Ergodicity of a system of interacting random walks with asymmetric interaction

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Abstract

We consider a model of interacting random walks on $\mathbb{N}$. We provide every particle with an intrinsic dynamics given by a biased random walk reflected in zero and with an asymmetric interaction that pushes each particle towards the origin and depends only on the fraction of particles *below* its position. We focused on the critical interaction strength above which the N particle system and its corresponding nonlinear limit have a stationary measure, balancing the tendency of the biased random walks to escape to infinity. This can be interpreted as N individuals, each associated with an integer valued *fitness*, that have an intrinsic tendency to improve their fitness in time. However, each individual mimicking only the *worse than him* may worsen his fitness. The question is whether a strong interaction can prevent some individuals from improving forever, i.e. escape towards infinity. A model with similar behavior has been studied in the continuous with diffusive dynamics, where particles interact through their CDF. The discrete model we consider displays a peculiar difference: the particles can form large clusters on a single site and, according to our description, they cannot interact. This gives rise to non-trivial expression for the critical interaction strength, unexpected from the analysis of the continuum model.

The N particle system

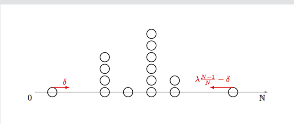
Fix a number $N \geq 2$ of particles on $\mathbb{N}$, each particle X_i^N, for $i = 1, \dots, N$, makes the following moves: if $X_i^N > 0$, then it goes to

$$X_i^N + 1 \text{ with rate } 1 + \delta, \tag{1}$$
$$X_i^N - 1 \text{ with rate } 1 + \lambda \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}(X_k^N < X_i^N),$$

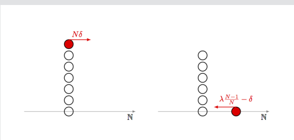
while when $X_i^N = 0$, the only allowed jump is the one on the right. It is clear that here $\delta \geq 0$ indicates a bias rightward, while $\lambda \frac{1}{N} \sum_{k=1}^{N} \mathbb{1}(X_k^N < X_i^N)$ is a bias leftward.

This gives origin to a system with the following peculiarities:

1) It is an **inhomogeneous system**, where the leftmost particle has a net drift δ and the rightmost has a net drift $\delta - \lambda \frac{N-1}{N}$.



2) Particles piled up at the same site **do not interact** and this produces a tendency for piles to spread rightward. However, as soon as a particle exits a pile from the right, it is strongly encouraged to go back.



The formation of these **clusters of particles** differentiates our model from its continuous analogue.

The mean field limit

We associate to (1) its correspondent *nonlinear Markov process*, that is a Markov process $\{X_t\}_{t \geq 0}$ whose possible transitions at time $t \geq 0$ are the following:

$$\begin{aligned} X_t + 1 \quad & \text{with rate } 1 + \delta, \\ X_t - 1 \quad & \text{with rate } 1 + \lambda \mu_t [0, X_t), \end{aligned} \tag{2}$$

where $\mu_t = Law(X_t)$ and, as in (1), when X_t=0, only the upward jump is allowed.

References

[1] L. Andreis, A. Asselah, and P. Dai Pra. Ergodicity of a system of interacting random walks with asymmetric interaction. *arXiv preprint arXiv:1702.02754*, 2017.

[2] J. R. Jackson. Jobshop-like queueing systems. *Management science*, 10(1):131–142, 1963.

[3] B. Jourdain and F. Malrieu. Propagation of chaos and poincaré inequalities for a system of particles interacting through their CDF. *The Annals of Applied Probability*, 18(5):1706–1736, 2008.

The critical interaction strength

When $\lambda = 0$ the particle system and its nonlinear limit have no stationary measure. We aim to estimate the **critical interaction strength** above which the system has a stationary measure, we indicate it as

$$\lambda_N^*(\delta) \text{ and } \lambda_\infty^*(\delta)$$

for the N particle system and the nonlinear process, respectively. If we slightly adapt the system of particles interacting through their cumulative density function (CDF) defined in [3], we get a continuous analogue of our model. In this case the *critical interaction strength* can be explicitly obtained:

$$\lambda_{N,cont}^*(\delta) = 2\delta \frac{N}{N-1} \text{ and } \lambda_{\infty,cont}^*(\delta) = 2\delta.$$

Bounds and conjectures on the critical interaction strength

By means of *ad hoc* Lyapunov functions, we get a lower and an upper bound on $\lambda_N^*(\delta)$:

$$2\delta \left[\frac{N^2(\delta+2)}{N(N-1)(\delta+2) - 2\delta} \right] \leq \lambda_N^*(\delta) \leq 8\delta^2 + 12\delta.$$

With the upper bound, which is *uniform in* N, we proved that for any $\delta \geq 0$ and any $\lambda > 8\delta^2 + 12\delta$ the Markov process X^N, defined in (1), is **exponentially ergodic**, for all $N \geq 2$. With the lower bound we highlight the difference with the *continuous analogue model*, since

$$\lambda_{N,cont}^*(\delta) < 2\delta \left[\frac{N^2(\delta+2)}{N(N-1)(\delta+2) - 2\delta} \right].$$

By means of a transformation Γ in the space $\mathcal{M}(\mathbb{N})$, for which every stationary distribution of the *nonlinear process* is a fixed point we give a lower and an upper bound on the critical value

$$2\delta \leq \lambda_\infty^*(\delta) \leq 4\delta.$$

We conjecture a more substantial difference between discrete and continuous models by means of a link with *Jackson's Networks* [2]. With a change of variables we study the dynamics of the *gaps* between successive particles and we compare it with this particular queueing system.

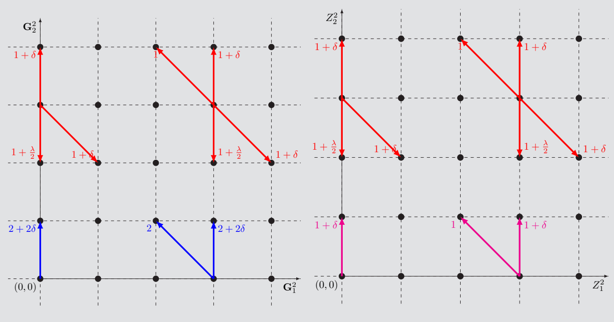


Figure 1: Rates of the *gaps process* $\mathbf{G}^2$ (left) and of the associated *Jackson network* Z^2 (right)

When $N = 2$, we define the *gaps process* $\mathbf{G}^2$ and we associate to it a particular *Jackson network*. The two processes have the same embedded Markov chain. This let us derive the exact form of

$$\lambda_2^*(\delta) = 2\delta^2 + 4\delta.$$

Moreover, this proves that, at equilibrium the position of the leftmost particle and the distance between it and the rightmost are **independent and geometrically distributed** (as in the continuous model).

For $N > 2$ the applicability of this method is still an open problem. The **piles of particles** modify the rates such that $\mathbf{G}^N$ and the *Jackson network* do not have the same embedded Markov chain anymore, however we make the following conjecture.

Conjecture. Fix $N \geq 3$, the process X^N is ergodic if, and only if,

$$(1+\delta)^N < \prod_{k=1}^{N-1} (1 + \lambda \frac{k}{N}).$$

Taking the limit as N goes to infinity, a natural conjecture is the critical interaction strength for the nonlinear process. Fix $\delta \geq 0$, then for all λ such that

$$(1 + \frac{1}{\lambda}) \ln (1 + \lambda) - 1 > \ln (1 + \delta),$$

the nonlinear process X has at least one stationary measure.