Rhythmic behavior in a two-population Curie-Weiss model

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Abstract

Complex systems are characterized by the emergence of recurrent dynamical patterns and self-organized behaviors are commonly observed in biology, ecology and socioeconomics. The attempt of modeling such complex systems leads naturally to consider large families of microscopic identical units. A fundamental problem is to understand how many interacting components organize in a coherent behavior at a macroscopic level. Examples include polarization (e.g. spin alignment) and synchronization (e.g. phase locking for rotators). A less understood phenomenon of self-organization consists in the emergence of periodic behavior in systems whose units have no tendency to evolve periodically. In the present poster we discuss some dynamical features of a two-population generalization of the Curie-Weiss model with the scope of investigating simple mechanisms capable to generate a rhythm in large groups of interacting individuals. In particular, we aim at understanding the role of interaction network topology and interaction delay in enhancing the creation of rhythms. Our main finding indicates that having two groups of spins with possibly different size and different inter- and intra-population interactions suffices for the emergence of macroscopic oscillations. Moreover, delay may produce periodic behavior in interaction network configurations where otherwise absent.

1. A two-population Curie-Weiss model

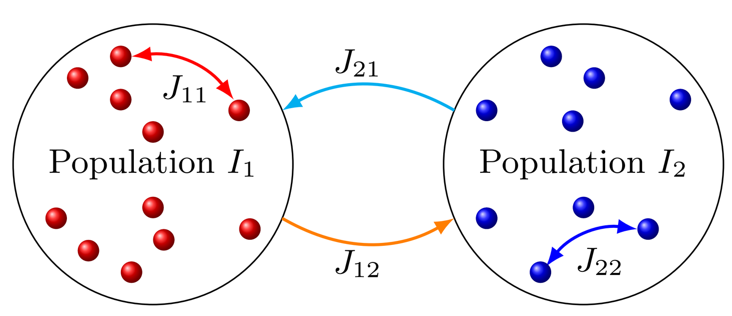
The two-population Curie-Weiss model is a mean field spin system where two types of spins are present.

- N particles on the *complete graph*.
- State of particle i: $\sigma_i \in \{-1, +1\}$.
- Populations:

$$\underbrace{(\sigma_1, \sigma_2, \dots, \sigma_{N_1}}_{\text{Population } I_1} \mid \underbrace{\sigma_{N_1+1}, \dots, \sigma_N)}_{\text{Population } I_2}$$

with $N_1 + N_2 = N$. Particles are differentiated by their mutual interactions.

- **Interaction network:** there are two *intra-group* interactions, tuning how strongly sites in the same group feel each other, and two *inter-group* interactions, giving the magnitude of the influence between particles of distinct populations.



Within I_1 (resp. I_2) particles feel a mean field interaction with coupling J_{11} (resp. J_{22}). Beside, population I_1 (resp. I_2) influences the dynamics of the other group through its magnetization with strength J_{12} (resp. J_{21}).
All the interactions can be either positive or negative allowing both ferromagnetic and anti-ferromagnetic interactions.

2. Microscopic Markovian evolutions

Microscopic dynamics without delay. At any time t the system may experience a transition whose rate depends on the magnetization vector at **time t only:**

$$\sigma_i \to -\sigma_i \text{ at rate } \begin{cases} e^{-\sigma_i R_1(m_{N_1}(t), m_{N_2}(t))} & \text{if } i \in I_1 \\ e^{-\sigma_i R_2(m_{N_1}(t), m_{N_2}(t))} & \text{if } i \in I_2, \end{cases} \qquad \text{(wD)}$$

where

- the magnetization of population I_i ($i = 1, 2$) at time t is

$$m_{N_i}(t) := \frac{1}{N_i} \sum_{j \in I_i} \sigma_j(t),$$

- the interaction functions are

$$R_1(x_1, x_2) = \alpha J_{11} x_1 + (1 - \alpha) J_{21} x_2$$
$$R_2(x_1, x_2) = (1 - \alpha) J_{22} x_2 + \alpha J_{12} x_1,$$

with α proportion of sites belonging to population I_1.

The R_i's are comprised of two terms: the first one tells how strong sites in the same population interact, while the second encodes the way one population influences the other.

