)$, $x \in \mathbb{R}^n$, $t > 0$:

$$\begin{aligned} f_t &= \operatorname{div}\Big(\mathbf{D}\nabla f + \mathbf{C}x\, f\Big) \\ f(x,0) &= f_0(x) \end{aligned} \tag{1}$$

$\mathbf{D} \in \mathbb{R}^{n\times n}$ ... symmetric, const in x, degenerate

$\mathbf{C} \in \mathbb{R}^{n\times n}$ ... const in x

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goals: existence & uniqueness of steady state $f_\infty(x)$;
convergence $f(t) \stackrel{t\to\infty}{\longrightarrow} f_\infty$ with sharp rates;
complete theory for the equation class (1)

hypocoercive example – from plasma physics

kinetic Fokker-Planck equation for $f(x,v,t)\,,\ x,v\in\mathbb{R}^n$:

$$f_t + \underbrace{v\cdot\nabla_x f}_{\text{free transport}} - \underbrace{\nabla_x V\cdot\nabla_v f}_{\text{influence of potential } V(x)} = \underbrace{\sigma\Delta_v f}_{\text{diffusion, } \sigma>0} + \underbrace{\nu\ \mathrm{div}_v(vf)}_{\text{friction, } \nu>0}$$

steady state: $f_\infty(x,v) = c\, e^{-\frac{\nu}{\sigma}\left[\frac{|v|^2}{2}+V(x)\right]}$

$V(x)$... given *confinement potential*

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rewritten (with x, v variables):

$$f_t \;=\; \mathrm{div}_{x,v}\Big[\underbrace{\begin{pmatrix}0 & 0\\ 0 & \sigma\,\mathbf{I}\end{pmatrix}}_{=:\mathbf{D}\ldots\text{diffusion}}\nabla_{x,v} f + \underbrace{\begin{pmatrix}-v\\ \nabla_x V+\nu\, v\end{pmatrix}}_{\text{drift}} f\Big]$$

- $\nexists$ explicit Green's function for V not quadratic!

Outline:

1. hypocoercivity, prototypic examples
2. review of standard entropy method for non-degenerate Fokker-Planck equations
3. decay of modified "entropy dissipation" functional
4. mechanism of new method

(hypo)coercivity 1

example 1: standard Fokker-Planck equation on $\mathbb{R}^n$:

$$f_t = \operatorname{div}\Big(\nabla f + x\, f\Big) =: Lf \quad \ldots \text{ symmetric on } \; H := L^2(f_\infty^{-1})$$

$$f_\infty(x) = c e^{-\frac{|x|^2}{2}}, \qquad \ker L = \operatorname{span}(f_\infty)$$

- L is dissipative, i.e. $\langle Lf, f\rangle_H \le 0 \quad \forall f \in \mathcal{D}(L)$
- $-L$ is coercive (has a spectral gap), in the sense:

$$\langle -Lf, f\rangle_H \ge \|f\|^2_{L^2(f_\infty^{-1})} \quad \forall\, f \in \{f_\infty\}^\perp$$

(hypo)coercivity 2

example 2:

$$f_t \quad = \quad \mathrm{div}\Big(\mathbf{D}\nabla f + \mathbf{C}x\, f\Big) =: Lf \tag{2}$$

with degenerate $\mathbf{D}$ is degenerate parabolic;
(symmetric part of) $-L$ is not coercive.

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Definition 1 (Villani 2009)

Consider L on Hilbert space H with $\mathcal{K} = \ker L$; let $\tilde{H} \hookrightarrow \mathcal{K}^\perp$ (densely)
(e.g. H ... weighted L^2, $\tilde{H}$ ... weighted H^1).
$-L$ is called hypocoercive on $\tilde{H}$ if $\exists\, \lambda > 0,\; c \geq 1$:

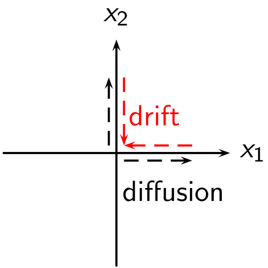
$$\|e^{Lt} f\|_{\tilde{H}} \leq c\, e^{-\lambda t} \|f\|_{\tilde{H}} \qquad \forall f \in \tilde{H}$$

- typically $c > 1$

Steady state of (non)degenerate FP equations:

standard Fokker-Planck equation $f_t = \text{div}(\nabla f + x\, f)$:
unique steady state $f_\infty(x) = c\, \mathrm{e}^{-|x|^2/2}$ as a balance of drift & diffusion;
sharp decay rate $= 1$

$n = 2$:



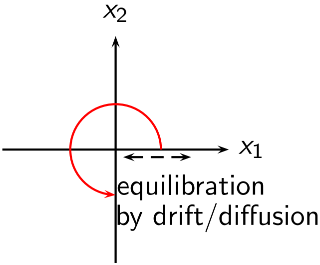
degenerate prototype:

- degenerate diffusion (1D Fokker-Planck) + rotation

$$f_t = \operatorname{div}\Big[\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{=\mathbf{D}} \nabla f + \underbrace{\begin{pmatrix} x_1 - \omega x_2 \\ \omega x_1 \end{pmatrix}}_{=\mathbf{C}x} f\Big]$$

$f_\infty(x) = c\, e^{-|x|^2/2} \qquad \forall \omega \in \mathbb{R}$ (unique for $\omega \neq 0$);

- sharp decay rate $= \frac{1}{2}$ ($= \min \Re \lambda_C$) for fast enough rotation ($|\omega| > \frac{1}{2}$)



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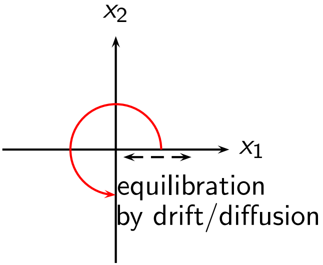
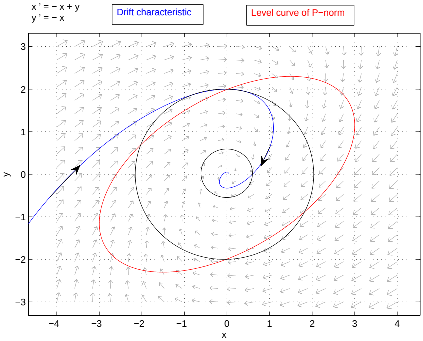


- $|\omega| = \frac{1}{2}$: **C** has a Jordan block $\Rightarrow$ (sharp) decay rate $= \frac{1}{2} - \varepsilon$

degenerate prototype with $\omega = 1$:

- $f_t = \operatorname{div}\Big[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \nabla f + \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} x\, f\Big]$
- x_2-axis: drift characteristics of $\dot{x} = -\mathbf{C}x$ tangent to level curve of $|x|$:



- level curve of "distorted" vector norm $\sqrt{x \cdot \mathbf{P} \cdot x}$; $\mathbf{P} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

Ref: [Dolbeault-Mouhot-Schmeiser] 2015

coefficients **C**, **D** in Fokker-Planck equation

$$f_t = \operatorname{div}\Big(\mathbf{D}\,\nabla f + \mathbf{C}\,x\,f\Big) =: Lf$$

Condition A: No (nontrivial) subspace of $\ker \mathbf{D}$ is invariant under $\mathbf{C}^\top$.
(equivalent: L is hypoelliptic.)

Proposition 1

Let Condition A hold.

a) *Let* $f_0 \in L^1(\mathbb{R}^d) \quad \Rightarrow \quad f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^+)$. *[Hörmander 1969]*

b) *Let* $f_0 \in L^1_+(\mathbb{R}^d) \quad \Rightarrow \quad f(x,t) > 0,\ \forall t > 0.$ *(Green's fct > 0)*

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Condition B: Condition A + let **C** be positively stable (i.e. $\Re\,\lambda_C > 0$)
$\rightarrow \exists$ confinement potential; drift towards $x = 0$.