Microscopic dynamics with delay. At any time t the influence of each population on the other is given by an average over the magnetization trajectory **up to time t** weighted through a delay kernel (idea borrowed from [2]):

$$\sigma_i \to -\sigma_i \text{ at rate } \begin{cases} e^{-\sigma_i R_1(m_{N_1}(t), \gamma^{(n)}_{N_2}(t))} & \text{if } i \in I_1 \\ e^{-\sigma_i R_2(\gamma^{(n)}_{N_1}(t), m_{N_2}(t))} & \text{if } i \in I_2, \end{cases} \qquad \text{(D)}$$

where, for $n \in \mathbb{N}$ and $k \in \mathbb{N} \setminus \{0\}$, we define

$$\gamma^{(n)}_{N_i}(t) = \int_0^t \frac{(t-s)^n}{n!} k^{n+1} e^{-k(t-s)} \, m_{N_i}(s) \, ds \quad (i = 1, 2).$$

3. Infinite volume dynamics

The infinite volume limits of the dynamics (wD) and (D) are deterministic. The limit as N goes to infinity must be taken in such a way the proportion α remains constant.

Macroscopic dynamics without delay. As $N \longrightarrow \infty$, the process $(m_{N_1}(t), m_{N_2}(t))_{t \geq 0}$ weakly converges to the solution of the system of ordinary differential equations

$$\begin{aligned} \dot{m}_1(t) &= 2\sinh\left[R_1\left(m_1(t), m_2(t)\right)\right] \\ &\quad - 2m_1(t)\cosh\left[R_1\left(m_1(t), m_2(t)\right)\right] \\ \dot{m}_2(t) &= 2\sinh\left[R_2\left(m_1(t), m_2(t)\right)\right] \\ &\quad - 2m_2(t)\cosh\left[R_2\left(m_1(t), m_2(t)\right)\right]. \end{aligned} \qquad \text{(MwD)}$$

Macroscopic dynamics with delay. We consider the process

$$\left(m_{N_1}(t), m_{N_2}(t), \left(\gamma^{(j)}_{N_1}(t)\right)^n_{j=0}, \left(\gamma^{(j)}_{N_2}(t)\right)^n_{j=0} \right)_{t \geq 0},$$

that, as $N \longrightarrow \infty$, weakly converges to the solution of the following system of ordinary differential equations

$$\begin{aligned} \dot{m}_1(t) &= 2\sinh\left[R_1\left(m_1(t), \gamma^{(n)}_2(t)\right)\right] \\ &\quad - 2m_1(t)\cosh\left[R_1\left(m_1(t), \gamma^{(n)}_2(t)\right)\right] \\ \dot{m}_2(t) &= 2\sinh\left[R_2\left(\gamma^{(n)}_1(t), m_2(t)\right)\right] \\ &\quad - 2m_2(t)\cosh\left[R_2\left(\gamma^{(n)}_1(t), m_2(t)\right)\right] \\ \dot{\gamma}^{(0)}_1(t) &= k\left[-\gamma^{(0)}_1(t) + m_1(t)\right] \\ \dot{\gamma}^{(n)}_1(t) &= k\left[-\gamma^{(n)}_1(t) + \gamma^{(n-1)}_1(t)\right], \text{ for } n > 0 \\ \dot{\gamma}^{(0)}_2(t) &= k\left[-\gamma^{(0)}_2(t) + m_2(t)\right] \\ \dot{\gamma}^{(n)}_2(t) &= k\left[-\gamma^{(n)}_2(t) + \gamma^{(n-1)}_2(t)\right], \text{ for } n > 0. \end{aligned} \qquad \text{(MD)}$$

Introducing delay through a kernel leads to a **finite dimensional macroscopic dynamics**. In contrast, if instead of $\gamma^{(n)}_{N_i}(t)$ we choose $\overline{\gamma}_{N_i} = m_{N_i}(t - \tau)$, with fixed $\tau > 0$ (*delayed rates*), the limiting dynamics are infinite dimensional. See for example [3] for an analysis of a mean field spin system with delayed rates.

4. Transition from disorder to rhythm

We want to detect the transition from a disordered behavior, where $m_{N_1}(\cdot)$ and $m_{N_2}(\cdot)$ fluctuate around zero, to a collective rhythmic behavior in which we have periodic motion of the magnetizations (see Figs. 1 and 2). To this aim we consider the limiting evolutions (MwD) and (MD) and derive the conditions for the presence of a **Hopf bifurcation** (see [1] for details).