- hypoelliptic + confinement = hypocoercive (for FP eq.)

steady state

$$f_t = \operatorname{div}\Big(\mathbf{D}\nabla f + \mathbf{C}x\, f\Big) \tag{3}$$

Theorem 2

(3) *has a unique (normalized) steady state* $f_\infty \in L^1(\mathbb{R}^n)$ *iff Condition B holds.*

Then: $f_\infty(x) = c_K e^{-\frac{x^\top \mathbf{K}^{-1} x}{2}}$ ... *non-isotropic Gaussian*

$0 < \mathbf{K} \in \mathbb{R}^{n\times n}$ ... *unique solution of* $2\mathbf{D} = \mathbf{C}\mathbf{K} + \mathbf{K}\mathbf{C}^\top$

(continuous Lyapunov equation)

normalization of Fokker-Planck equations

$$f_t = \operatorname{div}\big(\mathbf{D}\nabla f + \mathbf{C}x\, f\big) \quad \text{with } f_\infty(x) = c_K e^{-\frac{x^\top \mathbf{K}^{-1} x}{2}}$$

transformations:

1. $y := \sqrt{\mathbf{K}}^{-1} x \Rightarrow$

$$\begin{aligned} g_t &= \operatorname{div}_y\big(\tilde{\mathbf{D}}\nabla_y g + \tilde{\mathbf{C}} y\, g\big) \quad \text{with } g_\infty(x) = c e^{-\frac{|y|^2}{2}}, \\ \tilde{\mathbf{D}} &= \tilde{\mathbf{C}}_S \end{aligned}$$

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2. rotation of $y \Rightarrow \quad \check{\mathbf{D}} = \operatorname{diag}(d_1, \ldots, d_k, \underbrace{0, \ldots, 0}_{n-k})$

[normalization from now on assumed]

review of entropy method: linear symmetric Fokker-Planck equations

evolution of probability density $f(x,t)$, $x \in \mathbb{R}^n$, $t > 0$:

$$
\begin{aligned}
f_t &= \operatorname{div}\Big(\mathbf{D} \cdot [\nabla f + f \nabla A(x)]\Big) =: Lf \\
f(x,0) &= f_0(x); \quad f_0 \in L^1_+(\mathbb{R}^n),\ \int_{\mathbb{R}^n} f_0\, dx = 1 \quad \Rightarrow \quad f(x,t) \geq 0
\end{aligned}
$$

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$$f_\infty(x) = e^{-A(x)} \quad \ldots \text{ (unique) normalized steady state}$$

$$Lf = \operatorname{div}\Big(f_\infty \mathbf{D} \nabla \frac{f}{f_\infty}\Big) \quad \ldots \text{ symmetric in } L^2(\mathbb{R}^n, f_\infty^{-1})$$

$\mathbf{D} > 0$ ... positive definite matrix

$A(x)$ ... scalar confinement potential, i.e. $A(x) \to \infty$ as $|x| \to \infty$;
idea : $A(x) \gtrsim c|x|^2$

admissible relative entropies (for entropy method)

for probability densities $f_{1,2}$:

$$e_\psi(f_1|f_2) := \int_{\mathbb{R}^n} \psi\left(\frac{f_1}{f_2}\right) f_2 \, \mathrm{d}x \geq 0 \qquad \text{... relative entropy}$$

$\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ ... entropy generators

$\psi \geq 0, \quad \psi(1) = 0, \quad \psi'' > 0, \quad (\psi''')^2 \leq \frac{1}{2}\psi''\psi^{\mathrm{IV}}$

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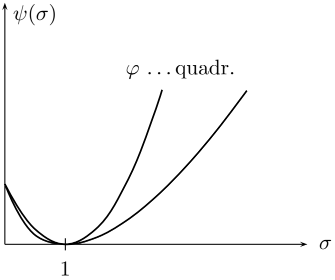
examples: 1) $\psi_1(\sigma) = \sigma \ln \sigma - \sigma + 1$

2) $\psi_p(\sigma) = \sigma^p - 1 - p\,(\sigma - 1), \quad 1 < p \leq 2$



$$e_\psi(f_1|f_2) = 0 \iff f_1 = f_2$$

entropy dissipation

Lemma 1

Let $f(t)$ solve Fokker-Planck equation $f_t = div\Big(\mathbf{D}\cdot[\nabla f + f\nabla A(x)]\Big)$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t} e_\psi(f(t)|f_\infty) \quad = \quad -\int_{\mathbb{R}^n} \psi''\left(\frac{f(t)}{f_\infty}\right) \nabla^\top \frac{f(t)}{f_\infty}\cdot \mathbf{D}\cdot\nabla\frac{f(t)}{f_\infty}\, f_\infty \,\mathrm{d}x$$

$$=: \quad -I_\psi(f(t)|f_\infty) \le 0 \ \ldots \ \textit{(negative) Fisher information}$$

Step 1: exponent. decay of entropy dissipation for $\mathbf{D} \equiv$const

$\mathbf{D} =$ const. in x, $\quad f_\infty(x) = e^{-A(x)}$

Theorem 3

Let $I_\psi(f_0|f_\infty) < \infty$. *Let* $\mathbf{D}, A$ *satisfy a*

$$\boxed{\textit{Bakry - Emery condition} \quad \frac{\partial^2 A(x)}{\partial x^2} \geq \underbrace{\lambda_1}_{>0} \mathbf{D}^{-1}} \quad \forall x \in \mathbb{R}^n \qquad (4)$$

$$\Rightarrow I_\psi(f(t)|f_\infty) \leq e^{-2\lambda_1 t} I_\psi(f_0|f_\infty), \qquad t \geq 0$$

A . . . uniformly convex if $\mathbf{D} = \mathbf{I}$

<u>Ref's:</u> [Bakry-Emery] 1984/85;
[Arnold-Markowich-Toscani-Unterreiter] Comm. PDE 2001

- robust w.r.t. many nonlinear perturbations

Step 2: exponential decay of relative entropy for $\mathbf{D} \equiv$const

Theorem 4

Let $\mathbf{D}$, *A satisfy BEC* $\quad \frac{\partial^2 A(x)}{\partial x^2} \geq \lambda_1 \mathbf{D}^{-1} \Rightarrow$

$$e_\psi(f(t)|f_\infty) \leq e^{-2\lambda_1 t} e_\psi(f_0|f_\infty), \qquad t \geq 0$$

Step 2: exponential decay of relative entropy for $\mathbf{D} \equiv$const

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Let $\mathbf{D}$, A *satisfy BEC* $\quad \frac{\partial^2 A(x)}{\partial x^2} \geq \lambda_1 \mathbf{D}^{-1} \Rightarrow$

$$e_\psi(f(t)|f_\infty) \leq e^{-2\lambda_1 t}\, e_\psi(f_0|f_\infty), \qquad t \geq 0$$

Proof: from proof of Theorem 3 :

$$\frac{\mathrm{d}}{\mathrm{d}t} I(t) \leq -2\lambda_1 \underbrace{I(t)}_{=-e'(t)} \qquad \Big| \int_t^\infty \ldots \,\mathrm{d}t$$

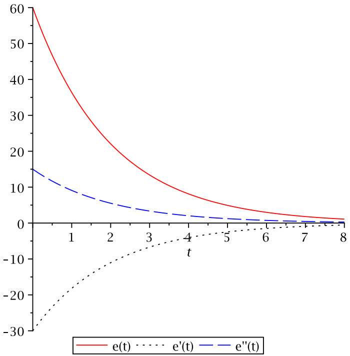
Since $I(t)$, $e(t) \xrightarrow{t\to\infty} 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t} e(t) \leq -2\lambda_1 e(t) \tag{5}$$

(+ density argument) □

problem of entropy decay (cp. standard entropy method)

decay of quadratic entropy $e_2(t) = \|f(t) - f_\infty\|^2_{L^2(f_\infty^{-1})}$:



standard Fokker-Planck equation:
non-degenerate $\rightarrow$ $e(t)$ is convex;
entropy dissip. $e'(t) < 0 \ \forall\ f \neq f_\infty$;
$e' \leq -\mu e$ possible (with $\mu > 0$)