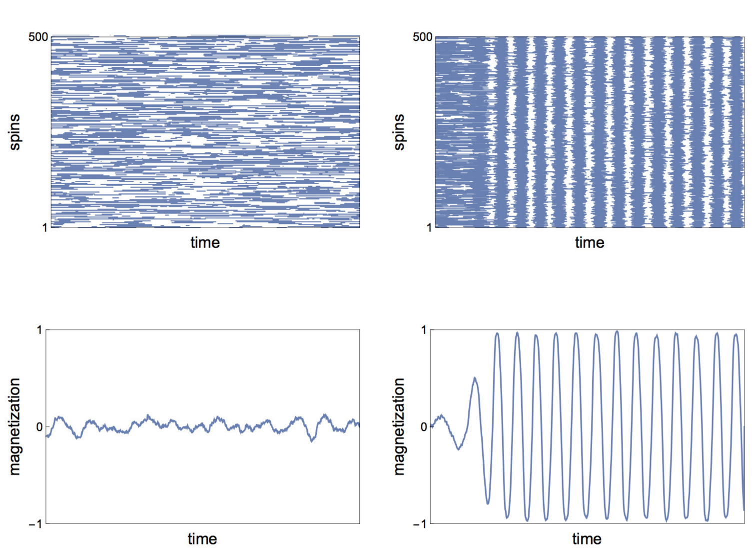


Figure 1: *Transition from disordered behavior (on the left) to collective rhythm (on the right) for the spin system* (wD). *Simulations have been run with $N = 1000$, $\alpha = 1/2$, $J_{21} = -6$, $J_{12} = 5$ and $J_{11} = J_{22} = 0.5$ on the left and $J_{11} = J_{22} = 3$ on the right. The top row shows the time evolution of all the spins belonging to population I_1. Spins are labelled from 1 to 500 on the y-axis. Blue spots represent $+1$ spins; whereas, white spots stand for -1. In the bottom line the corresponding evolution for the magnetization is depicted.*

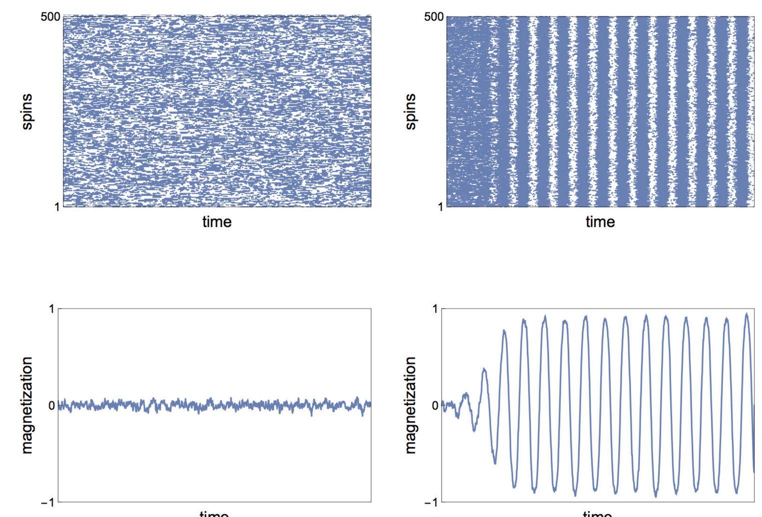


Figure 2: *Transition from disordered behavior (on the left) to collective rhythm (on the right) for the spin system* (D) *when both the intra-group interactions are negative (i.e. $J_{11}, J_{22} < 0$). Simulations have been run with $N = 1000$, $\alpha = 1/2$, $n = 2$, $J_{21} = 5$, $J_{12} = -6$ and $J_{11} = J_{22} = -4$ with $k = 6$ on the left and $J_{11} = J_{22} = -1$ with $k = 3$ on the right. The top row shows the time evolution of all the spins belonging to population I_1. Spins are labelled from 1 to 500 on the y-axis. Color convention as in Fig. 1.*

5. Conclusions (cfr. [1])

We investigated the emergence of collective periodic behavior in a two-population generalization of the Curie-Weiss model. We analyzed the role of interaction network and delay in enhancing an oscillatory evolution for the magnetization vector. We were interested in showing that it is possible to induce a transition from a disordered phase, where m_{N_1} and m_{N_2} fluctuate closely around zero, to a phase in which they both display a macroscopic regular rhythm. In particular we have proven that **a robust choice of the coupling constants and of the population sizes is sufficient for a limit cycle to arise**. Moreover, **in the case when the choice of the parameters does not suffice to favor the transition, delay may help** in this respect. See Table 1.

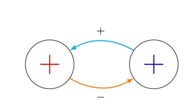
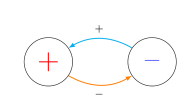
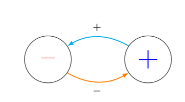
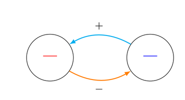
Interactions \ Dynamics	*without* delay	*with* delay
	Rhythmic behavior	Rhythmic behavior
	Rhythmic behavior	Rhythmic behavior
	Rhythmic behavior	Rhythmic behavior
	~~Rhythmic behavior~~	Rhythmic behavior

Table 1: *Qualitative summary of the results. In the left column a schematic representation of the considered interaction network is displayed. For each interaction network we highlight the possibility of observing or not observing periodic behavior when considering the dynamics* (MwD) *(central column) or* (MD) *(right column).*

References

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