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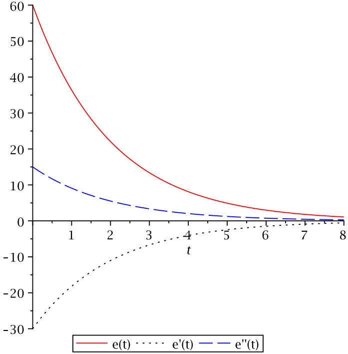

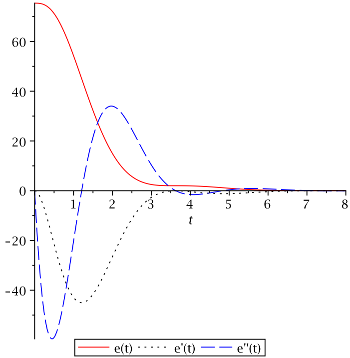

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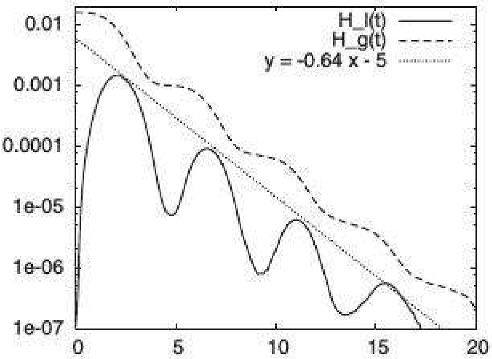
degenerate prototype ex.:
$\to$ $e(t)$ is not convex;
$e'(t) = 0$ for some $f \neq f_\infty$;
$e' \leq -\mu e$ wrong (in general)

entropy decay in inhomogeneous Boltzmann equation

- simulation of 1+2 D Boltzmann equation: wavy entropy decay
- $- - -$ $H_g(t)$: relative entropy w.r.t. the global Maxwellian



Ref: [Filbet-Mouhot-Pareschi] 2006

new entropy method for degen. FP: $f_t = \mathrm{div}(\mathbf{D}\nabla f + \mathbf{C}x\, f)$

- $e'(t) = 0$ for some $f \neq f_\infty \quad \Rightarrow \quad$ entropy dissipation:

$$\frac{\mathrm{d}}{\mathrm{d}t} e_\psi = -\int_{\mathbb{R}^n} \psi''\left(\frac{f}{f_\infty}\right) \nabla^\top \frac{f}{f_\infty} \cdot \underbrace{\mathbf{D}}_{\geq 0} \cdot \nabla \frac{f}{f_\infty}\, f_\infty \,\mathrm{d}x =: -I_\psi(f) \leq 0$$

is "useless" as Lyapunov functional.

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$\Rightarrow$ define modified “entropy dissipation” as auxiliary functional:

$$S_\psi(f) := \int_{\mathbb{R}^n} \psi''\left(\frac{f}{f_\infty}\right) \nabla^\top \frac{f}{f_\infty} \cdot \underbrace{\mathbf{P}}_{>0} \cdot \nabla \frac{f}{f_\infty} f_\infty \,\mathrm{d}x \geq 0$$

goal: estimate between $S(f(t))$, $\frac{\mathrm{d}}{\mathrm{d}t} S(f(t))$ for “good” choice of $\mathbf{P} > 0$.
Then:

$$\mathbf{P} \geq c_P\, \mathbf{D} \quad \Rightarrow \quad S_\psi(f) \geq c_P\, I_\psi(f) \quad \searrow 0$$

modified "entropy dissipation" $S_\psi(f)$: choice of $\mathbf{P}$

Lemma 2

Let $\mathbf{Q}$ *be positively stable, i.e.* $\mu := \min\{\Re\,\lambda_Q\} > 0\,.$

1. *If all* $\lambda_Q^{\min} \in \{\lambda \in \sigma(\mathbf{Q})\,|\,\Re\lambda = \mu\}$ *are non-defective (i.e. geometric = algebraic multiplicity)*
   $\Rightarrow \quad \exists\,\mathbf{P} \in \mathbb{R}^{n\times n},\ \mathbf{P} > 0\ : \quad \mathbf{P}\mathbf{Q} + \mathbf{Q}^\top\mathbf{P} \ge 2\mu\mathbf{P}\,.$

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2. *If (at least) one* $\lambda_Q^{\min}$ *is defective* $\Rightarrow$
   $\forall\, \varepsilon > 0 \quad \exists\, \mathbf{P} = \mathbf{P}(\varepsilon) > 0 \ : \quad \mathbf{P}\mathbf{Q} + \mathbf{Q}^\top \mathbf{P} \geq 2(\mu - \varepsilon)\mathbf{P}$ *.*

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2. *If (at least) one* $\lambda_Q^{\min}$ *is defective* $\Rightarrow$
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Proof: $\mathbf{P}$ can be constructed explicitly; e.g. for $\mathbf{Q}$ non-defective / diagonalizable:

$$\mathbf{P} := \sum_{j=1}^{n} z_j \otimes \bar{z}_j^\top \ ; \qquad z_j \text{ ... eigenvectors of } \mathbf{Q}^\top$$

□

- $\mathbf{P}$ not unique; but the decay rates μ (or $\mu - \varepsilon$) are independent of $\mathbf{P}$.
- application with $\mathbf{Q} := \mathbf{C}$.

Step 1: exponential decay of auxiliary functional $S_\psi(f)$

$$S_\psi(f) := \int_{\mathbb{R}^n} \psi''\left(\frac{f}{f_\infty}\right) \nabla^\top \frac{f}{f_\infty} \cdot \mathbf{P} \cdot \nabla \frac{f}{f_\infty}\, f_\infty \,\mathrm{d}x \geq 0$$

Proposition 2

$\mu := \min\{\Re\,\lambda_C\}$. *Let f_0 satisfy:*

$$\int \psi''\left(\frac{f_0}{f_\infty}\right) \left|\nabla \frac{f_0}{f_\infty}\right|^2 f_\infty \,\mathrm{d}x < \infty \quad (\sim \textit{ weighted } H^1\textit{–seminorm})$$

1. *If all $\lambda_C^{\min}$ are non-defective* $\Rightarrow \quad S(f(t)) \leq e^{-2\mu t} S(f_0)\,, \quad t \geq 0;$

Step 1: exponential decay of auxiliary functional $S_\psi(f)$

$$S_\psi(f) := \int_{\mathbb{R}^n} \psi''\left(\frac{f}{f_\infty}\right) \nabla^\top \frac{f}{f_\infty} \cdot \mathbf{P} \cdot \nabla \frac{f}{f_\infty} \, f_\infty \, \mathrm{d}x \geq 0$$

Proposition 2

$\mu := \min\{\Re \lambda_C\}$. *Let f_0 satisfy:*

$$\int \psi''\left(\frac{f_0}{f_\infty}\right) \left|\nabla \frac{f_0}{f_\infty}\right|^2 f_\infty \, \mathrm{d}x < \infty \quad (\sim \textit{ weighted } H^1\textit{–seminorm})$$

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2. *If one $\lambda_C^{\min}$ is defective* $\Rightarrow \quad S(f(t), \varepsilon) \leq e^{-2(\mu-\varepsilon)t} S(f_0, \varepsilon)\,, \quad t \geq 0.$

Proof of Proposition 2 – modified entropy method

$$\frac{\mathrm{d}}{\mathrm{d}t} S(f(t)) = -\int \psi''\Big(\frac{f}{f_\infty}\Big) u^\top \underbrace{\Big[\mathbf{PC} + \mathbf{C}^\top \mathbf{P}\Big]}_{\geq 2\mu \mathbf{P} \text{ ... replaces BEC}} u f_\infty \, \mathrm{d}x$$

$$-2 \int \underbrace{\mathrm{Tr}\,(\mathbf{XY})}_{\geq 0} f_\infty \, \mathrm{d}x \leq -2\mu \, S(f(t)) \, ; \quad u = \nabla \frac{f}{f_\infty}$$

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with Cauchy-Schwarz

Step 2: exponential decay of relative entropy

Theorem 5

Let f_0 *satisfy:*

$$\int \psi''\left(\frac{f_0}{f_\infty}\right) |u_0|^2 \, f_\infty \, \mathrm{d}x < \infty \, .$$

$$\Rightarrow \quad \boxed{e(f(t)|f_\infty) \le c\, S(f(t)) \le c\, e^{-2\mu t} S(f_0) \, , \quad t \ge 0}$$

(reduced rate for a defective $\lambda_C^{\min}$*:* $2(\mu - \varepsilon)$ *)*

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Proof: Consider non-degenerate (auxiliary) symmetric FP equation:

$$g_t = \operatorname{div}\Big(\underbrace{\mathbf{P}}_{>0}(\nabla \frac{g}{f_\infty}) f_\infty\Big) \, ; \quad g_\infty = f_\infty = c\, e^{-|x|^2/2} = c\, e^{-A(x)} \qquad (6)$$

It satisfies the Bakry-Emery condition $\frac{\partial^2 A}{\partial x^2} = \mathbf{I} \geq \lambda_P \mathbf{P}^{-1}$.

$$\Rightarrow \quad \text{convex Sobolev inequality:} \quad e_\psi(g|f_\infty) \leq \frac{1}{2\lambda_P} S_\psi(g) \quad \forall\, g$$

Remark: $S_\psi(g)$ is the true entropy dissipation for (6) ! $\square$

(parabolic) regularization of semigroup e^{Lt}

Proposition 3

The *Hörmander order* $m \in [1, n-k]$ ($k =$*rank* $\mathbf{D}$) *is the minimum such that*

$$\sum_{j=0}^{m} \mathbf{C}^j \mathbf{D} (\mathbf{C}^\top)^j \geq \kappa \, \mathbf{I} \quad \textit{for some } \kappa > 0 \, .$$

(Existence of m is equivalent to Condition A, i.e. hypoellipticity of L.)

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Ref's:

Prop. 3 is generalization to all admissible relative entropies of:

[Hérau] JFA 2007;

[Villani] book 2009 (only for quadratic & logarithmic entropies)

exp. decay of rel. entropy for $f_t = \mathrm{div}(\mathbf{D}\nabla f + \mathbf{C}x\, f) =: Lf$

combination of regularization for initial time with Th.5 (entropy decay) $\Rightarrow$

Theorem 6 (Arnold-Erb 2014)

Let L satisfy Condition B; $\mu := \min\{\Re\,\lambda_C\}$*.* $\Rightarrow \exists\, c > 0$*:*

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<u>Proof:</u>

$$e(t) \overset{\text{CSI}}{\le} \frac{1}{2\lambda_P} S(f(t)) \overset{\text{decay}}{\le} \frac{1}{2\lambda_P} \mathrm{e}^{-2\mu(t-\delta)} S(f(\delta)) \overset{\text{regularization}}{\le} c(\delta)\, e^{-2\mu t} \mathrm{e}\,(0)$$

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□

Remark: Rate μ is sharp, but constant c is not.

kinetic Fokker-Planck eq. with non-quadratic potential

$$f_t + v \cdot \nabla_x f - \nabla_x V(x) \cdot \nabla_v f = \sigma \Delta_v f + \nu \, \mathrm{div}_v(vf)\,; \quad x, v \in \mathbb{R}^n$$

steady state factors in x, v: $\quad f_\infty(x,v) = c\, e^{-\frac{\nu}{\sigma}\left[\frac{|v|^2}{2} + V(x)\right]}$

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Theorem 7 (Arnold-Erb, Achleitner-Arnold 2015)

Let $n = 1$; $V(x) = \frac{\omega_0^2}{2}|x|^2 + \tilde{V}(x)$ *... given confinement potential with* $\sqrt{\max V''(x)} - \sqrt{\min V''(x)} \le \nu$

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- $\tilde{V}$ ... $\mathcal{O}(1)$ perturbation
- Greens function not explicit

local vs. global decay rate in non-symmetric FP equations

decay of logarithmic entropy for the non-symmetric FP equation

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with $f_\infty(x) = c\, \mathrm{e}^{-|x|^2/2} = \mathrm{e}^{-A(x)}$: find $e(f(t)|f_\infty) \leq c\, e(0)\, \mathrm{e}^{-\lambda t}$

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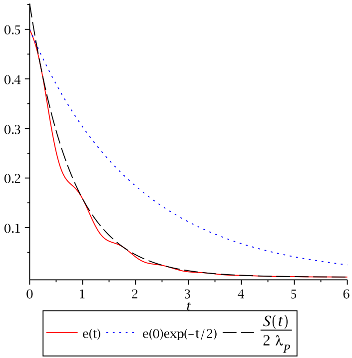

standard entropy method:
BEC $\frac{\partial^2 A}{\partial x^2} = \mathbf{I} \ge \lambda \mathbf{D}^{-1}$ yields sharp
local decay rate: $\lambda_l = \frac{1}{4}$; $c = 1$

"Hypocoercive method" yields sharp
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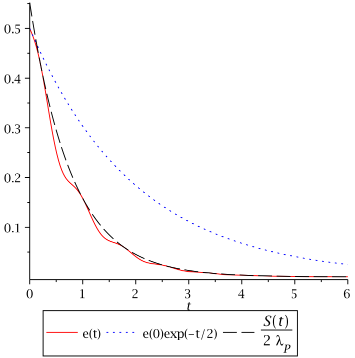

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Lemma: 2D, $\forall$ admissible $\mathbf{P}$:
multiplicative constant $\frac{S(0)}{2\lambda_P}$ is sharp.

Why does the "hypocoercive method" work?

algebraic essence: comparison of the spectral gaps of a non-symmetric matrix $\mathbf{Q}$ and its symmetric part $\mathbf{Q}_s := \frac{1}{2}(\mathbf{Q} + \mathbf{Q}^T)$.

motivation: entropy method operates mostly with quadratic functionals (e.g. $e_2(f|f_\infty)$; $I_\psi = \int \psi''(\frac{f}{f_\infty})\nabla^T \frac{f}{f_\infty} \cdot \mathbf{D} \cdot \nabla \frac{f}{f_\infty} f_\infty \, \mathrm{d}x$)

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Let $\mathbf{Q}$ *be positively stable, i.e.* $\mu := \min\{\Re \lambda_Q\} > 0$ *.*

$\Rightarrow \quad \lambda_{\min}(\mathbf{Q}_s) \le \mu \qquad$ *("typically" strict inequality)*

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Lemma 4 ($=$ Lemma 2 for choice of $\mathbf{P}$)

$\exists\, \mathbf{P} > 0$ *such that the similar matrix* $\tilde{\mathbf{Q}} := \sqrt{\mathbf{P}}\mathbf{Q}\sqrt{\mathbf{P}}^{-1}$ *satisfies:*

$$\lambda_{\min}(\tilde{\mathbf{Q}}_s) = \mu$$

($\lambda_{\min}(\tilde{\mathbf{Q}}_s) = \mu - \varepsilon \qquad$ *in the defective case*)

- So: $\exists$ a similarity transformation such that $\mathbf{Q}$, $\tilde{\mathbf{Q}}$, $\tilde{\mathbf{Q}}_s$ have the same spectral gap.

Application to $f_t = \mathrm{div}(\mathbf{D}\nabla f + \mathbf{C}xf) = Lf$

- Decay estimate of the drift characteristics $\dot{x} = -\mathbf{C}x$:

Let $\|x\|_P^2 := \langle x, \mathbf{P}x \rangle$.

$$\begin{aligned} \frac{d}{dt} \|x\|_P^2 &= -2x^T \mathbf{P}\mathbf{C}x \\ &= -(\sqrt{\mathbf{P}}x)^T \big(\underbrace{\sqrt{\mathbf{P}}\mathbf{C}\sqrt{\mathbf{P}}^{-1} + \sqrt{\mathbf{P}}^{-1}\mathbf{C}^T\sqrt{\mathbf{P}}}_{:=2\tilde{\mathbf{C}}_s \geq 2\mu \mathbf{I}} \big) (\sqrt{\mathbf{P}}x) \\ &\leq -2\mu \|x\|_P^2 \end{aligned}$$

- carries over to all invariant eigenspaces of L

Conclusion

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